On Antimagic Labeling For Power of Cycles

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Abstract

We prove that the power of cycles C_n^2 for odd n are antimagic. We provide explicit constructions to demonstrate all power of cycles C_n^2 for odd n are antimagic and its vertex sums form a set of successive integers.

1 Introduction and preliminaries

An antimagic labeling of a graph with m edges and n vertices is a bijective map from the set of edges to the integers $\{1, 2, ..., m\}$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it accommodate an antimagic labeling.

The concept of antimagic graph is introduced by Hartsfield and Ringel [2]. They conjectured that every connected graphs but K_2 are antimagic.

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They showed the paths, cycles, and complete graphs besides K_2 are antimagic. In [1], N.Alon et al showed that the conjecture is true for dense graphs and the graph with $n(\geq 4)$ vertices and minimum degree $\Omega(\log n)$. They also proved that if G has $n(\geq 4)$ vertices and $\Delta(G) \geq n-2$, then G is antimagic and all complete partite graphs except K_2 are antimagic. Latterly D.Hefetz [3] proved that a graph with 3^k vertices, $k \in N$ is antimagic if it admits a K_3-factor. Recently, T.Wang [4] showed that the Cartesian products of cycles and regular graphs are antimagic graphs. In [5], he introduced new classes of antimagic graphs, called k-antimagic, through Cartesian and lexicographic products.

Suppose G is a graph. We define G^k to be the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{uv : u, v \in V(G), 1 \leq d_G(u,v) \leq k\}$, where $d_G(u,v)$ denote the distance between u and v in G. Let C_n be a n-vertices cycle with vertex set $\{x_1, x_2, ..., x_n\}$ and k be a positive integer with $n \geq 2k+1$. We use C_n^k to denote $(C_n)^k$. Hence C_n^k is the graph with vertex set $\{x_1, x_2, ..., x_n\}$ and edge set $\{x_i x_{i+j} : i = 1, 2, ..., n; j = 1, 2, ..., k\}$, where the subscript of x is taken modulo n. In this paper, we consider n is odd and $k = 2$. Our proof combines simple tools from analytic number theory and combinatorial techniques. The following is our main conclusion.

Theorem 1. Let n be an odd integer. Then the graph C_n^2 is antimagic. Moreover in our construction, the vertex sums form a set of successive integers.

2 Proof of Theorem 1

It is trivial that C_n^2 can be decomposed into 2 edge disjoint cycles C' and C'' where $V(C') = V(C'') = V(C_n^2) = \{x_1, x_2, ..., x_n\}$ and $E(C') = \{x_i x_{i+1} | i = 1, 2, ..., n\}$, $E(C'') = \{x_i x_{i+2} | i = 1, 2, ..., n\}$ where the subscript of x is modulo n. The method of the following proof is given two bijective maps α', α'' on $E(C')$ and $E(C'')$ respectively, than combine α', α'' together to form the vertex sums $w(x_i)$ of C_n^2, where $w(x_i) = \alpha'(x_i, x_{i+1}) + \alpha'(x_i, x_{i-1}) + \alpha''(x_i, x_{i+2}) + \alpha''(x_i, x_{i-2})$.

Define

$$\alpha' : E(C') \to \{1, 2, ..., n\}$$

by

$$\begin{cases} \alpha'(x_{2i-1}, x_{2i}) = i & for\ 1 \leq i \leq \frac{n+1}{2} \\ \alpha'(x_{2i}, x_{2i+1}) = \frac{n+1}{2} + i & for\ 0 \leq i \leq \frac{n-1}{2}. \end{cases}$$

Then for $i = \frac{n-1}{2} + j$ where $j = 1, 2, ..., \frac{n-1}{2}$ we have that

$$\begin{aligned}
\alpha(v_{2i-1}, v_{2i}) &= \alpha(v_{n-1+2j-1}, v_{n-1+2j}) \\
&= \alpha(v_{2(j-1)}, v_{2j-1}) \\
&= \frac{n+1}{2} + j - 1 \\
&= \frac{n+1}{2} + i - \frac{n-1}{2} - 1 \\
&= i,
\end{aligned}$$

and

$$\begin{aligned}
\alpha(v_{2i}, v_{2i+1}) &= \alpha(v_{n+(2j-1)}, v_{n+(2j-1)+1}) \\
&= \alpha(v_{2j-1}, v_{2j}) \\
&= j \\
&= i - \frac{n-1}{2}.
\end{aligned}$$

So, the definition of α' is the same as following:

$$\begin{cases} \alpha'(x_{2i}, x_{2i+1}) = i - \frac{n-1}{2} & for \ \ i = \frac{n+1}{2}, \frac{n+3}{2}, ..., n \\ \alpha'(x_{2i-1}, x_{2i}) = i & for \ \ i = \frac{n+1}{2}, \frac{n+3}{2}, ..., n. \end{cases}$$

Next, define

$$\alpha'' : E(C'') \to \{n+1, n+2, ..., 2n\}$$

by

$$\begin{cases} \alpha''(x_{2i}, x_{2i+2}) = 2n - \frac{i-1}{2} & for \ \ i = 1, 3, ..., n \\ \alpha''(x_{2i}, x_{2i+2}) = \frac{3n}{2} - \frac{i-1}{2} & for \ \ i = 2, 4, ..., n-1. \end{cases}$$

Then we calculate the vertex sums of C_n^2. For odd i and $1 \leq i \leq \frac{n-1}{2}$, we have

$$\begin{aligned}
w(x_{2i}) &= \alpha'(x_{2i-1}, x_{2i}) + \alpha'(x_{2i}, x_{2i+1}) \\
&\quad +\alpha''(x_{2i}, x_{2i+2}) + \alpha''(x_{2i}, x_{2i-2}) \\
&= i + (\frac{n+1}{2} + i) + (2n - \frac{i-1}{2}) + (\frac{3n}{2} - \frac{(i-1)-1}{2}) \\
&\quad (\text{since } \alpha''(x_{2i}, x_{2i-2}) = \alpha''(x_{2(i-1)}, x_{2i}) \\
&\qquad = \frac{3n}{2} - \frac{(i-1)-1}{2} \ \ \text{for} \ \ i-1 \ \ \text{even}) \\
&= 4n + i + 2. \cdots\cdots\cdots\cdots\cdots\cdots (1)
\end{aligned}$$

For odd i and $\frac{n+1}{2} \le i \le n$, we have

$$\begin{aligned}
w(x_{2i}) &= \alpha'(x_{2i-1}, x_{2i}) + \alpha'(x_{2i}, x_{2i+1}) + \\
&\quad \alpha''(x_{2i}, x_{2i+2}) + \alpha''(x_{2i}, x_{2i-2}) \\
&= i + (i - \frac{n-1}{2}) + (2n - \frac{i-1}{2}) + (\frac{3n}{2} - \frac{(i-1)-1}{2}) \\
&= 3n + i + 2. \cdots\cdots\cdots\cdots\cdots\cdots (2)
\end{aligned}$$

For even i and $1 \le i \le \frac{n-1}{2}$, we have

$$\begin{aligned}
w(x_{2i}) &= \alpha'(x_{2i-1}, x_{2i}) + \alpha'(x_{2i}, x_{2i+1}) \\
&\quad +\alpha''(x_{2i}, x_{2i+2}) + \alpha''(x_{2i}, x_{2i-2}) \\
&= i + (\frac{n+1}{2} + i) + (\frac{3n}{2} - \frac{i-1}{2}) + (2n - \frac{(i-1)-1}{2}) \\
&\quad (\text{since } \alpha''(x_{2i}, x_{2i-2}) = \alpha''(x_{2(i-1)}, x_{2i}) \\
&\qquad = 2n - \frac{(i-1)-1}{2} \quad \text{for} \quad i-1 \quad \text{odd}) \\
&= 4n + i + 2. \cdots\cdots\cdots\cdots\cdots\cdots (3)
\end{aligned}$$

And for even i and $\frac{n+1}{2} \le i \le n$, we have

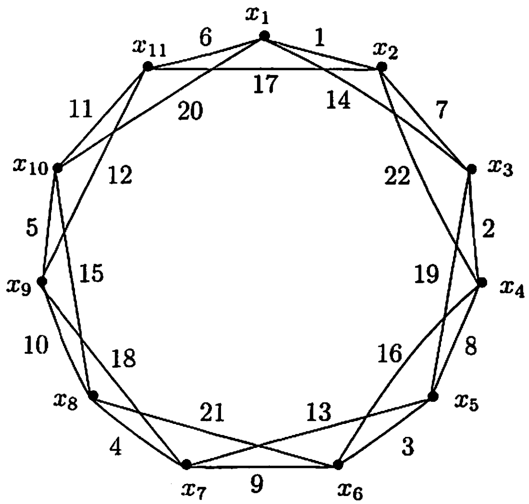
$$\begin{aligned}
w(x_{2i}) &= \alpha'(x_{2i-1}, x_{2i}) + \alpha'(x_{2i}, x_{2i+1}) \\
&\quad +\alpha''(x_{2i}, x_{2i+2}) + \alpha''(x_{2i}, x_{2i-2}) \\
&= i + (i - \frac{n-1}{2}) + (\frac{3n}{2} - \frac{i-1}{2}) + (2n - \frac{(i-1)-1}{2}) \\
&= 3n + i + 2. \cdots\cdots\cdots\cdots\cdots\cdots (4)
\end{aligned}$$

Now combine (1), (2), (3)and (4) together, we have

$$w(x_{2i}) = \begin{cases} 4n + i + 2 & if \;\; 1 \le i \le \frac{n-1}{2} \\ 3n + i + 2 & if \;\; \frac{n+1}{2} \le i \le n. \end{cases}$$

Thus C_n^2 is antimagic and the set of vertex sums is $\{\frac{7n+5}{2}, \frac{7n+7}{2}, \cdots, 4n + 2, 4n+3, \cdots, \frac{9n+3}{2}\}$, a set of successive integers. □

Following is our construction for C_{11}^2.



We hope the following conjecture is true :

Conjecture : Let n, k be any integer with $n \geq 2k+1$. The power of cycle C_n^k is antimagic.

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