Skolem-Gracefulness of k-Stars *

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Abstract

A graph $G = (V, E)$ is Skolem-graceful if its vertices can be labelled $1, 2, \ldots, |V|$, so that the edges are labelled $1, 2, \ldots, |E|$, where each edge label is the absolute difference of the labels of the two end-vertices. It is shown that a k-star is Skolem-graceful only if at least one star has even size or $k \equiv 0$ or $1 (\mathrm{mod}\ 4)$, and for $k \leq 5$, a k-star is Skolem-graceful if at least one star has even size or $k \equiv 0$ or $1 (\mathrm{mod}\ 4)$. In this paper we show that k-stars are Skolem-graceful if at least one star has even size or $k \equiv 0$ or $1 (\mathrm{mod}\ 4)$ for all positive integer k.
Keywords: *Skolem-graceful graph, k-stars, vertex labeling, edge labeling*

1 Introduction

A graph $G = (V, E)$ is defined to be Skolem-graceful if there exists a one-to-one mapping $\theta : V \to \{1, 2, \ldots, |V|\}$ such that the induced mapping $\lambda : E \to \{1, 2, \ldots, |E|\}$, defined by $\lambda(uv) = |\theta(u) - \theta(v)|$ is a bijection, where $uv \in E$.

A k-star $St(n_1, n_2, \ldots, n_k)$ is a disconnected graph with k components $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_k}$, where K_{1,n_j} denotes a star with $n_j + 1$ vertices $(1 \leq j \leq k)$.

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In [4], Lee and Shee mention some necessary conditions for Skolem-graceful graphs. If G is Skolem-graceful then $|E| \leqslant |V|-1$ and moreover, it is possible to partition v into even vertices and odd vertices such that the number of edges connecting even vertices with odd vertices in G is exactly $\lfloor (q+1)/2 \rfloor$. A tree is Skolem-graceful if and only if it is graceful.

In [6], Rosa proves caterpillars are graceful. Lee and Wui [5] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. Obviously, all 1-stars are Skolem graceful. In [5], Lee and Wui prove that 2-stars and 3-stars are Skolem-graceful if and only if at least one star has even size. In [2], Denham prove that all 4-stars are Skolem-graceful. Choudum and Kishore [1] prove that all 5-stars are Skolem graceful.

For the literature on Skolem-graceful graphs we refer the reader to [3] and the relevant references given in it.

In [1], Choudum and Kishore show a necessary condition for a k-star to be Skolem-graceful as follows:

Theorem 1.1. A k-star is Skolem-graceful only if at least one star has even size or $k \equiv 0$ or $1 \pmod 4$.

In this paper, we show that a k-star is Skolem-graceful if at least one star has even size or $k \equiv 0$ or $1 \pmod 4$ for all positive integer k.

2 Main Result

Theorem 2.1. The k-stars $St(n_1, n_2, \ldots, n_k)$ are Skolem-graceful if at least one star has even size or $k \equiv 0$ or $1 \pmod 4$ for all positive integer k.
Proof. Let $G = St(n_1, n_2, \ldots, n_k)$ be a k-stars graph with n vertices and $n-k$ edges. Let $a_1 = 1$. For $1 \leq j \leq k$, let $St_j \cong K_{1,n_j}$ be the j-th star in G, let

$$\begin{aligned} a_{j+1} &= 1 + \textstyle\sum_{t=1}^{j} (n_t + 1), \\ V(St_j) &= \{v_{a_j+t} \ : \ 0 \leq t \leq n_j\}, \\ E(St_j) &= \{v_{a_j} v_{a_j+t} \ : \ 1 \leq t \leq n_j\}, \end{aligned}$$

then

$$\begin{aligned} V(G) &= \textstyle\bigcup_{j=1}^{k} (V(St_j)), \\ E(G) &= \textstyle\bigcup_{j=1}^{k} (E(St_j)). \end{aligned}$$

Let $S_{odd}(S_{even})$ be the set of odd(even) stars of G, let S_{one} be the set of stars with one leaf, i.e.

$$\begin{aligned} S_{odd} &= \{St_j \ : \ n_j \text{ is odd}\}, \\ S_{even} &= \{St_j \ : \ n_j \text{ is even}\}, \\ S_{one} &= \{St_j \ : \ n_j = 1\}. \end{aligned}$$

In some of the following cases, we need rearrange the stars (by defining special sets S_{odd}, S_{even} and S_{one}) in a convenient form in order to obtain their labeling, note that this produces new values for a_j, $V(St_j)$ and $E(St_j)$.

If $|S_{odd}| = u$, then $|S_{even}| = k - u$. Let $u_2 = u \bmod 2$, $j_2 = j \bmod 2$.

Case 1. $u < k$, i.e. at least one star has even size.

Case 1.1. k is even. When u is odd, let $S_{odd} = \{S_j \ : \ 2 \le j \le u\} \cup \{S_k\}$. When u is even, let $S_{odd} = \{S_j \ : \ 2 \le j \le u+1\}$. Then, we can label the vertices as follows (see Figure 2.1-2.2):

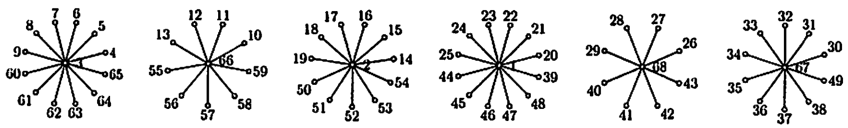

Figure 2.1. Skolem-graceful labelings of $St(12,9,11,12,8,10)$

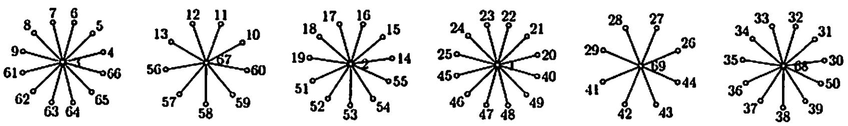

Figure 2.2: Skolem-graceful labelings of $St(12,9,11,12,8,11)$

$$f(v_i)=\begin{cases} k/2-1+i, \\ \qquad 1\le i\le a_2/2, \\ n-k/2+1-a_2+i, \\ \qquad a_2/2+1\le i\le a_2-1, \\ n-(k-j)/2, \\ \qquad i=a_j \text{ and } 2\le j\le u-u_2 \text{ and } j \text{ is even }, \\ (k-j+1)/2, \\ \qquad i=a_j \text{ and } 3\le j\le u-u_2+1 \text{ and } j \text{ is odd }, \\ (k-a_j-j-j_2)/2+i, \\ \qquad a_j+1\le i\le (a_j+a_{j+1})/2+j_2-1 \text{ and } 2\le j\le u-u_2+1, \\ n-(k+a_j-j+j_2)/2-a_{j+1}+1+i, \\ \qquad (a_j+a_{j+1})/2+j_2\le i\le a_{j+1}-1 \text{ and } 2\le j\le u-u_2+1, \\ (k+u-u_2)/2-j+1, \\ \qquad i=a_j \text{ and } u-u_2+2\le j\le (k+u-u_2)/2, \\ n+(k+u-u_2)/2-j+1, \\ \qquad i=a_j \text{ and } (k+u-u_2)/2+1\le j\le k, \\ (k-a_j-j)/2+i, \\ \qquad a_j+1\le i\le (a_j+a_{j+1}-1)/2 \text{ and } u-u_2+2\le j\le k-1, \\ n-(k+a_j-j)/2-a_{j+1}+1+i, \\ \qquad (a_j+a_{j+1}+1)/2\le i\le a_{j+1}-2 \text{ and } u-u_2+2\le j\le k-1, \\ i-a_k/2, \\ \qquad a_k+1\le i\le n-1, \\ n-(a_{k+2-j+u-u_2}+j-u+u_2)/2+1, \\ \qquad i=a_{j+1}-1 \text{ and } u-u_2+2\le j\le k. \end{cases}$$

Firstly, we show that f is a bijective mapping from $V(G)$ onto $\{1,2,\ldots,n\}$.
Denote by

$$\begin{aligned} S_j &= \{f(v_i)\mid a_j\le i\le a_{j+1}-1\}, \quad 1\le j\le k, \\ S &= \textstyle\bigcup_{j=1}^{k}(S_j). \end{aligned}$$

Then,

$$S_1=S_{1,1}\cup S_{1,2},$$

where

$$\begin{aligned} S_{1,1} &= \{k/2,k/2+1,\ldots,(k+a_2)/2-1\}, \\ S_{1,2} &= \{n-(k+a_2)/2+2,n-(k+a_2)/2+3,\ldots,n-k/2\}. \end{aligned}$$

For $2\le j\le u-u_2+1$,

$$S_j=S_{j,1}\cup S_{j,2}\cup S_{j,3},$$

where

$$\begin{aligned} S_{j,1} &= \begin{cases} \{n-(k-j)/2\}, & j \text{ is even }, \\ \{(k+1-j)/2\}, & j \text{ is odd }, \end{cases} \\ S_{j,2} &= \{(k+a_j-j-j_2)/2+1,(k+a_j-j-j_2)/2+2,\ldots, \\ &\qquad (k+a_{j+1}-j+j_2)/2-1\}, \\ S_{j,3} &= \{n-(k+a_{j+1}-j-j_2)/2+1,n-(k+a_{j+1}-j-j_2)/2+2,\ldots, \\ &\qquad n-(k+a_j-j+j_2)/2\}. \end{aligned}$$

For $u - u_2 + 2 \le j \le k - 1$,

$$S_j = S_{j,1} \cup S_{j,2} \cup S_{j,3} \cup S_{j,4},$$

where

$$S_{j,1} = \begin{cases} \{(k+u-u_2)/2+1-j\}, & u-u_2+2 \le j \le (k+u-u_2)/2, \\ \{n+(k+u-u_2)/2+1-j\}, & (k+u-u_2)/2+1 \le j \le k-1, \end{cases}$$
$$S_{j,2} = \{(k+a_j-j)/2+1, (k+a_j-j)/2+2, \ldots, (k+a_{j+1}-j-1)/2\},$$
$$S_{j,3} = \{n-(k+a_{j+1}-j-1)/2+1, n-(k+a_{j+1}-j-1)/2+2, \ldots, n-(k+a_j-j)/2-1\},$$
$$S_{j,4} = \{n-(a_{k+2-j+u-u_2}+j-u+u_2)/2+1\}.$$

$$S_k = S_{k,1} \cup S_{k,2} \cup S_{k,3},$$

where

$$\begin{aligned} S_{k,1} &= \{n-(k-u+u_2)/2+1\}, \\ S_{k,2} &= \{a_k/2+1, a_k/2+2, \ldots, n-a_k/2-1\}, \\ S_{k,3} &= \{n-(k+a_{(u-u_2+2)}-u+u_2)/2+1\}. \end{aligned}$$

For example, when $G = St(12,9,11,12,8,10)$, we have

$$\begin{aligned} S_1 &= \{3,4,5,6,7,8,9\} \cup \{60,61,62,63,64,65\}, \\ S_2 &= \{66\} \cup \{10,11,12,13\} \cup \{55,56,57,58,59\}, \\ S_3 &= \{2\} \cup \{14,15,16,17,18,19\} \cup \{50,51,52,53,54\}, \\ S_4 &= \{1\} \cup \{20,21,22,23,24,25\} \cup \{44,45,46,47,48\} \cup \{39\}, \\ S_5 &= \{68\} \cup \{26,27,28,29\} \cup \{40,41,42\} \cup \{43\}, \\ S_6 &= \{67\} \cup \{30,31,32,33,34,35,36,37,38\} \cup \{49\}. \end{aligned}$$

When $G = St(12,9,11,12,8,11)$, we have

$$\begin{aligned} S_1 &= \{3,4,5,6,7,8,9\} \cup \{61,62,63,64,65,66\}, \\ S_2 &= \{67\} \cup \{10,11,12,13\} \cup \{56,57,58,59,60\}, \\ S_3 &= \{2\} \cup \{14,15,16,17,18,19\} \cup \{51,52,53,54,55\}, \\ S_4 &= \{1\} \cup \{20,21,22,23,24,25\} \cup \{45,46,47,48,49\} \cup \{40\}, \\ S_5 &= \{69\} \cup \{26,27,28,29\} \cup \{41,42,43\} \cup \{44\}, \\ S_6 &= \{68\} \cup \{30,31,32,33,34,35,36,37,38,39\} \cup \{50\}. \end{aligned}$$

Hence,

$$\begin{aligned} S = \; & S_1 \cup \bigcup_{t=2}^{u-u_2+1}(S_j) \cup \bigcup_{t=u-u_2+2}^{(k+u-u_2)/2}(S_j) \cup \bigcup_{t=(k+u-u_2)/2+1}^{k-1}(S_j) \cup S_k \\ = \; & S_{(k+u-u_2)/2,1} \cup S_{(k+u-u_2)/2-1,1} \cup \ldots \cup S_{u-u_2+2,1} \\ & \cup S_{u-u_2+1,1} \cup S_{u-u_2-1,1} \cup \ldots \cup S_{3,1} \\ & \cup S_{1,1} \cup S_{2,2} \cup S_{3,2} \cup \ldots \cup S_{k,2} \\ & \cup S_{u-u_2+2,4} \cup S_{k-1,3} \cup S_{u-u_2+3,4} \cup S_{k-2,3} \cup \ldots \\ & \cup S_{k-1,4} \cup S_{u-u_2+2,3} \\ & \cup S_{k,3} \cup S_{u-u_2+1,3} \cup S_{u-u_2,3} \cup \ldots \cup S_{2,3} \cup S_{1,2} \\ & \cup S_{2,1} \cup S_{4,1} \cup \ldots \cup S_{u-u_2,1} \\ & \cup S_{k,1} \cup S_{k-1,1} \cup \ldots \cup S_{(k+u-u_2)/2+1,1} \\ = \; & \{1, 2, \ldots, n-1, n\}. \end{aligned}$$

Therefore, f is a bijection from $V(G)$ onto $\{1, 2, \ldots, n\}$.

Secondly, we show that f is a bijective mapping from $E(G)$ onto $\{1, 2, \ldots, n-k\}$.
Denote by

$$\begin{aligned} D_j &= \{|f(v_i) - f(v_{a_j})| \mid a_j + 1 \le i \le a_{j+1} - 1\}, \quad 1 \le j \le k, \\ D &= \bigcup_{j=1}^{k}(D_j). \end{aligned}$$

Then,

$$D_1 = D_{1,1} \cup D_{1,2},$$

where

$$\begin{aligned} D_{1,1} &= \{1, 2, \ldots, a_2/2 - 1\}, \\ D_{1,2} &= \{n - k - a_2/2 + 2, n - k - a_2/2 + 3, \ldots, n - k\}. \end{aligned}$$

For $2 \le j \le u - u_2 + 1$,

$$D_j = D_{j,1} \cup D_{j,2},$$

When j is even,

$$\begin{aligned} D_{j,1} &= \{n - k + j - a_j/2 - 1, n - k + j - a_j/2 - 2, \ldots, n - k + j - a_{j+1}/2 + 1\} \\ &= \{n - k + j - a_{j+1}/2 + 1, \ldots, n - k + j - a_j/2 - 2, n - k + j - a_j/2 - 1\}, \\ D_{j,2} &= \{a_{j+1}/2 - 1, a_{j+1}/2 - 2, \ldots, a_j/2\} \\ &= \{a_j/2, \ldots, a_{j+1}/2 - 2, a_{j+1}/2 - 1\}. \end{aligned}$$

When j is odd,

$$\begin{aligned} D_{j,1} &= \{a_j/2, a_j/2 + 1, \ldots, a_{j+1}/2 - 1\}, \\ D_{j,2} &= \{n - k + j - a_{j+1}/2 + 1, n - k + j - a_{j+1}/2 + 2, \ldots, n - k + j - a_j/2 - 1\}. \end{aligned}$$

For $u - u_2 + 2 \le j \le (k + u - u_2)/2$,

$$D_j = D_{j,1} \cup D_{j,2} \cup D_{j,3},$$

where

$$\begin{aligned} D_{j,1} &= \{(a_j + j - u + u_2)/2, (a_j + j - u + u_2)/2 + 1, \ldots, \\ &\qquad (a_{j+1} + j - u + u_2 - 1)/2 - 1\}, \\ D_{j,2} &= \{n - k + j - (a_{j+1} - j + u - u_2 - 1)/2, n - k + j - (a_{j+1} - j + u - u_2 \\ &\qquad -1)/2 + 1, \ldots, n - k + j - (a_j - j + u - u_2)/2 - 2\}, \\ D_{j,3} &= \{n - (k + a_{k+2-j+u-u_2} - j)/2\}. \end{aligned}$$

For $(k + u - u_2)/2 + 1 \le j \le k - 1$,

$$D_j = D_{j,1} \cup D_{j,2} \cup D_{j,3},$$

where

$$\begin{aligned} D_{j,1} &= \{n - (a_j + j - u + u_2)/2, n - (a_j + j - u + u_2)/2 - 1, \ldots, \\ &\qquad n - (a_{j+1} + j - u + u_2 - 1)/2 + 1\} \\ &= \{n - (a_{j+1} + j - u + u_2 - 1)/2 + 1, \ldots, n - (a_j + j - u + u_2)/2 - 1, \\ &\qquad n - (a_j + j - u + u_2)/2\}, \\ D_{j,2} &= \{k - j + (a_{j+1} - j + u - u_2 - 1)/2, k - j + (a_{j+1} - j + u - u_2 - 1)/2 - 1, \\ &\qquad \ldots, k - j + (a_j - j + u - u_2)/2 + 2\} \\ &= \{k - j + (a_j - j + u - u_2)/2 + 2, \ldots, k - j + (a_{j+1} - j + u - u_2 - 1)/2 - 1, \\ &\qquad k - j + (a_{j+1} - j + u - u_2 - 1)/2\}, \\ D_{j,3} &= \{(k + a_{k+2-j+u-u_2} - j)/2\}. \end{aligned}$$

$$D_k = D_{k,1} \cup D_{k,2},$$

where

$$\begin{aligned} D_{k,1} &= \{n-(a_k+k-u+u_2)/2, n-(a_k+k-u+u_2)/2-1, \ldots, \\ &\quad (a_k-k+u-u_2)/2+2\} \\ &= \{(a_k-k+u-u_2)/2+2, \ldots, n-(a_k+k-u+u_2)/2-1, \\ &\quad n-(a_k+k-u+u_2)/2\}, \\ D_{k,2} &= \{a_{u-u_2+2/2}\}. \end{aligned}$$

For example, when $G = St(12,9,11,12,8,10)$, we have

$$\begin{aligned} D_1 &= \{1,2,3,4,5,6\} \cup \{57,58,59,60,61,62\}, \\ D_2 &= \{53,54,55,56\} \cup \{7,8,9,10,11\}, \\ D_3 &= \{12,13,14,15,16,17\} \cup \{48,49,50,51,52\}, \\ D_4 &= \{19,20,21,22,23,24\} \cup \{43,44,45,46,47\} \cup \{38\}, \\ D_5 &= \{39,40,41,42\} \cup \{26,27,28\} \cup \{25\}, \\ D_6 &= \{29,30,31,32,33,34,35,36,37\} \cup \{18\}. \end{aligned}$$

When $G = St(12,9,11,12,8,11)$, we have

$$\begin{aligned} D_1 &= \{1,2,3,4,5,6\} \cup \{58,59,60,61,62,63\}, \\ D_2 &= \{54,55,56,57\} \cup \{7,8,9,10,11\}, \\ D_3 &= \{12,13,14,15,16,17\} \cup \{49,50,51,52,53\}, \\ D_4 &= \{19,20,21,22,23,24\} \cup \{44,45,46,47,48\} \cup \{39\}, \\ D_5 &= \{40,41,42,43\} \cup \{26,27,28\} \cup \{25\}, \\ D_6 &= \{29,30,31,32,33,34,35,36,37,38\} \cup \{18\}. \end{aligned}$$

Hence,

$$\begin{aligned} D = \quad & D_1 \cup \bigcup_{t=2}^{u-u_2+1}(D_j) \cup \bigcup_{t=u-u_2+2}^{(k+u-u_2)/2}(D_j) \cup \bigcup_{t=(k+u-u_2)/2+1}^{k-1}(D_j) \\ & \cup D_k \\ = \quad & D_{1,1} \cup D_{2,2} \cup D_{3,1} \cup D_{4,2} \cup D_{5,1} \cup \ldots \cup D_{u-u_2,2} \cup D_{u-u_2+1,1} \\ & \cup D_{k,2} \cup D_{u-u_2+2,1} \cup D_{k-1,3} \cup D_{u-u_2+3,1} \cup D_{k-2,3} \cup \ldots \\ & \cup D_{(k+u-u_2)/2,1} \cup D_{(k+u-u_2)/2+1,3} \\ & \cup D_{(k+u-u_2)/2+1,2} \cup D_{(k+u-u_2)/2+2,2} \cup \ldots \cup D_{k-1,2} \\ & \cup D_{k,1} \cup D_{u-u_2+2,3} \cup D_{k-1,1} \cup D_{u-u_2+3,3} \cup \ldots \\ & \cup D_{(k+u-u_2)/2+2,1} \cup D_{(k+u-u_2)/2,3} \\ & \cup D_{(k+u-u_2)/2+1,1} \cup D_{(k+u-u_2)/2,2} \cup D_{(k+u-u_2)/2-1,2} \cup \ldots \\ & \cup D_{u-u_2+2,2} \\ & \cup D_{u-u_2+1,2} \cup D_{u-u_2,1} \cup D_{u-u_2-1,2} \cup D_{u-u_2-2,1} \cup \ldots \\ & \cup D_{3,2} \cup D_{2,1} \cup D_{1,2} \\ = \quad & \{1,2,\ldots,n-k-1,n-k\}. \end{aligned}$$

Since f satisfies the conditions to be a Skolem-graceful labeling, we conclude that the graph G is Skolem-graceful when k is even and $u < k$.

Case 1.2. k is odd.

Case 1.2.1. $u = 0$.

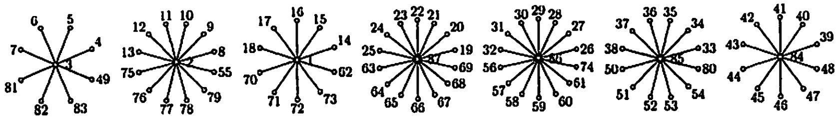

Figure 2.3: Skolem-graceful labelings of $St(8,12,10,14,14,12,10)$

We label the vertices as follows (see Figure 2.3):

$$f(v_i) = \begin{cases} (k+1)/2 - j, \\ \qquad i = a_j \text{ and } 1 \le j \le (k-1)/2, \\ n + (k+1)/2 - j, \\ \qquad i = a_j \text{ and } (k+1)/2 \le j \le k, \\ (k - a_j - j - 1)/2 + i, \\ \qquad a_j + 1 \le i \le (a_j + a_{j+1} - 1)/2 \text{ and } 1 \le j \le k-1, \\ n - (k + a_j - j - 1)/2 - a_{j+1} + 1 + i, \\ \qquad (a_j + a_{j+1} + 1)/2 \le i \le a_{j+1} - 2 \text{ and } 1 \le j \le k-1, \\ n - (a_{k+1-j} + j)/2 + 1, \\ \qquad i = a_{j+1} - 1 \text{ and } 1 \le j \le k-1, \\ i - (a_k + 1)/2, \\ \qquad a_k + 1 \le i \le n. \end{cases}$$

Case 1.2.2. $1 \le u \le k-1$. When u is odd, let $S_{odd} = \{S_j \ : \ 2 \le j \le u+1\}$. When u is even, let $S_{odd} = \{S_j \ : \ 2 \le j \le u\} \cup \{S_k\}$. We can label the vertices as follows (see Figure 2.4-2.5):

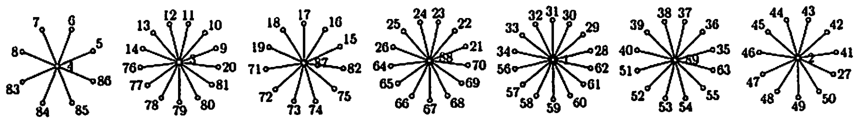

Figure 2.4: Skolem-graceful labelings of $St(8,13,11,13,14,12,11)$

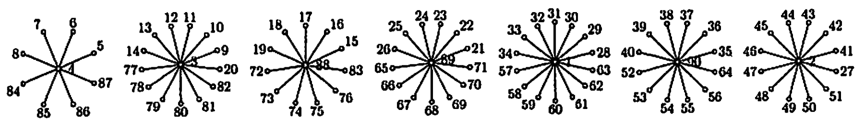

Figure 2.5: Skolem-graceful labelings of $St(8,13,11,13,14,12,12)$

$$f(v_i) = \begin{cases} (k-1)/2+i, \\ \qquad 1 \le i \le a_2/2, \\ n-(k+1)/2-a_2+2+i, \\ \qquad a_2/2+1 \le i \le a_2-1, \\ (k-j+1)/2, \\ \qquad i=a_j \text{ and } 2 \le j \le u+u_2-2 \text{ and } j \text{ is even }, \\ n-(k-j)/2, \\ \qquad i=a_j \text{ and } 3 \le j \le u+u_2-1 \text{ and } j \text{ is odd }, \\ (k-a_j-j-j_2+1)/2+i, \\ \qquad a_j+1 \le i \le (a_j+a_{j+1})/2-1 \text{ and } 2 \le j \le u+u_2-1, \\ n-(k+a_j-j-j_2-1)/2-a_{j+1}+1+i, \\ \qquad (a_j+a_{j+1})/2 \le i \le a_{j+1}-2 \text{ and } 2 \le j \le u+u_2-1, \\ (k+a_{j+2}-j-1)/2, \\ \qquad i=a_{j+1}-1 \text{ and } 2 \le j \le u+u_2-2 \text{ and } j \text{ is even }, \\ n-(k+a_{j-1}-j)/2, \\ \qquad i=a_{j+1}-1 \text{ and } 3 \le j \le u+u_2-1 \text{ and } j \text{ is odd }, \\ n-(k-j-1)/2, \\ \qquad i=a_j \text{ and } j=u+u_2, \\ (k-a_j-j-j_2+1)/2+i, \\ \qquad a_j+1 \le i \le (a_j+a_{j+1})/2-1 \text{ and } j=u+u_2, \\ n-(k+a_j-j-j_2-1)/2-a_{j+1}+1+i, \\ \qquad (a_j+a_{j+1})/2 \le i \le a_{j+1}-1 \text{ and } j=u+u_2, \\ (k+u+u_2+1)/2-j, \\ \qquad i=a_j \text{ and } u+u_2+1 \le j \le (k+u+u_2-1)/2, \\ n+(k+u+u_2+1)/2-j, \\ \qquad i=a_j \text{ and } (k+u+u_2+1)/2 \le j \le k-1, \\ (k-a_j-j)/2+1+i, \\ \qquad a_j+1 \le i \le (a_j+a_{j+1}-1)/2 \text{ and } u+u_2+1 \le j \le k-1, \\ n-(k+a_j-j)/2-a_{j+1}+i, \\ \qquad (a_j+a_{j+1}+1)/2 \le i \le a_{j+1}-1 \text{ and } j=(k+u+u_2-1)/2, \\ n-(k+a_j-j)/2-a_{j+1}+1+i, \\ \qquad (a_j+a_{j+1}+1)/2 \le i \le a_{j+1}-2 \text{ and } u+u_2+1 \le j \le k-1 \\ \qquad\quad \text{and } j \ne (k+u+u_2-1)/2, \\ n-(a_{(k-j+u+u_2)}+j-u-u_2)/2, \\ \qquad i=a_{j+1}-1 \text{ and } u+u_2+1 \le j \le k-1 \text{ and} \\ \qquad\quad j \ne (k+u+u_2-1)/2, \\ (k-u-u_2+1)/2, \\ \qquad i=a_k, \\ i-a_k/2+1, \\ \qquad a_k+1 \le i \le n-1 \text{ and } u \le k-3, \\ (k+a_{(u+u_2+1)}-u-u_2+1)/2, \\ \qquad i=n \text{ and } u \le k-3, \\ i-a_k/2, \\ \qquad a_k+1 \le i \le n \text{ and } u \ge k-2. \end{cases}$$

Since the proof in this case is similar to the one in Case 1.1, we omit it. Thus, we have that this assignment provides a Skolem-graceful labeling when k is odd and $u < k$.

Case 2. $u = k$, i.e. all the stars of G have odd size, $k \equiv 0$ or $1 \pmod 4$. Let $|S_{one}| = u_1$, $w = k - u_1$. Let $k_2 = k \bmod 2$, $w_2 = w \bmod 2$.

Case 2.1. $u_1 \geq k-(k-k_2)/4+2$. Let $S_{one} = \{St_j \;:\; w+1 \leq j \leq k\}$. We label the vertices as follows (see Figure 2.6):

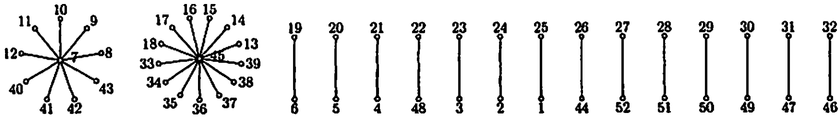

Figure 2.6: Skolem-graceful labelings of $St(9,13,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$

$$f(v_i) = \begin{cases} (k+k_2)/2-(j+1)/2, \\ \qquad i=a_j \text{ and } 1 \leq j \leq w \text{ and } j \text{ is odd }, \\ n-(k-k_2)/2+j/2, \\ \qquad i=a_j \text{ and } 2 \leq j \leq w \text{ and } j \text{ is even }, \\ (k+k_2+w-w_2)/2-j, \\ \qquad i=a_j \text{ and } w+1 \leq j \leq ((k-k_2)/2+w-w_2)/2, \\ n-(k-k_2)/4, \\ \qquad i=a_j \text{ and } j=((k-k_2)/2+w-w_2)/2+1, \\ (k+k_2+w-w_2)/2+1-j, \\ \qquad i=a_j \text{ and } ((k-k_2)/2+w-w_2)/2+2 \leq j \leq \\ \qquad\qquad (k-k_2+w-w_2)/2+k_2, \\ n-(k-k_2)/2, \\ \qquad i=a_j \text{ and } j=(k-k_2+w-w_2)/2+k_2+1, \\ n+(k+w-w_2)/2+k_2+2-j, \\ \qquad i=a_j \text{ and } (k-k_2+w-w_2)/2+k_2+2 \leq j \leq \\ \qquad\qquad k+1-(k-k_2)/4+(w-w_2)/2, \\ n+(k+w-w_2)/2+k_2+1-j, \\ \qquad i=a_j \text{ and } k+2-(k-k_2)/4+(w-w_2)/2 \leq j \leq k, \\ (k-k_2-a_j-j-j_2-1)/2+k_2+i, \\ \qquad a_j+1 \leq i \leq (a_j+a_{j+1})/2+j_2-1 \text{ and } j \leq w, \\ n-(k-k_2+a_j-j+j_2-1)/2-a_{j+1}+i, \\ \qquad (a_j+a_{j+1})/2+j_2 \leq i \leq a_{j+1}-1 \text{ and } j \leq w, \\ (k+k_2-w+w_2)/2-1+i/2, \\ \qquad i=a_{j+1}-1 \text{ and } w+1 \leq j \leq k. \end{cases}$$

Case 2.2. $u_1 \leq k-(k-k_2)/4+1$. Let

$$S_{one} = \begin{cases} \{St_{k-4\lfloor (k-j)/3 \rfloor-(k-j) \bmod 3} : w+2 \leq j \leq k\}, \\ \qquad k \equiv 0 \pmod 4 \text{ and } u_1 \leq k-k/4, \\ \{St_1\} \cup \{St_{k-4\lfloor (k-j)/3 \rfloor-(k-j) \bmod 3} : w+2 \leq j \leq k\}, \\ \qquad k \equiv 0 \pmod 4 \text{ and } u_1 = k-k/4+1, \\ \{St_k\}, \\ \qquad k \equiv 1 \pmod 4 \text{ and } u_1 = 1, \\ \{St_{k+3-4\lfloor (k-j+2)/3 \rfloor-(k-j+2) \bmod 3} : w+1 \leq j \leq k-1\} \cup \{St_k\}, \\ \qquad k \equiv 1 \pmod 4 \text{ and } 2 \leq u_1 \leq k-(k-1)/4, \\ \{St_1\} \cup \{St_{k+3-4\lfloor (k-j+2)/3 \rfloor-(k-j+2) \bmod 3} : w+1 \leq j \leq k-1\} \cup \{St_k\}, \\ \qquad k \equiv 1 \pmod 4 \text{ and } u_1 = k-(k-1)/4+1. \end{cases}$$

We label the vertices as follows (see Figure 2.7):

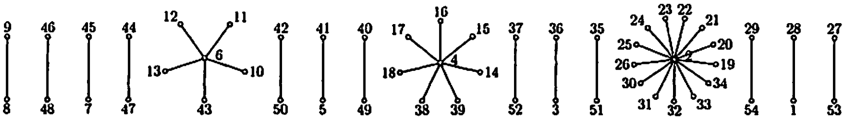

Figure 2.7: Skolem-graceful labelings of St(1,1,1,1,5,1,1,1,7,1,1,1,13,1,1,1)

$$f(v_i)=\begin{cases}(k+k_2)/2-(j-1)/2, & \\ \quad i=a_j \text{ and } 1\le j\le k \text{ and } j \text{ is odd}, & \\ n+1-(k-k_2-j)/2-(j-2) \bmod 4, & \\ \quad i=a_j \text{ and } 2\le j\le k \text{ and } j \text{ is even}, & \\ (k+k_2)/2-1+i, & \\ \quad 2\le i\le (a_2+1)/2, & \\ n-(k-k_2)/2-a_2+1+i, & \\ \quad (a_2+3)/2\le i\le a_2-1, & \\ i-(a_j+j-k-k_2+2)/2, & \\ \quad a_j+1\le i\le (a_j+a_{j+1})/2+1 \text{ and } j \bmod 4=1 \text{ and } 2\le j\le k-1, & \\ n-(k-k_2)/2-(a_j-(j-2)/4\times 4-5)/2-a_{j+1}+i, & \\ \quad (a_j+a_{j+1})/2+2\le i\le a_{j+1}-1 \text{ and } j \bmod 4=1 \text{ and } 2\le j\le k-1, & \\ i-(a_j+j+(j-2) \bmod 4-k-k_2-1)/2, & \\ \quad a_j+1\le i\le (a_j+a_{j+1})/2-1 \text{ and } j \bmod 4\ne 1 \text{ and } 2\le j\le k-1, & \\ n-(k-k_2)/2-(a_j-(j-2)/4\times 4-5)/2-a_{j+1}+i, & \\ \quad (a_j+a_{j+1})/2\le i\le a_{j+1}-1 \text{ and } j \bmod 4\ne 1 \text{ and } 2\le j\le k-1, & \\ i-(a_k+(k-2) \bmod 4-k_2-1)/2, & \\ \quad a_k+1\le i\le n. & \end{cases}$$

Since the proof in this case is similar to the one in Case 1.1, we omit it. Thus, we have that this assignment provides a Skolem-graceful labeling for $u=k$, i.e., when all the components of G have odd size and $k\equiv 0$ or $1 \pmod 4$.

Therefore, we can conclude that the graph $St(n_1,n_2,\ldots,n_k)$ is Skolem-graceful if at least one star has even size or $k\equiv 0$ or $1 \pmod 4$. □

From Theorem 1.1 and Theorem 2.1, we have

Theorem 2.2. The k-stars $St(n_1,n_2,\ldots,n_k)$ are Skolem-graceful if and only if at least one star has even size or $k\equiv 0$ or $1 \pmod 4$ for all positive integer k. □

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References

[1] S. A. Choudum and S. P. M. Kishore, All 5-stars are Skolem-graceful, *Indian J. Pure Appl. Math.*, 27 (1996) 1101–1105.

[2] G. Denham, M. G. Leu, and A. Liu, All 4-stars are Skolem-graceful, *Ars Combin.*, 36 (1993) 183–191.

[3] J. A. Gallian, A Survey: A Dynamic Survey on Graph Labeling, *Electron. J. Combin.*, #DS6 (2005).

[4] S. M. Lee and S. C. Shee, On Skolem-graceful graphs, *Discrete Math.*, 93 (1991) 195–200.

[5] S. M. Lee and I. Wui , On Skolem-gracefulness of 2-stars and 3-stars, *Bull. Malaysian Math. Soc.*, 10 (1987) 15-20.

[6] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)* , Gordon and Breach, N. Y. and Dunod Paris, (1967) 349–355.