A construction of distance-regular graphs from subspaces in $d-$bounded distance-regular graphs *

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Abstract

Let Γ be a $d-$bounded distance-regular graph with diameter $d \geq 3$ and with geometric parameters (d, b, α). Pick $x \in V(\Gamma)$, and let $P(x)$ be the set of all subspaces containing x. Suppose $P(x, m)$ is the set of all subspaces in $P(x)$ with diameter m, where $1 \leq m < d$. Define a graph Γ' whose vertex-set is $P(x, m)$, and in which Δ_1 is adjacent to Δ_2 if and only if $d(\Delta_1 \cap \Delta_2) = m - 1$. We prove that Γ' is a distance-regular graph and compute its intersection numbers.

Key words: Distance-regular graph, Strongly closed subgraphs, $d-$bounded.
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1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For two vertices $u, v \in \Gamma$, let $\partial_\Gamma(u, v)$ denote the distance between u and v in Γ, i.e. the length of a shortest path connecting u and v. We also write

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$\partial(u, v)$ when no confusion occurs. Let

$$d(\Gamma) = \max\{\partial(u, v) | u, v \in v(\Gamma)\}$$

and call $d(\Gamma)$ the diameter of Γ. For $u \in V(\Gamma)$, set

$$\Gamma_i(u) \quad = \quad \{v \in V(\Gamma) | \partial_\Gamma(u, v) = i\}, \ \Gamma(u) = \Gamma_1(u).$$

For vertices $u, v \in \Gamma$ with $\partial(u, v) = i$, set

$$\begin{aligned} C(u,v) &= C_i(u,v) = \Gamma_{i-1}(u) \cap \Gamma(v), \\ A(u,v) &= A_i(u,v) = \Gamma_i(u) \cap \Gamma(v), \\ B(u,v) &= B_i(u,v) = \Gamma_{i+1}(u) \cap \Gamma(v). \end{aligned}$$

For the cardinalities we use lower case letters, i.e.

$$\begin{aligned} c_i &= c_i(u,v) = |C_i(u,v)|, \\ a_i &= a_i(u,v) = |A_i(u,v)|, \\ b_i &= b_i(u,v) = |B_i(u,v)|. \end{aligned}$$

A connected graph Γ is said to be *distance-regular* if c_i, a_i, b_i are well-defined for all $i, 0 \leq i \leq d$, i.e. these numbers depend only on i rather than the individual choice of vertices. The constants c_i, a_i and b_i $(0 \leq i \leq d)$ are known as the *intersection numbers* of Γ.

The reader is referred to [1,2,3] for general theory of distance-regular graphs.

For a subset $\Delta \subset V(\Gamma)$, we identify Δ with the induced subgraph on Δ and write $\Delta = (V(\Delta), E(\Delta))$. Denote by $d(\Delta)$ the diameter of a subgraph Δ.

A subgraph Δ of Γ is said to be *strongly closed* if $C(u, v) \cup A(u, v) \subset \Delta$ for every pair of vertices $u, v \in \Delta$. Properties of strongly closed subgraphs of distance-regular graphs are discussed first by H. Suzuki in [9]. A *subspace* of Γ is a regular strongly closed subgraph ([11]). It is obvious the strongly closed subgraphs are connected and for all $u, v \in \Delta$, $\partial_\Gamma(u, v) = \partial_\Delta(u, v)$.

We use $\langle\langle x, y\rangle\rangle$ to denote the smallest strongly closed subgraph containing x and y for $x, y \in V(\Gamma)$.

Let Γ be a distance-regular graph with diameter d. Γ is said to be *$d-$bounded,* if the following two conditions hold:

(i) Every strongly closed subgraph of Γ is regular,

(ii) For all $x, y \in V(\Gamma)$, x and y are contained in a common strongly closed subgraph of diameter $\partial(x, y)$.

It is clear that every strongly closed subgraph in a $d-$bounded distance-regular graph is a subspace.

A distance-regular graph Γ is said to have *classical parameters* (d, b, α, β) whenever the diameter of Γ is d, and the intersection numbers of Γ satisfy

$$\begin{aligned} c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right), 0 \le i \le d, \\ b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), 0 \le i \le d, \end{aligned}$$

where

$$\begin{bmatrix} i \\ l \end{bmatrix}_b = \begin{cases} \prod_{j=0}^{l-1} \frac{i-j}{l-j} = \binom{i}{l} & \text{if } b = 1, \\ \prod_{j=0}^{l-1} \frac{b^i - b^j}{b^l - b^j} & \text{if } b \neq 1 \end{cases}$$

are Gaussian binomial coefficients with basis b.

A distance-regular graph Γ with classical parameters (d, b, α, β) is said to have *geometric parameters* (d, b, α) if $\beta = \alpha(1 + b^d)/(1 - b)$, $b \neq -1$.

The following two classes of distance-regular graphs have geometric parameters.

Example 1. Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let b denote a complex number. Then the following (a)-(b) are equivalent [6].

(a) $-b$ is a power of a prime, and Γ is the dual polar graph ${}^2A_{2d-1}(-b)$.

(b) Γ has geometric parameters (d, b, α), where $\alpha = b(b-1)/(b+1)$.

Example 2. Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let b denote a complex number. Then the following (a)-(b) are equivalent [7, 8, 10].

(a) $-b$ is a power of a prime, and Γ is the Hermitian forms graph $Her_{-b}(d)$.

(b) Γ has geometric parameters (d, b, α), where $\alpha = b - 1$.

Let Γ be a $d-$bounded distance-regular graph with diameter $d \geq 3$ and with geometric parameters (d, b, α). Pick $x \in V(\Gamma)$, and let $P(x)$ be the set of all subspaces containing x. Suppose $P(x, m)$ is the set of all subspaces in $P(x)$ with diameter m, where $1 \leq m < d$. Define a graph Γ' whose vertex-set is $P(x, m)$, and in which Δ_1 is adjacent to Δ_2 if and only if $\mathrm{d}(\Delta_1 \cap \Delta_2) = m - 1$.

Remark: The construction itself is directly analogous to the construction of the Grassmann graph.

The following is our main result.

Theorem 1.1. *Let $\Gamma' = (V', E')$ be the graph constructed above. Then Γ' is a distance-regular graph with diameter* $\min(d - m, m)$ *and intersection numbers*

$$\begin{aligned} k' &= b'_0 = b^2 \begin{bmatrix} m \\ 1 \end{bmatrix}_{b^2} \begin{bmatrix} d-m \\ 1 \end{bmatrix}_{b^2}, \\ c'_t &= \left(\begin{bmatrix} t \\ 1 \end{bmatrix}_{b^2} \right)^2, \\ b'_t &= b^{4t+2} \begin{bmatrix} m-t \\ 1 \end{bmatrix}_{b^2} \begin{bmatrix} d-m-t \\ 1 \end{bmatrix}_{b^2}, \end{aligned}$$

where $1 \leq t \leq \min(d - m, m)$ *and* $\begin{bmatrix} h \\ 1 \end{bmatrix}_{b^2}$ *are Gaussian binomial coefficients with basis b^2.*

2 Proof of Theorem 1.1

Let Γ be a $d-$bounded distance-regular graph, and let Δ, Δ' be two subspaces in Γ. The smallest subspace containing Δ and Δ' is called the *join* of Δ and Δ' and denoted by $\Delta + \Delta'$.

In [11], Chih-wen Weng obtained the following two important results.

Proposition 2.1. *([11] Lemma 4.2, 4.5) Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a $d-$bounded distance-regular graph with diameter d. Then the following (i)-(iii) hold.*

(i) The intersection of two subspaces is either a subspace or the empty set.

(ii) Let Δ be a subspace of Γ, and $0 \le i \le \mathrm{d}(\Delta)$. Then Δ is distance-regular with intersection numbers

$$\begin{aligned} c_i(\Delta) &= c_i, \\ a_i(\Delta) &= a_i, \\ b_i(\Delta) &= b_i - b_{\mathrm{d}(\Delta)}. \end{aligned}$$

(iii) For any $x, y \in V(\Gamma)$, the subspace of diameter $\partial(x,y)$ containing x, y is unique.

Proposition 2.2. *([11] Lemma 5.5) Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d–bounded distance-regular graph with geometric parameters (d, b, α) and with diameter $d \ge 3$. For any subspaces Δ and Δ' in Γ, if $\Delta \cap \Delta' \neq \emptyset$, then*

$$\mathrm{d}(\Delta) + \mathrm{d}(\Delta') = \mathrm{d}(\Delta \cap \Delta') + \mathrm{d}(\Delta + \Delta').$$

Lemma 2.3. *Let Γ be a d–bounded distance-regular graph with geometric parameters (d, b, α) and with diameter $d \ge 2$. Suppose Δ and Δ' are strongly closed subgraphs with diameter i and $i + s + t \le d$, respectively, and with $\Delta \subset \Delta'$. Then the number of the strongly closed subgraphs $\widetilde{\Delta}$ with diameter $i + s$ satisfying $\Delta \subset \widetilde{\Delta} \subset \Delta'$, denoted by $N(i, i+s; i+s+t)$, is determined by i, s and t, independent of the choice of Δ and Δ' and is given by*

$$\begin{bmatrix} s+t \\ s \end{bmatrix}_{b^2},$$

where $\begin{bmatrix} s+t \\ s \end{bmatrix}_{b^2}$ is a Gaussian binomial coefficient with basis b^2.

Proof. By Lemma 2.1 of [4], we have $N(i, i+s; i+s+t)$ is independent of the choice of Δ and Δ', and

$$N(i, i+s; i+s+t) = \frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}.$$

Since Γ is a d–bounded distance-regular graph with geometric parameters (d, b, α), we have

$$b \neq -1, \quad \beta = \alpha\frac{1+b^d}{1-b}$$

and

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\alpha\frac{1+b^d}{1-b} - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}_b\right) \\ &= -\alpha\frac{b^{2d}-b^{2i}}{(b-1)^2}. \end{aligned}$$

It follows that

$$\begin{aligned} & N(i, i+s; i+s+t) \\ = & \frac{(b^{2(i+s+t)} - b^{2i})(b^{2(i+s+t)} - b^{2(i+1)}) \cdots (b^{2(i+s+t)} - b^{2(i+s-1)})}{(b^{2(i+s)} - b^{2i})(b^{2(i+s)} - b^{2(i+1)}) \cdots (b^{2(i+s)} - b^{2(i+s-1)})} \\ = & \frac{(b^{2(s+t)} - b^0)(b^{2(s+t)} - b^2) \cdots (b^{2(s+t)} - b^{2(s-1)})}{(b^{2s} - b^0)(b^{2s} - b^2) \cdots (b^{2s} - b^{2(s-1)})} \\ = & \begin{bmatrix} s+t \\ s \end{bmatrix}_{b^2}. \end{aligned}$$

□

Lemma 2.4. *Let Γ be a $d-$bounded distance-regular graph with diameter $d \geq 3$ and with geometric parameters (d, b, α). Pick $x \in V(\Gamma)$, and let $P(x)$ be the set of all subspaces containing x. Let Δ_1, Δ and $\overline{\Delta}$ be subspaces in $P(x)$ such that $\Delta_1 \subset \Delta \subset \overline{\Delta}$ with diameter t, $i+t$ and d_1, respectively, where $0 \leq t \leq i+t, j+t \leq i+j+t \leq d_1 \leq d$. Then the number of subspaces Δ' in $\overline{\Delta}$ with diameter $j+t$ such that $\Delta \cap \Delta' = \Delta_1$ is independent of the choice of Δ and Δ_1, is denoted by $M_1(t, i+t, j+t; d_1)$, and is given by*

$$M_1(t, i+t, j+t; d_1) = b^{2ij}\begin{bmatrix} d_1 - i - t \\ j \end{bmatrix}_{b^2}.$$

Furthermore, the number of subspaces Δ' in $\overline{\Delta}$ with diameter $j+t$ such that $\mathrm{d}(\Delta \cap \Delta') = t$ is independent of the choice of Δ, is denoted by $M(t, i+t, j+t; d_1)$, and is given by

$$M(t, i+t, j+t; d_1) = b^{2ij}\begin{bmatrix} d_1 - i - t \\ j \end{bmatrix}_{b^2}\begin{bmatrix} i+t \\ t \end{bmatrix}_{b^2},$$

where $\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}_{b^2}$ are Gaussian binomial coefficients with basis b^2.

Proof. By a similar argument to the proof of Lemma 2.1 of [5], we have $M_1(t, i+t, j+t; d_1)$ is independent of the choice of Δ and Δ_1, and

$$\begin{aligned} & M_1(t, i+t, j+t; d_1) \\ = & \frac{(b_{i+t} - b_{d_1})(b_{i+t+1} - b_{d_1}) \cdots (b_{i+t+j-1} - b_{d_1})}{(b_t - b_{j+t})(b_{t+1} - b_{j+t}) \cdots (b_{t+j-1} - b_{j+t})} \\ = & \frac{(b^{2d_1} - b^{2(i+t)})(b^{2d_1} - b^{2(i+t+1)}) \cdots (b^{2d_1} - b^{2(i+t+j-1)})}{(b^{2(j+t)} - b^{2t})(b^{2(j+t)} - b^{2(t+1)}) \cdots (b^{2(j+t)} - b^{2(j+t-1)})} \\ = & b^{2ij} \frac{(b^{2d_1-i-t} - b^0)(b^{2d_1-i-t} - b^2) \cdots (b^{2d_1-i-t} - b^{2(j-1)})}{(b^{2j} - b^0)(b^{2j} - b^2) \cdots (b^{2j} - b^{2(j-1)})} \\ = & b^{2ij} \begin{bmatrix} d_1 - i - t \\ j \end{bmatrix}_{b^2} . \end{aligned}$$

It follows that

$$M(t, i+t, j+t; d_1) = b^{2ij} \begin{bmatrix} d_1 - i - t \\ j \end{bmatrix}_{b^2} \begin{bmatrix} i+t \\ t \end{bmatrix}_{b^2} .$$

□

Lemma 2.5. *Let $\Gamma' = (V', E')$ be the graph constructed above. For any Δ_1, $\Delta_2 \in V'$, $\partial_{\Gamma'}(\Delta_1, \Delta_2) = i$ if and only if $\mathrm{d}(\Delta_1 \cap \Delta_2) = m - i$, where $0 \le i \le \min(m, d-m)$.*

Proof. Suppose that $\mathrm{d}(\Delta_1 \cap \Delta_2) = m - i$. Then from Proposition 2.2 $\mathrm{d}(\Delta_1 + \Delta_2) = m + i$. Take y in $\Delta_1 \cap \Delta_2$ such that $\partial_\Gamma(x, y) = m - i$. Then from Proposition 2.1, $\Delta_1 \cap \Delta_2 = \langle\langle x, y \rangle\rangle$. Thus there exists z in Δ_1 such that $\partial_\Gamma(x, z) = i$, $\partial_\Gamma(z, y) = m$ and $\langle\langle z, y \rangle\rangle = \Delta_1$; similarly, there exists w in Δ_2 such that $\partial_\Gamma(y, w) = i$, where $\partial_\Gamma(x, w) = m$, and $\langle\langle x, w \rangle\rangle = \Delta_2$. We first show that $\partial_\Gamma(z, w) = m + i$ and $\Delta_1 + \Delta_2 = \langle\langle z, w \rangle\rangle$. Pick a shortest path connecting z and x in Δ_1, $z = v_i$, v_{i-1}, $\cdots$, v_1, $v_0 = x$, where $\partial_\Gamma(v_l, v_{l-1}) = 1$, $1 \le l \le i$; pick a shortest path connecting y and w in Δ_2, $y = u_0$, u_1, $\cdots$, u_{i-1}, $u_i = w$, where $\partial_\Gamma(u_l, u_{l-1}) = 1$, $1 \le l \le i$. In the following, we prove that $\partial_\Gamma(z, u_l) = m + l$, where $0 \le l \le i$.

The assertion is clearly true when $l = 0$. Suppose it is true when $l - 1$. Then $\partial_\Gamma(z, u_{l-1}) = m + l - 1$. It follows that $\partial_\Gamma(z, u_l) = m + l - 2$, $m + l - 1$ or $m + l$. Suppose that $\partial_\Gamma(z, u_l) = m + l - 2$ or $m + l - 1$. Then

$u_l \in C(z, u_{l-1}) \cup A(z, u_{l-1}) \subset \langle\!\langle z, u_{l-1}\rangle\!\rangle$. Since $\langle\!\langle z, u_{l-1}\rangle\!\rangle$ is the subspace containing Δ_1 and $\langle\!\langle x, u_l\rangle\!\rangle$, and

$$\langle\!\langle x, y\rangle\!\rangle \subset \Delta_1 \cap \langle\!\langle x, u_l\rangle\!\rangle \subset \Delta_1 \cap \Delta_2,$$

we have

$$\Delta_1 \cap \langle\!\langle x, u_l\rangle\!\rangle = \langle\!\langle x, y\rangle\!\rangle.$$

From Proposition 2.2,

$$\begin{aligned} & \mathrm{d}(\Delta_1 + \langle\!\langle x, u_l\rangle\!\rangle) \\ = \; & \mathrm{d}(\Delta_1) + \mathrm{d}(\langle\!\langle x, u_l\rangle\!\rangle) - \mathrm{d}(\Delta_1 \cap \langle\!\langle x, u_l\rangle\!\rangle) \\ = \; & m + l, \end{aligned}$$

contradicting the fact that $\langle\!\langle z, u_{l-1}\rangle\!\rangle$ is a subspace with diameter $m+l-1$. So $\partial_\Gamma(z, u_l) = m + l$, where $0 \le l \le i$. It implies that $\partial_\Gamma(z, w) = m + i$, and hence $\Delta_1 + \Delta_2 = \langle\!\langle z, w\rangle\!\rangle$.

Next, we show that $\partial_{\Gamma'}(\Delta_1, \Delta_2) \le i$. Set $\Delta^{(i-t)} = \langle\!\langle v_{i-t}, u_t\rangle\!\rangle$, where $0 \le t \le i$. Then $\Delta^{(i)} = \Delta_1$ and $\Delta^{(0)} = \Delta_2$. Since v_l and u_l, where $0 \le l \le i-1$, are the vertices on a shortest path connecting z and w, we have $\mathrm{d}(\Delta^{(i-t)}) = m$, where $0 \le t \le i$. From Proposition 2.2, $\mathrm{d}(\Delta^{(i-t)} \cap \Delta^{(i-t+1)}) = m - 1$, that is, $\partial_{\Gamma'}(\Delta^{(i-t)}, \Delta^{(i-t-1)}) = 1$, $0 \le t \le i-1$. Thus $\partial_{\Gamma'}(\Delta_1, \Delta_2) \le i$.

Finally, we show $\partial_{\Gamma'}(\Delta_1, \Delta_2) = t \ge i$. Let $\partial_{\Gamma'}(\Delta_1, \Delta_2) = t$, and let

$$\Delta_1 = \Delta^{(t)}, \Delta^{(t-1)}, \cdots, \Delta^{(1)}, \Delta^{(0)} = \Delta_2$$

be the vertices on a shortest path connecting Δ_1 and Δ_2, where $\partial_{\Gamma'}(\Delta^{(l)}, \Delta^{(l-1}$ 1, $1 \le l \le t$. We claim that $\mathrm{d}(\Delta_1 + \Delta^{(t-1)} + \cdots + \Delta^{(1)} + \Delta_2) \le m + t$. Indeed, from Proposition 2.2, we have $\mathrm{d}(\Delta_1 + \Delta^{(t-1)}) = m + 1$ and

$$\begin{aligned} & \mathrm{d}(\Delta_1 + \Delta^{(t-1)} + \Delta^{(t-2)}) \\ = \; & \mathrm{d}((\Delta_1 + \Delta^{(t-1)}) + \Delta^{(t-2)}) \\ = \; & \mathrm{d}(\Delta_1 + \Delta^{(t-1)}) + \mathrm{d}(\Delta^{(t-2)}) - \mathrm{d}((\Delta_1 + \Delta^{(t-1)}) \cap \Delta^{(t-2)}) \\ \le \; & \mathrm{d}(\Delta_1 + \Delta^{(t-1)}) + \mathrm{d}(\Delta^{(t-2)}) - \mathrm{d}(\Delta^{(t-1)} \cap \Delta^{(t-2)}) \\ = \; & m + 2. \end{aligned}$$

So we may assume that $\mathrm{d}(\Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)}) \le m+t-1$. Then from Proposition 2.2,

$$\begin{aligned} & \mathrm{d}(\Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)}+\Delta_2) \\ = & \mathrm{d}((\Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)})+\Delta_2) \\ = & \mathrm{d}(\Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)})+\mathrm{d}(\Delta_2)-\mathrm{d}((\Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)})\cap\Delta_2) \\ \le & \mathrm{d}(\Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)})+\mathrm{d}(\Delta_2)-\mathrm{d}(\Delta^{(1)}\cap\Delta_2) \\ \le & m+t. \end{aligned}$$

Since $\mathrm{d}(\Delta_1+\Delta_2)=m+i$ and $\Delta_1+\Delta_2 \subset \Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)}+\Delta_2$, we have $m+i \le m+t$. Thus $\partial_{\Gamma'}(\Delta_1,\Delta_2)=t \ge i$. It follows that $\partial_{\Gamma'}(\Delta_1,\Delta_2)=i$.

Conversely, let $\partial_{\Gamma'}(\Delta_1,\Delta_2)=i$ and let $\Delta_1=\Delta^{(i)}, \Delta^{(i-1)}, \cdots, \Delta^{(1)}, \Delta^{(0)}=\Delta_2$ be the vertices on a shortest path connecting Δ_1 and Δ_2, where $\partial_{\Gamma'}(\Delta^{(t)},\Delta^{(t-1)})=1$, $1 \le t \le i$. In the following we show $\mathrm{d}(\Delta_1+\Delta_2)=m+i$. Note that $\mathrm{d}(\Delta^{(t)}\cap\Delta^{(t-1)})=m-1$, $1 \le t \le i$. Thus from Proposition 2.2 and the proof similar to that above

$$\mathrm{d}(\Delta_1+\Delta^{(i-1)}+\cdots+\Delta^{(1)}+\Delta_2) \le m+i.$$

Consequently $\mathrm{d}(\Delta_1+\Delta_2) \le m+i$, since $\Delta_1+\Delta_2 \subset \Delta_1+\Delta^{(t-1)}+\cdots+\Delta^{(1)}+\Delta_2$. Suppose that $\mathrm{d}(\Delta_1+\Delta_2)=m+l<m+i$. Then from Proposition 2.2, $\mathrm{d}(\Delta_1\cap\Delta_2)=m-l$. By the proof of sufficiency, we obtain that $\partial_{\Gamma'}(\Delta_1,\Delta_2)=l<i$, a contradiction. Thus $\mathrm{d}(\Delta_1+\Delta_2)=m+i$. Furthermore, from Proposition 2.2, $\mathrm{d}(\Delta_1\cap\Delta_2)=m-i$. □

Lemma 2.6. *Let $\Gamma'=(V',E')$ be the graph constructed above. Let Δ_1, $\Delta_2 \in V'$ such that $\partial_{\Gamma'}(\Delta_1,\Delta_2)=t$, where $1 \le t \le \min(d-m,m)$, and let Δ_3 be a subspace with diameter m and $\partial_{\Gamma'}(\Delta_3,\Delta_2)=1$. Then $\mathrm{d}(\Delta_1\cap\Delta_2\cap\Delta_3)=m-t-1$ or $m-t$. Furthermore, if $\mathrm{d}(\Delta_1\cap\Delta_2\cap\Delta_3)=m-t-1$, then $\mathrm{d}(\Delta_1\cap\Delta_3)=m-t-1$ or $m-t$; if $\mathrm{d}(\Delta_1\cap\Delta_2\cap\Delta_3)=m-t$, then $\mathrm{d}(\Delta_1\cap\Delta_3)=m-t$ or $m-t+1$.*

Proof. Let Δ_1, $\Delta_2 \in V'$ such that $\partial_{\Gamma'}(\Delta_1,\Delta_2)=t$, where $1 \le t \le \min(d-m,m)$, and let Δ_3 be a subspace with diameter m and $\partial_{\Gamma'}(\Delta_3,\Delta_2)=1$.

Then from Lemma 2.5, $d(\Delta_1 \cap \Delta_2) = m - t$ and $d(\Delta_2 \cap \Delta_3) = m - 1$. We claim that $d(\Delta_1 \cap \Delta_2 \cap \Delta_3) = m - t - 1$ or $m - t$. Indeed, since $d(\Delta_1 + (\Delta_2 \cap \Delta_3)) \leq d(\Delta_1 + \Delta_2) = m + t$, it follows from Proposition 2.2 that

$$\begin{aligned} & d(\Delta_1 \cap \Delta_2) \\ \geq\ & d(\Delta_1 \cap (\Delta_2 \cap \Delta_3)) \\ =\ & d(\Delta_1) + d(\Delta_2 \cap \Delta_3)) - d(\Delta_1 + (\Delta_2 \cap \Delta_3)) \\ \geq\ & d(\Delta_1) + d(\Delta_2 \cap \Delta_3) - d(\Delta_1 + \Delta_2) \\ =\ & m - t - 1. \end{aligned}$$

It implies that $d(\Delta_1 \cap \Delta_2 \cap \Delta_3) = m - t - 1$ or $m - t$. Note that

$$\begin{aligned} & d(\Delta_1 \cap \Delta_2 \cap \Delta_3) \\ \leq\ & d(\Delta_1 \cap \Delta_3) \\ =\ & d(\Delta_1) + d(\Delta_3) - d(\Delta_1 + \Delta_3) \\ \leq\ & m + m - d(\Delta_1 + (\Delta_2 \cap \Delta_3)) \\ =\ & 2m - (d(\Delta_1) + d(\Delta_2 \cap \Delta_3) - d(\Delta_1 \cap \Delta_2 \cap \Delta_3)). \end{aligned}$$

So when $d(\Delta_1 \cap \Delta_2 \cap \Delta_3) = m - t - 1$, we have $d(\Delta_1 \cap \Delta_3) = m - t - 1$ or $m - t$; when $d(\Delta_1 \cap \Delta_2 \cap \Delta_3) = m - t$, we have $d(\Delta_1 \cap \Delta_3) = m - t$ or $m - t + 1$.

□

Proof of Theorem 1.1. By Lemma 2.4, Γ' is a regular graph with valency

$$k' = M(m-1, m, m; d) = b^2 \begin{bmatrix} m \\ 1 \end{bmatrix}_{b^2} \begin{bmatrix} d-m \\ 1 \end{bmatrix}_{b^2}.$$

Let $\Delta_1, \Delta_2 \in V'$ such that $\partial_{\Gamma'}(\Delta_1, \Delta_2) = t$, where $1 \leq t \leq \min(d-m, m)$, and let Δ_3 be a subspace with diameter m and $\partial_{\Gamma'}(\Delta_3, \Delta_2) = 1$. To prove Γ' is a distance-regular graph, it suffices to prove b_t' and c_t' are independent of the choice of Δ_1 and Δ_2.

By Lemmas 2.5 and 2.6, to compute b_t' we only consider the case $d(\Delta_1 \cap \Delta_2) = m - t$, $d(\Delta_2 \cap \Delta_3) = m - 1$ and $d(\Delta_1 \cap \Delta_2 \cap \Delta_3) = m - t - 1$.

Note that, for a given $\Delta_1 \cap \Delta_2 \cap \Delta_3$, $\Delta_2 \cap \Delta_3$ is a subspace with diameter $m-1$, contained in Δ_2 and intersect $\Delta_1 \cap \Delta_2$ at $\Delta_1 \cap \Delta_2 \cap \Delta_3$. Thus, from Lemma 2.4, the number of subspaces of $\Delta_2 \cap \Delta_3$ with diameter $m-1$ which intersect $\Delta_1 \cap \Delta_2$ at subspace with diameter $m-t-1$ is

$$M(m-t-1, m-t, m-1; m).$$

From Lemma 2.4 again, for the given subspace $\Delta_2 \cap \Delta_3$, the number of subspaces Δ_3 with diameter m containing $\Delta_2 \cap \Delta_3$ and intersect $\Delta_1 \cap \Delta_2$ at $\Delta_1 \cap \Delta_2 \cap \Delta_3$ is

$$M_1(m-1, m, m; d).$$

So the number of subspaces Δ_3 such that $\mathrm{d}(\Delta_1 \cap \Delta_2 \cap \Delta_3) = m-t-1$, $\mathrm{d}(\Delta_1 \cap \Delta_2) = m-t$ and $\mathrm{d}(\Delta_2 \cap \Delta_3) = m-1$ is

$$M(m-t-1, m-t, m-1; m) M_1(m-1, m, m; d).$$

Clearly, the subspaces Δ_3 above contain the subspaces Δ_3 with $\mathrm{d}(\Delta_1 \cap \Delta_3) = m-t$. In the following we compute the number of such subspaces.

We claim that, for a given $\Delta_2 \cap \Delta_3$, Δ_3 is the subspace with diameter m such that $\mathrm{d}(\Delta_1 \cap \Delta_3) = m-t$ if and only if there exists a subspace Δ_4 with diameter $m-t$ in Δ_1 containing $\Delta_1 \cap \Delta_2 \cap \Delta_3$ such that $\Delta_3 = \Delta_4 + (\Delta_2 \cap \Delta_3)$. Indeed, let $\Delta_3 = \Delta_4 + (\Delta_2 \cap \Delta_3)$ where Δ_4 is a subspace with diameter $m-t$ in Δ_1 containing $\Delta_1 \cap \Delta_2 \cap \Delta_3$. Since

$$\Delta_4 \cap (\Delta_2 \cap \Delta_3) \subset \Delta_1 \cap \Delta_2 \cap \Delta_3,$$

we have

$$\Delta_4 \cap \Delta_2 \cap \Delta_3 = \Delta_1 \cap \Delta_2 \cap \Delta_3.$$

From Proposition 2.2,

$$\mathrm{d}(\Delta_1 + \Delta_2 \cap \Delta_3) = m+t.$$

So from Proposition 2.2 again,

$$\mathrm{d}(\Delta_1 \cap \Delta_3) = \mathrm{d}(\Delta_1 \cap (\Delta_4 + \Delta_2 \cap \Delta_3)) = 2m - \mathrm{d}(\Delta_1 + \Delta_2 \cap \Delta_3) = m-t.$$

It implies that Δ_3 is the subspace with diameter m satisfying $\mathrm{d}(\Delta_1 \cap \Delta_3) = m - t$.

Conversely, let Δ_3 be a subspace with diameter m satisfying $\mathrm{d}(\Delta_1 \cap \Delta_3) = m - t$. Then $\Delta_1 \cap \Delta_3$ is a subspace with diameter $m - t$ in Δ_1 containing $\Delta_1 \cap \Delta_2 \cap \Delta_3$. From Proposition 2.2,

$$\mathrm{d}((\Delta_1 \cap \Delta_3) + (\Delta_2 \cap \Delta_3)) = m - t + m - 1 - (m - t - 1) = m.$$

It follows from Proposition 1.1(iii) that $\Delta_3 = (\Delta_1 \cap \Delta_3) + (\Delta_2 \cap \Delta_3)$. Set $\Delta_4 = \Delta_1 \cap \Delta_3$, as desired.

From proof above, we know that for a given $\Delta_2 \cap \Delta_3$, the number of the subspaces Δ_3 with diameter m satisfying $\mathrm{d}(\Delta_1 \cap \Delta_3) = m - t$ is equal to the number of the subspaces Δ_4 with diameter $m - t$ in Δ_1 containing $\Delta_1 \cap \Delta_2 \cap \Delta_3$ such that $\Delta_3 = \Delta_4 + (\Delta_2 \cap \Delta_3)$. The latter is $N(m - t - 1, m - t; m)$ by Lemma 2.3. Note that $\Delta_1 \cap \Delta_2$ is a subspace with diameter $m - t$ in Δ_1 containing $\Delta_1 \cap \Delta_2 \cap \Delta_3$ such that

$$(\Delta_1 \cap \Delta_2) + (\Delta_2 \cap \Delta_3) = \Delta_2.$$

So for a given $\Delta_2 \cap \Delta_3$, the number of the subspaces Δ_3 with diameter m satisfying $\mathrm{d}(\Delta_1 \cap \Delta_3) = m - t$, is

$$N(m - t - 1, m - t; m) - 1 = (b_{m-t} - b_m)/(b_{m-t-1} - b_{m-t}).$$

Thus, for a given subspace $\Delta_2 \cap \Delta_3$, the number of subspace Δ_3 with diameter m satisfying $\mathrm{d}(\Delta_1 \cap \Delta_3) = m - t - 1$ is

$$\begin{aligned} & M_1(m-1, m, m; d) - N(m - t - 1, m - t; m) + 1 \\ = {} & b_m/(b_{m-1} - b_m) - (b_{m-t} - b_m)/(b_{m-t-1} - b_{m-t}). \end{aligned}$$

It follows that

$$\begin{aligned} b'_t &= \left(\frac{b_m}{b_{m-1} - b_m} - \frac{b_{m-t} - b_m}{b_{m-t-1} - b_{m-t}} \right) M(m - t - 1, m - t, m - 1; m) \\ &= b^{4t+2} \begin{bmatrix} m - t \\ 1 \end{bmatrix}_{b^2} \begin{bmatrix} d - m - t \\ 1 \end{bmatrix}_{b^2}. \end{aligned}$$

Similarly,

$$c'_t = \frac{b_{m-t} - b_m}{b_{m-t} - b_{m-t+1}} N(m-t, m-1; m) = \left(\begin{bmatrix} t \\ 1 \end{bmatrix}_{b^2} \right)^2 .$$

Clearly c'_t and b'_t, where $1 \le t \le \min(d-m, m)$, are independent of the choice of Δ_1 and Δ_2. So Γ' is a distance-regular graph. □

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