Edge colorings of planar graphs with maximum degree five *

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Abstract

A finite simple graph is of class one if its edge chromatic number is equal to the maximum degree of this graph. It is proved here that every planar graph with the maximum degree 5 and without 4 or 5-cycles is of class one. One of Zhou's results is improved.

Key words: Edge coloring; Planar graph; Class one

1 Introduction

All graphs considered here are finite and simple. Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of G by $\Delta(G)$. If $v \in V(G)$, then its neighbor set $N_G(v)$ (or simply $N(v)$) is the set of the vertices in G adjacent to v. For $V' \subseteq V(G)$, denote $N(V') = \cup_{u \in V'} N(u)$. Given a plane graph G, let $F(G)$ be the face set of G. A face of a graph is said to be *incident* with all edges and vertices in its boundary. Two faces sharing an edge e are said to be *adjacent* at e. The *degree* of x, denoted by $d(x)$, is the number of vertices adjacent to x if $x \in V(G)$, or the number of edges incident with x where each cut edge is counted twice, if $x \in F(G)$.

A graph is *k-edge-colorable*, if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The *edge chromatic number* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. In 1964, Vizing showed that if G is a graph with maximum degree Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$. A graph G is said to be of *class one* if $\chi'(G) = \Delta$, and of *class two* if $\chi'(G) = \Delta + 1$. A graph G is

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critical if it is connected and of class two, and $\chi'(G-e) < \chi'(G)$ for every edge e of G. A critical graph with maximum degree Δ is called a Δ-*critical graph*. It is clear that every critical graph is 2-connected.

For planar graphs, more is known. As noted by Vizing [1], if C_4, K_4, the octahedron, and the icosahedron have one edge subdivided each, class two planar graphs are produced for $\Delta \in \{2,3,4,5\}$. He proved that every planar graph with $\Delta \geq 8$ is of class one (There are more general results, see [2] and [5]) and then conjectured that every planar graph with maximum degree 6 or 7 is of class one. The case $\Delta = 7$ for the conjecture has been verified by Zhang [8] and, independently, by Sanders and Zhao [6]. The case $\Delta = 6$ remains open, but some partial results are obtained. Theorem 16.3 [1] stated that a planar graph with the maximum degree Δ and the girth g is of class one if $\Delta \geq 3$ and $g \geq 8$, or $\Delta \geq 4$ and $g \geq 5$, or $\Delta \geq 5$ and $g \geq 4$. Lam, Liu, Shiu and Wu [7] proved that a planar graph G is of class one if it satisfies one of the following conditions: (1) $\Delta \geq 6$ and G contains no 4-cycles; (2) $\Delta \geq 6$ and no two 3-cycles of G sharing a common vertex; (3) $\Delta \geq 5$ and G contains no 4-cycles and 5-cycles; (4) $\Delta \geq 5$ and G contains no 4-cycles and has no two 3-cycles sharing a common vertex; (5) $\Delta \geq 4$ and G contains no i-cycles, where $4 \leq i \leq 14$; (6) $\Delta \geq 4$ and G contains no 4-through 6-cycle and has no two cycles sharing a common vertex. Zhou [9] obtained that every planar graph with $\Delta = 6$ and without 4 or 5-cycles is of class one. Recently, We proved in [7] that every planar graph with $\Delta \geq 6$ and without 6-cycles is of class one. In the paper, we shall improve some above results by proving that every planar graph with $\Delta = 5$ and without 4 or 5-cycles is of class one.

2 The main result and its proof

To prove our result, we will introduce some known lemmas.

Lemma 2.1. [1] *If G is a graph of class two, then G contains a k-critical subgraph for each k satisfying* $2 \leq k \leq \Delta(G)$.

Lemma 2.2. (*Vizing's Adjacency Lemma* [1]). *Let G be a Δ-critical graph, and let u and v be adjacent vertices of G with $d(v) = k$. Then*

(*a*) *if $k < \Delta$, then u is adjacent to at least $\Delta - k + 1$ vertices of degree* Δ;

(*b*) *if $k = \Delta$, then u is adjacent to at least two vertices of degree Δ.*

From the Vizing's Adjacency Lemma, it is easy to get the following corollary.

Corollary 2.3. *Let G be a Δ-critical graph. Then*

(a) *every vertex is adjacent to at most one* 2*-vertex and at least two* Δ*-vertices*;

(b) *the sum of the degree of any two adjacent vertices is at least* $\Delta + 2$;

(c) *if* $uv \in E(G)$ *and* $d(u)+d(v) = \Delta+2$, *then every vertex of* $N(\{u,v\}) \setminus \{u,v\}$ *is a* Δ*-vertex.*

Lemma 2.4. [8] *Let* G *be a* Δ*-critical graph,* $uv \in E(G)$ *and* $d(u)+d(v) = \Delta + 2$. *Then*

(a) *every vertex of* $N(N(\{u,v\})) \setminus \{u,v\}$ *is of degree at least* $\Delta - 1$;

(b) *if* $d(u), d(v) < \Delta$, *then every vertex of* $N(N(\{u,v\})) \setminus \{u,v\}$ *is a* Δ*-vertex.*

Lemma 2.5. [6] *No* Δ*-critical graph has distinct vertices* x, y, z *such that* x *is adjacent to* y *and* z, $d(z) < 2\Delta - d(x) - d(y) + 2$, *and* xz *is in at least* $d(x) + d(y) - \Delta - 2$ *triangles not containing* y.

To be convenient, for a plane graph G, a k-vertex or k^+-vertex is a vertex of degree k or at least k, respectively. Similarly, we define a k-face or k^+-face. A k-face of G is called an $(i_1, i_2, \cdots, i_k)$-face if the vertices in its boundary are of degrees $i_1, i_2, \cdots, i_k$ respectively. A 3-face is denoted by $[x, y, z]$ if it is incident with distinct vertices x, y, z and $d(x) \le d(y) \le d(z)$.

Theorem 2.6. *Let* G *be a planar graph with* $\Delta = 5$. *If* G *contains no* 4*-cycle or* 5*-cycle, then* G *is of class one.*

Proof. Suppose that G is a counterexample to our theorem with the minimum number of edges and suppose that G is embedded in the plane. Then G is a 5-critical graph by Lemma 2.1, and it is 2-connected. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have

$$\sum_{x \in V(G)} (2d(x) - 6) + \sum_{x \in F(G)} (d(x) - 6) = -12. \tag{1}$$

We define ch to be the *initial charge.* Let $ch(x) = 2d(x) - 6$ for each $x \in V(G)$ and $ch(x) = d(x) - 6$ for each $x \in F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12. \tag{2}$$

If we can show that $ch'(x) \ge 0$ for each $x \in V(G) \cup F(G)$, then we obtain a contradiction to (2), completing the proof.

First, we assume that G contains no 4-cycle. Then the discharging rules are defined as follows.

R1-1. For each 2-vertex v, every 5-vertex adjacent to v sends 1 to v.

R1-2. For each a 3-face $f = [x, y, z]$, (i) if f is a $(k, 5, 5)$-face, k=2, 3 or 4, then each of y, z sends $\frac{3}{2}$ to f; (ii) if f is a (k, k, k)-face, k=4 or 5, then each of x, y, z sends 1 to f; (iii) if f is a $(3, 4, 5)$-face, then y sends 1 to f and z sends 2 to f; (iv) if f is a $(4, 4, 5)$-face, then each of x, y sends $\frac{3}{4}$ to f and z sends $\frac{3}{2}$ to f.

R1-3. For each 5-face f, if each vertex incident with f is of degree at least 4, then each of them sends $\frac{1}{4}$ to f, otherwise every 4-vertex incident with f sends $\frac{1}{4}$ to it and every 5-vertex incident with f sends $\frac{1}{3}$ to it.

Let f be a 3-face. Then $ch(f) = -3$, and by Lemma 2.2, each 3-face must be one of the types $(2, 5, 5)$, $(3, 4, 5)$, $(3, 5, 5)$ and $(4^+, 4^+, 4^+)$-face. Thus, by R1-2, $ch'(f) = -3 + 3 = 0$.

Let f be a 5-face. Then $ch(f) = -1$. If there is a 2-vertex incident with f, by Corollary 2.3, f is a $(2, 5, 5, 5, 5)$-face. Thus, $ch'(f) = -1 + \frac{1}{3} \times 4 > 0$. If there is a 3-vertex incident with f, then f must be one of the types $(3, 3^+, 5, 5, 5)$ and $(3, 4, 4, 5, 5)$-face by Lemma 2.4. By R1-3, we have $ch'(f) \geq 0$ in each type. Otherwise, every vertex incident with f is of degree at least 4. It follows that $ch'(f) = -1 + \frac{1}{4} \times 5 > 0$.

Let f be a 6^+-face. Then $ch'(f) = ch(f) \geq 0$.

Let v be a 2-vertex. Then $ch(v) = -2$, and by Corollary 2.3, v is adjacent to two 5-vertices, so $ch'(v) = -2 + 1 \times 2 = 0$.

Let v be a 3-vertex. Then $ch'(v) = ch(v) = 0$.

Let v be a 4-vertex. Then $ch(v) = 2$, v is adjacent to at most one 3-vertex and $\min\{d(u)|u \in N(v)\} \geq 3$ by Lemma 2.2. Since G contains no 4-cycle, v can be incident with at most two 3-faces. By R1-2 and R1-3, we know that v sends at most 1 to each incident 3-face and $\frac{1}{4}$ to each incident 5-face. If v is incident with no 3-faces, then v is incident with at most four 5-faces. Thus $ch'(v) \geq 2 - \frac{1}{4} \times 4 > 0$. If v is incident with exactly one 3-face, then v can be incident with at most three 5-faces. Thus $ch'(v) \geq 2 - 1 - \frac{1}{4} \times 3 > 0$. Now, assume that v is incident with exactly two 3-faces. If there is a 3-face f incident with v which receives 1 from v by R1-2, then f is a $(4, 4, 4)$-face or $(3, 4, 5)$-face. In each case, the other incident 3-face of v must be a $(4, 5, 5)$-face by corollary 2.3. It follows that $ch'(v) \geq 2 - 1 - 0 - \frac{1}{4} \times 2 > 0$. Otherwise, v sends at most $\frac{3}{4}$ to each 3-face by R1-2. So we have $ch'(v) \geq 2 - \frac{3}{4} \times 2 - \frac{1}{4} \times 2 = 0$.

Let v be a 5-vertex. Then $ch(v) = 4$, v is adjacent to at most one 2-vertex, and $\min\{d(u)|u \in N(v)\} \geq 2$ by Lemma 2.2. Since G contains no 4-cycle, there are at most two 3-faces incident with v. If v is not incident with 3-faces, then v can be incident with at most five 5-faces. Thus $ch'(v) \geq 4 - 1 - \frac{1}{3} \times 5 > 0$ by R1-1 and R1-3. If v is incident with exactly one 3-face, then there are at most four 5-faces incident with v. Whether v is adjacent to a 2-vertex or not, we have $ch'(v) \geq 4 - \max\{1 + \frac{3}{2} + \frac{1}{3} \times 4, 2 + \frac{1}{3} \times 4\} > 0$.

Now, assume that v is incident with exactly two 3-faces. We consider the following cases.

Case 1-1. $\min\{d(u)|u \in N(v)\} = 2$. Let u be the 2-vertex which receives 1 from v by R1-1. Then the other vertices adjacent to v are of degree 5 by corollary 2.3, and each of the 3-faces incident with v could only be a $(2,5,5)$-face or a $(5,5,5)$-face. If the edge vu is incident with a $(2,5,5)$-face, then the other face incident with vu is a 6^+-face since G contains no 4-cycle. Thus, we have that v is incident with at most two 5-faces each of which is a $(4^+,4^+,5,5,5)$-face by Lemma 2.4, and then $ch'(v) \geq 4-1-\frac{3}{2}-1-\frac{1}{4}\times 2 = 0$. Otherwise, v sends 1 to each of its incident 3-faces by R1-2. It follows that $ch'(v) \geq 4 - 1 - 1 \times 2 - \frac{1}{3} \times 3 = 0$.

Case 1-2. $\min\{d(u)|u \in N(v)\} = 3$. Then v is adjacent to at least three 5-vertices and at most two 3-vertices. If v sends 2 to a $(3,4,5)$-face by R1-2, then the other 3-face incident with v could only be a $(5,5,5)$-face by Lemma 2.4. Thus, $ch'(v) \geq 4-2-1-\frac{1}{3}\times 3 = 0$. Otherwise, v sends at most $\frac{3}{2}$ to each of its incident 3-faces, and then $ch'(v) \geq 4 - \frac{3}{2} \times 2 - \frac{1}{3} \times 3 = 0$.

Case 1-3. $\min\{d(u)|u \in N(v)\} \geq 4$. Then v sends at most $\frac{3}{2}$ to each of its incident 3-faces, and we also have $ch'(v) \geq 4 - \frac{3}{2} \times 2 - \frac{1}{3} \times 3 = 0$.

Now we assume that G contains no 5-cycle. Then the discharging rules are defined as follows.

R2-1. For each 2-vertex v, every 5-vertex adjacent to v sends 1 to v.

R2-2. Let $f = [x, y, z]$ be a 3-face.

R2-21. Suppose f is a $(2,5,5)$-face. If x is incident with a 7^+-face f', then each of y, z sends 1 to f and f' sends 1 to f via x, too. Otherwise, x is incident with a $(2,5,5,5)$-face f' by Corollary 2.3 (see Figure 1(a)). Then each of y, z sends $\frac{4}{3}$ to f and the 5-vertex which is incident with f' and not adjacent to x sends $\frac{1}{3}$ to f through f'.

R2-22. Suppose f is a $(3,4,5)$-face. Then y sends 1 to f and z sends 2 to f.

R2-23. Suppose f is a $(3,5,5)$-face. If f is adjacent to just one $(3,4,5)$-face, and assume that they are adjacent at xy, then y send 1 to f and z sends 2 to f. If f is adjacent to two $(3,4,5)$-faces (see Figure 1(b)), let v be the other 5-vertex which is adjacent to the 4-vertex here, then each of y, z, v sends 1 to f. Otherwise each of y, z sends $\frac{3}{2}$ to f.

R2-24. Suppose f is a $(4,4,5)$-face. Then each of x, y sends $\frac{4}{5}$ to f and z sends $\frac{7}{5}$ to f.

R2-25. Suppose f is a $(4,4,4)$-face, $(4,5,5)$-face, or $(5,5,5)$-face. Then each of x, y, z sends 1 to f.

R2-3. For each 4-face f, if f is a $(k,5,5,5)$-face, k=2 or 3, then each of the 5-vertices sends $\frac{2}{3}$ to f; if f is a $(3,3,5,5)$-face or $(3,4,5,5)$-face, then each of the 5-vertices sends 1 to f; otherwise each 4^+-vertex incident with f sends $\frac{1}{2}$ to f.

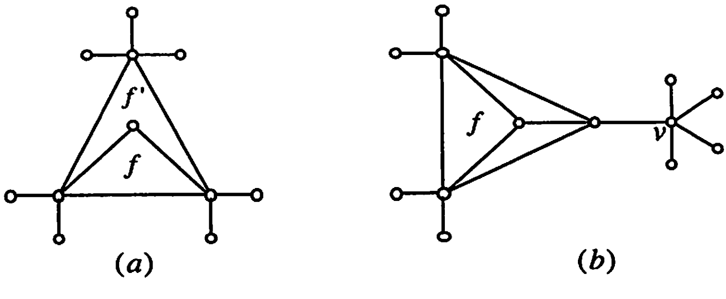

Figure 1: The two special cases of R2-21 and R2-23 respectively.

Let f be a 3-face. Then $ch(f) = -3$, and each 3-face must be one of the types $(2,5,5)$, $(3,4,5)$, $(3,5,5)$ and $(4^+,4^+,4^+)$-face by Lemma 2.2. Thus, by R2-2, $ch'(f) = -3+3 = 0$.

Let f be a 4-face. Then $ch(f) = -2$, and by Corollary 2.3, each 4-face must be one of the types $(2,5,5,5)$, $(3,3^+,5,5,)$, and $(4^+,4^+,4^+,4^+)$-face. Thus, f receives exactly 2 from its incident vertices by R2-3 and $ch'(f) = 0$.

Let f be a 6-face. Then $ch'(f) = ch(f) = 0$.

Let f be a 7^+-face. Then f just sends charges out via 2-vertices by R2-21. By Corollary 2.3 and Lemma 2.4, there are at most $\lceil \frac{d(f)}{7} \rceil$ 2-vertices in the boundary of f. It follows that $ch'(f) \geq d(f) - 6 - \lceil \frac{d(f)}{7} \rceil \geq 0$.

Let v be a 2-vertex. Then $ch(v) = -2$, and by Corollary 2.3, v is adjacent to two 5-vertices, so $ch'(v) = -2 + 1 \times 2 = 0$.

Let v be a 3-vertex. Then $ch'(v) = ch(v) = 0$.

Let v be a 4-vertex. Then $ch(v) = 2$, and $\min\{d(u) | u \in N(v)\} \geq 3$ by Lemma 2.2. Let s and t be the number of 3-faces and 4-faces incident with v, respectively. Then we have $s \leq 2$ since G contains no 5-cycle. If $s = 0$, then $t \leq 4$ and v sends at most $\frac{1}{2} \times 4$ out by R2-3. If $s = 1$, then $t \leq 1$ and v sends at most $1 + \frac{1}{2}$ out by R2-2 and R2-3. If $s = 2$, then $t = 0$ and v sends at most 1×2 out by R2-2. Thus, we have $ch'(v) \geq 0$.

Let v be a 5-vertex. Then $ch(v) = 4$, v is adjacent to at most one 2-vertex and at most two 3-vertices and $\min\{d(u) | u \in N(v)\} \geq 2$ by Lemma 2.2. Suppose v sends charges to its non-incident 3-faces just through its incident $(2,5,5,5)$-faces by R2-21. Then all vertices adjacent to v are of degree at least 4 by Lemma 2.4. Let s be the number of such $(2,5,5,5)$-faces incident with v. We have $1 \leq s \leq 2$ since G contains no 5-cycle, and each of such 4-faces receives $\frac{2}{3}$ from v by R2-3. If $s = 1$, then v sends at most $\frac{7}{5} \times 2$ to its other incident 3-faces and 4-faces. It follows that $ch'(v) \geq 4 - \frac{1}{3} - \frac{2}{3} - \frac{7}{5} \times 2 > 0$. If $s = 2$, then there is no other 3-face or

4-face incident with v, and so $ch'(v) \geq 4 - \frac{1}{3} \times 2 - \frac{2}{3} \times 2 > 0$. Suppose v sends 1 to a non-incident 3-face by R2-23. Then it is adjacent to four 5-vertices and a 4-vertex by Lemma 2.4. Whether v sends charges to other non-incident 3-faces by R2-21 or not, we have that v sends out at most 3 in all. It follows that $ch'(v) > 0$. Now we assume that v only sends charges to its incident faces and adjacent 2-vertices.

Suppose $\min\{d(u)|u \in N(v)\} = 2$. Let u be the 2-vertex adjacent to v. Then v is adjacent to four 5-vertices, and v sends 1 to u by R2-1. Each 3-face incident with v must be a $(2,5,5)$-face or a $(5,5,5)$-face, and by Lemma 2.4, each 4-face incident with v must be a $(2,5,5,5)$-face or a $(4^+,5,5,5)$-face. We use f_1 and f_2 to denote the two faces incident with uv, respectively. (i) If $d(f_1), d(f_2) \geq 6$, then v sends at most $\max\{1 \times 2, \frac{1}{2} \times 3\}$ to its incident faces. It follows that $ch'(v) \geq 4 - 1 - 2 > 0$. (ii) If $d(f_1) = 3$ and $d(f_2) \geq 6$, we have $d(f_2) \geq 7$ and v is incident with at most another two 3-faces or two 4-faces, since there is no 5-cycle in G. Thus $ch'(v) \geq 4 - 1 - 1 \times 3 = 0$. (iii) If $d(f_1) = 3$ and $d(f_2) = 4$, then v sends $\frac{4}{3}$ to f_1 by R2-1 and $\frac{2}{3}$ to f_2 by R2-3. Additionally, v sends at most 1 to the other 3-face or 4-face incident with it. So $ch'(v) \geq 4 - 1 - \frac{4}{3} - \frac{2}{3} - 1 = 0$. (iv) Otherwise, we have $d(f_1) = 4$ and $d(f_2) \geq 6$ or $d(f_1) = d(f_2) = 4$. In each case, v sends no more than 3 to its incident 3-faces and 4-faces. Thus $ch'(v) \geq 0$.

Let s and t be the number of 3-faces and 4-faces incident with v, respectively. Then we have $s \leq 3$ since G contains no 5-cycle. Suppose $\min\{d(u)|u \in N(v)\} = 3$. We consider the following three cases.

Case 2-1. v is adjacent to two 3-vertices u_1 and u_2. Then the other vertices adjacent to v are of degree 5, and each of vu_1 and vu_2 is not incident with 3-faces because of Lemma 2.5. So in this case, the 3-faces incident with v could only be $(5,5,5)$-faces and then $0 \leq s \leq 2$. If $s = 0$, then $t \leq 5$. There are at most one 4-face incident with v which receives 1 from it, and the other incident 4-faces of v are $(k,5,5,5)$-faces where $k \geq 3$ by Lemma 2.4. Thus $ch'(v) \geq 4 - 1 - \frac{2}{3} \times 4 > 0$ by R2-3. If $s = 1$, then $t \leq 2$. Since v sends at most 1 to each 4-face incident with it, we have $ch'(v) \geq 4 - 1 - 1 \times 2 > 0$. If $s = 2$, then $t \leq 1$, and it follows that $ch'(v) \geq 4 - 1 \times 2 - 1 > 0$.

Case 2-2. v is adjacent to a 3-vertex and a 4-vertex. Then the other vertices adjacent to v are of degree 5. If $s = 0$, then $t \leq 5$. There are at most one incident 4-face of v which receives 1 from v by R2-3, and each of the other 4-faces incident with v receives at most $\frac{2}{3}$ from it by Lemma 2.4. Thus $ch'(v) \geq 4 - 1 - \frac{2}{3} \times 4 > 0$. If $s = 1$, then $t \leq 2$. Since v sends at most 2 to the 3-face by R2-2 and at most 1 to each of these 4-faces by R2-3, we have $ch'(v) \geq 4 - 2 - 1 \times 2 = 0$. If $s = 2$, then $t \leq 1$. We know that v can not send 2 to each $(3,5,5)$-face by R2-23, otherwise it is adjacent to one 3-vertex and four 5-vertices by Lemma 2.4, which is a contradiction. Thus

v sends at most 3 to its incident 3-faces by R2-22, R2-23, and R2-25, and we have $ch'(v) \geq 4 - 3 - 1 = 0$. If $s = 3$, then $t = 0$, and we also have that v sends at most 4 out by R2-22, R2-23, and R2-25. Thus $ch'(v) \geq 0$.
Case 2-3. v is adjacent to a 3-vertex and four 5-vertices. Then each 3-face incident with v could only be a $(3,5,5)$-face or a $(5,5,5)$-face. If $s = 0$, then $t \leq 5$. There are at most one 4-face incident with v which receives 1 from v by R2-3 and the other 4-faces are $(k,5,5,5)$-faces where $k \geq 3$ by Lemma 2.4. Thus $ch'(v) \geq 4 - 1 - \frac{2}{3} \times 3 > 0$. If $s = 1$, then $t \leq 2$. By R2-23, R2-25, and R2-3, it follows that $ch'(v) \geq 4 - 2 - 1 \times 2 = 0$. If $s = 2$, then $t \leq 1$. In the case that v sends out 2 by R2-23, the other 3-face must be a $(5,5,5)$-face by Lemma 2.4. And in the other case v sends at most $\frac{3}{2} \times 2$ to its incident 3-faces. Thus $ch'(v) \geq 4 - 3 - 1 = 0$. If $s = 3$, then $t = 0$. We also have $ch'(v) \geq 0$ by R2-23 and R2-25.

Suppose $\min\{d(u) | u \in N(v)\} \geq 4$. Then each 3-face incident with v could only be a $(4,4,5)$-face, $(4,5,5)$-face or $(5,5,5)$-face. If $s = 0$, then $t \leq 5$. In the case that there is a 4-face which receives 1 from v, this 4-face must be a $(3,4,5,5)$-face and the other 4-faces incident with v are $(k,5,5,5)$-faces where $k \geq 2$. Thus $ch'(v) \geq 4 - 1 - \frac{2}{3} \times 4 > 0$. In the other case, v sends at most $\frac{2}{3}$ to each of its incident 4-faces, so we also have $ch'(v) > 0$. If $s = 1$, then $t \leq 2$, and $ch'(v) \geq 4 - \frac{7}{5} - 1 \times 2 > 0$ by R2-24, R2-25, and R2-3. If $s = 2$, then $t \leq 1$, and $ch'(v) \geq 4 - \frac{7}{5} \times 2 - 1 > 0$. If $s = 3$, then $t = 0$. We also have $ch'(v) \geq 4 - \frac{7}{5} \times 2 - 1 > 0$ by R2-24 and R2-25 because there are at most two $(4,4,5)$-faces incident with v. □

Remarks: (1) Li and Luo in [4] proved that a simple graph G with the maximum degree Δ and the girth g that is embeddable in a surface Σ of characteristic $\chi(\Sigma) \geq 0$ satisfies $\chi'(G) = \Delta$ if $\Delta \geq 3$ and $g \geq 9$, or $\Delta \geq 4$ and $g \geq 5$, or $\Delta \geq 5$ and $g \geq 4$. Here, if we make a further discussion in the above proof, it is easy to get the stronger result about the case $\Delta = 5$ and 4-cycle-free, with a surface of nonnegative characteristic instead of the plane.

(2) It is proved in [7] that every planar graph with $\Delta \geq 6$ and without 6-cycles is of class one. we conjecture that it is also true for $\Delta = 5$.

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