On 2-Factors with Chorded Quadrilaterals in Graphs*

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Abstract

Let k be a positive integer and G a graph with order $n \geq 4k + 3$. It is proved that if the minimum degree sum of any two nonadjacent vertices is at least $n + k$, then G contains a 2-factor with $k + 1$ disjoint cycles $C_1, \ldots, C_{k+1}$ such that C_i are chorded quadrilateral for $1 \leq i \leq k - 1$ and the length of C_k is at most 4.

Key words: Degree condition; Vertex-disjoint; Disjoint cycle.
AMS subject classification: 05C35, 05C38.

1 Terminology and Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges and we use Bondy and Murty [1] for terminology and notation not defined here. Let $G = (V, E)$ be a graph, the order of G is $|G| = |V|$ and its size is $e(G) = |E|$. A set of subgraphs is said to be vertex-disjoint or independent if no two of them have common vertex in G, and we use disjoint or independent to stand for vertex-disjoint throughout this paper. Let G_1 and G_2 be two subgraphs of G or subsets of $V(G)$. If G_1 and G_2 have no common vertex in G, we define $E(G_1, G_2)$ to be the set of edges of G between G_1 and G_2, and let $e(G_1, G_2) = |E(G_1, G_2)|$. Let H be a subgraph of G and $u \in V(G)$ a vertex, $N(u, H)$ is the set of neighbors of u contained in H. We write $d(u, H) = d_H(u) = |N(u, H)|$. Clearly, $d(u, G)$ is the degree of u in G, and we write $d(x)$ to replace $d(x, G)$. If there is no fear of confusion, we often identify a subgraph H of G with its vertex set $V(H)$. For a subset U of $V(G)$, we denote by $G[U]$ the subgraph of G induced by U and write $d_H(U) = \sum_{x \in U} d_H(x)$ for a subgraph H of G. Let C be a cycle,

*This work is supported by research grants from Ningxia University under grant number: (E)ndzr09-1 and Scientific research project in Xinjiang under grant number: XJEDU2009S101.

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we use $l(C)$ to denote the length of C. That is, $l(C) = |C|$. A Hamiltonian cycle of G is a cycle which contains all vertices of G, and a Hamiltonian path of G is a path of G which contains every vertex in G. A cycle of length 4 is called a quadrilateral. For a noncomplete graph G, let $\sigma_2(G) = \min\{d(x) + d(y) | xy \notin E(G)\}$; if G is a complete graph, let $\sigma_2(G) := \infty$. Let D be the graph obtained from K_4 by removing exactly one edge.

In his very excellent paper [3], Enomoto proposed the following interesting conjecture.

Conjecture 1.1 *[3] Let s and k be two positive integers with $1 \leq s \leq k$ and G be graph with order $n \geq 3s + 4(k - s) + 3$. Suppose $\sigma_2(G) \geq n + s$. Then G can be partitioned into $k + 1$ disjoint cycles $H_1, \ldots, H_{k+1}$ satisfying $|H_i| = 3$ for $1 \leq i \leq s$ and $|H_i| \leq 4$ for $s < i \leq k$.*

It is probably the first step to specify the length of $|H_i|$ for $s < i \leq k$ to solve Enomoto's Conjecture. The following result obtained by Yan stated that the length of these cycles is four.

Theorem 1.2 *[4] Let s and k be two positive integers with $1 \leq s \leq k$ and G be graph with order $n \geq 3s + 4(k - s) + 3$. Suppose $\sigma_2(G) \geq n + s$. Then G contains k disjoint cycles $H_1, \ldots, H_k$ satisfying $|H_i| = 3$ for $1 \leq i \leq s$ for $1 \leq i \leq s$ and $|H_i| = 4$ for $s < i \leq k$.*

Very recently, we improve the condition $n \geq 3s+4(k-s)+3$ of Theorem 1.2 to $n \geq 3s + 4(k - s) + 1$, and it seems that the following conjecture is the further step to solve Enomoto's Conjecture.

Conjecture 1.3 *Let s and k be two positive integers with $1 \leq s \leq k$ and G be graph with order $n \geq 3s + 4(k - s) + 1$. Suppose $\sigma_2(G) \geq n + s$. Then G contains k disjoint cycles $H_1, \ldots, H_k$ satisfying $|H_i| = 3$ for $1 \leq i \leq s$ and H_i contains D as a spanning subgraph for $s < i \leq k$.*

The main purpose of this paper is to prove the following theorem.

Theorem 1.4 *Suppose G is a graph of order $n \geq 4k + 3$ with $\sigma_2(G) \geq n + k$. Then G contains a 2-factor with $k + 1$ disjoint cycles $C_1, \ldots, C_{k+1}$ such that C_i are chorded quadrilateral for $1 \leq i \leq k - 1$ and $l(C_k) \leq 4$.*

In the proof of Theorem 1.4, we make use of the following theorem which solves the packing problem for D.

Theorem 1.5 *[6] Let k be a positive integer and G a graph of order $n \geq 4k + 1$. If $\sigma_2(G) \geq n + k$, then G contains k disjoint D.*

2 Lemmas

In the following, G is a graph of order $n \geq 3$.

Lemma 2.1 *[5] Let $P = x_1x_2\ldots x_m$ be a path of G with $m \geq 2$ and $y \in V(G) - V(P)$. If $d(y,P) + d(x_m,P) \geq m+1$, then G has a path P' from x_1 to y such that $V(P') = V(P) \cup \{y\}$.*

Lemma 2.2 *Let $P = x_1\ldots x_p$ be a path with $p \geq 1$, $M = y_1z_1$ be an edge and S such that all of them are disjoint, where S is isomorphic to D or K_4. Suppose $e(\{x_1,x_p\}\cup M, S) \geq 11$, then $G[V(M\cup P\cup S)]$ contains two disjoint subgraphs S' and P' such that P' is a path of order $p+1$, where S' is isomorphic to D or K_4.*

Proof For convenience, we write $V(S) = \{a,b,c,d\}$ so that $d_S(a) \geq d_S(b) \geq d_S(c) \geq d_S(d)$. Note that $d_S(a) = d_S(b) = 3$ and $d_S(c) = d_S(d) \geq 2$.

First suppose that $P = x_1$. Then the condition implies that $e(M,S) + 2d(x_1,S) \geq 11$. Since $e(M,S) \leq 8$, thus, $d(x_1,S) \geq 2$. If $d(x_1,S) = 4$, then $e(M,S) = 0$, so $e(M,S) + 2d(x_1,S) = 8 < 11$, a contradiction. Hence, $d(x_1,S) \leq 3$. If $d(x_1,S) = 3$, denote $S' = G[N(x_1,S)\cup\{x_1\}]$, we see that $G[V(M\cup P\cup S)]$ contains two required subgraphs S' and $P' = y_1z_1$. So, it remains the case $d(x_1,S) = 2$. Then we obtain $e(M,S) \geq 7$. By symmetry, we may assume that $d(y_1,S) = 4$ and $d(z_1,S) \geq 3$. If $az_1 \notin E(G)$, then $G[\{y_1,z_1,b,d\}] \supseteq D$, which disjoints from the path $P' = ca$. Consequently, $az_1 \in E(G)$ and $bz_1 \in E(G)$ by symmetry. Furthermore, we may assume that $cz_1 \in E(G)$ by symmetry. Then $G[\{y,a,d,b\}] \supseteq D$, which disjoints from the path $P' = cz_1$.

Hence, we may assume that the order of the path P is at least 2. Note that

$$11 \leq e(\{x_1,x_p\}\cup M,S) = e(\{x_p,y_1\},S) + e(\{x_1,z_1\},S),$$

and $e(\{x_p,y_1\},S) \leq 8$, we may assume that $e(\{x_p,y_1\},S) \geq 6$ and then $e(\{x_1,z_1\},S) \geq 3$. Furthermore, we observe that $e(\{x_p,y_1\},S) \leq 6$. Otherwise, it is easy to see that $G[V(M\cup S\cup P)]$ contains two required disjoint subgraphs. Then it follows that $e(\{x_p,y_1\},S) = 6$ and $e(\{x_1,z_1\},S) \geq 5$. If $d(y_1,S) = 4$, then we have nothing to prove as $d(x_p,S) \geq 2$. So, we assume $2 \leq d(y_1,S) \leq 3$ and then $d(x_p,S) \geq 3$.

Case 1. $d(y_1,S) = 3$. Then $d(x_p,S) = 3$.

Suppose that $N(y_1,S) = \{a,b,c\}$. If $x_pd \in E(G)$, then $G[V(M\cup S\cup P)]$ contains two required subgraphs $S' = G[\{y_1,a,b,c\}]$ and $P' = P+d$. Therefore, $x_pd \notin E(G)$ and then $\{a,b,c\} = N(x_p,S)$. However, we observe $N(z_1,S)\cap N(x_1,S) = \emptyset$, which contradicts the fact that $e(\{x_1,z_1\},S) \geq 5$. Hence, by symmetry, we may assume that $N(y_1, S) = \{d, b, c\}$. As $G[\{y_1, d, b, c\}] \supseteq D$, $x_1 a \notin E(G)$ and $x_p a \notin E(G)$. Then $N(x_p, S) = \{d, b, c\}$. Note that $N(x_1, S) \cap N(z_1, S) \subseteq \{b\}$, it follows from $e(\{x_1, z_1\}, S) \geq 5$ that $\{z_1 b, x_1 b\} \subseteq E(G)$. Hence, $G[V(M \cup S \cup P)]$ contains two required disjoint subgraphs $S' = G[\{y_1, z_1, b, d\}]$ and $P' = P + c$.

Case 2. $d(y_1, S) = 2$. Then $d(x_p, S) = 4$.

Suppose $N(y_1, S) = \{c, d\}$. If $d(z_1, S) \geq 3$, then we have nothing to prove. Hence, we may assume that $d(z_1, S) \leq 2$ and so $d(x_1, S) \geq 3$. By symmetry, say $cx_1 \in E(G)$. Then $G[V(M \cup S \cup P)]$ contains two required subgraphs $S' = G[\{x_p, a, b, d\}$ and $P' = x_{p-1} \dots x_1 c y_1$. Hence, by symmetry, we may assume that $N(y_1, S) = \{a, b\}$ or $N(y_1, S) = \{c, a\}$. In both cases, since $G[\{y_1, a, c, b\}] \supseteq D$, then we can choose $P' = P + d$. The proof is complete. □

Lemma 2.3 *[2] Let $P = x_1 \dots x_k$ be a path of G with $k \geq 3$. If $d(x_1, P) + d(x_k, P) \geq k$, then $G[V(P)]$ contains a cycle C such that $V(C) = V(P)$.*

3 Proof of Theorem 1.4

Let G be a graph of order $n \geq 4k + 3$ with $\sigma_2(G) \geq n + k$. Suppose that Theorem 1.4 is false. According to Theorem 1.5, G contains k vertex-disjoint subgraphs $S_1, \dots, S_k$ such that S_i is isomorphic D or K_4 for each $i \in \{1, \dots, k\}$. We choose k disjoint $S_1, \dots, S_k$ in G such that

$$\text{The length of a longest path in } G - V\left(\bigcup_{i=1}^{k} S_i\right) \text{ is maximized.} \tag{1}$$

Let $P = x_1 \cdots x_p$ be a longest path of $G - V(\bigcup_{i=1}^{k} C_i)$. Subject to (1), we choose k disjoint subgraphs $S_1, \dots, S_k$ and P in G such that

$$\text{Size of the maximum matching in } G - V\left(\bigcup_{i=1}^{k} S_i\right) \cup V(P) \text{ is maximum.} \tag{2}$$

Let $H = \bigcup_{i=1}^{k} S_i$, $F = G - V(H)$ and $|F| = f$. Clearly, $f \geq 3$ as $n \geq 4k + 3$. Furthermore, let $M = \{y_1 z_1, \dots, y_r z_r\}$ be a maximum matching of $F - V(P)$. We suppose that F contains no hamiltonian cycle. Our proof includes several claims.

For convenience, for $i = 1, \dots, k$, we write $V(S_i) = \{a_i, b_i, c_i, d_i\}$ so that $d_{S_i}(a_i) \geq d_{S_i}(b_i) \geq d_{S_i}(c_i) \geq d_{S_i}(d_i)$. Note that $d_{S_i}(a_i) = d_{S_i}(b_i) = 3$ and $d_{S_i}(c_i) = d_{S_i}(d_i) \geq 2$.

Claim 1. $p + 2r \geq f - 1$.

Proof On the contrary, suppose that $p + 2r \leq f - 2$. Let w_1 and w_2 be two non-adjacent vertices in $F - V(P) \cup V(M)$ subject to (2). Then $e(\{w_1, w_2\}, y_i z_i) \leq 2$ for each $i \in \{1, 2, \ldots, r\}$ by the maximality of M. We prove that $e(\{w_1, w_2\}, P) \leq p$. If $p = 1$, then by (2), we see that $e(\{w_1, w_2\}, P) = 0 < 1$. Thus, it remains the case that $p \geq 2$, by the maximality of P and Lemma 2.1, we see that $e(\{w_1, w_2\}, P) \leq p$. Thus, $e(\{w_1, w_2\}, F) \leq p + 2r \leq f - 2$. It follows that

$$e(\{w_1, w_2\}, H) \geq n + k - (f - 2) = 5k + 2.$$

This implies that there exists $S_i \in H$ such that $e(\{w_1, w_2\}, S_i) \geq 6$. Without loss of generality, say $d(w_1, S_i) \geq d(w_2, S_i)$. Then $d(w_1, S_i) \geq 3$ and $d(w_2, S_i) \geq 2$. We will show that $G[V(S_i) \cup \{w_1, w_2\}]$ contains a subgraph S_i' and an edge e such that they are disjoint, where S_i' is isomorphic D or K_4.

If $d(w_1, S_i) = 4$, it is obvious as $d(w_2, S_i) \geq 2$. So we may assume that $N(w_1, S_i) = N(w_2, S_i)$ and $d(w_1, S_i) = d(w_2, S_i) = 3$. By symmetry, we may assume that $\{d_i, b_i, c_i\} = N(w_1, S_i)$ or $\{a_i, b_i, c_i\} = N(w_1, S_i)$. In both cases, we can choose $S_i' = G[\{w_1, c_i, w_2, b_i\}]$ and $e = a_i d_i$.

In both cases, replace S_i with S_i' resulting a contradiction with the maximality of M while (1) still holds. Thus, $p + 2r \geq f - 1$. □

Claim 2. $p \geq f - 1$.

Proof By contradiction, suppose that $p \leq f-2$. According to Claim 1, we see that $M \neq \emptyset$. Since P is a longest path in F, let $R = \{x_1, x_p, y_1, z_1\}$. By the maximality of P and Lemma 2.1, we obtain $e(\{x_1, y_1\}, P) \leq p$ and $e(\{x_p, z_1\}, P) \leq p$. Note that $e(\{x_1, x_p\}, F - V(P)) = 0$, Thus, $e(R, F) \leq 2p + 2(f - p - 1) = 2f - 2$. As $x_1 y_1 \notin E(G)$ and $x_p z_1 \notin E(G)$, we obtain

$$e(R, H) \geq 2(n + k) - (2f - 2) = 10k + 2.$$

This implies that there exists some $S_i \in H$ such that $e(R, S_i) \geq 11$. By Lemma 2.2, $G[V(S_i \cup P) \cup \{y_1, z_1\}]$ contains a subgraph $S_i' \supseteq D$ and a path P' of order $p + 1$ such that S_i' and P' are disjoint. Replace S_i with S_i', we obtain a contradiction with (1). Thus, $p \geq f - 1$. □

Claim 3. We can properly choose $S_1, \ldots, S_k$ such that P is a hamiltonian path in F.

Proof Otherwise, suppose $p < f$. By Claim 2, $p = f - 1$. Take $y \in V(F - P)$. By Lemma 2.1, $d(x_p, P) + d(y, P) \leq p$. So, $d(x_p, F) + d(y, F) \leq p + d(y, F - P) \leq p + f - p - 1 = f - 1$. It follows that $d(x_1, H) + d(x_p, H) + 2d(y, H) \geq 2(n + k) - 2(f - 1) = 10k + 2$. This implies that there exists $S_i \in H$ such that $d(x_1, S_i) + d(x_p, S_i) + 2d(y, S_i) \geq 11$.

Now we will show that $G[V(S_i \cup F)]$ can be partitioned into a subgraph $S_i' \supseteq D$ and P' of order f such that they are disjoint, a contradiction. Clearly, $d(y, S_i) \geq 2$. If $d(y, S_i) = 4$, as $d(x_1, S_i) + d(x_p, S_i) \geq 3$, we may assume that $zx_1 \in E(G)$ with $z \in V(S_i)$. Then $G[V(S_i - z) \cup \{y\}] \supseteq S_i' \supseteq D$, which disjoints the path $P' = P + z$. Hence, we have $d(y, S_i) \leq 3$ and so $d(x_1, S_i) + d(x_p, S_i) \geq 5$. Without loss of generality, assume $d(x_1, S_i) \geq d(x_p, S_i)$. Then $d(x_1, S_i) \geq 3$ and $d(x_p, S_i) \geq 1$.

Subclaim 3.1. $d(y, S_i) \leq 2$.

Otherwise, suppose $d(y, S_i) = 3$. We observe that $G[N(y, S_i) \cup \{y\}] \supseteq D$, thus, it follows that $N(y, S_i) = N(x_1, S_i)$ and so $d(x_p, S_i) \geq 2$. If $N(y, S_i) = \{a_i, b_i, c_i\}$, then $x_p d_i \notin E(G)$. If $a_i x_p \in E(G)$, then we can choose $S_i' = G[\{y, b_i, x_1, c_i\}]$ and $P' = P - x_1 + a_i d_i$. Hence, $a_i x_p \notin E(G)$ and so $b_i x_p \notin E(G)$ by symmetry. It follows that $d(x_p, S_i) \leq 1$, a contradiction. Therefore, by symmetry, we assume $N(y, S_i) = \{d_i, b_i, c_i\}$. Clearly, $x_p c_i \notin E(G)$ and $x_p d_i \notin E(G)$. Consequently, $N(x_p, S_i) = \{a_i, b_i\}$. Then we can choose $S_i' = G[\{x_1, b_i, y, d_i\}]$ and $P' = P - x_1 + a_i c_i$ such that they are disjoint.

By Subclaim 3.1, we obtain $d(y, S_i) = 2$ and so $d(x_1, S_i) = 4$ and $d(x_p, S_i) \geq 3$. Furthermore, we observe that $N(y, S_i) = \{c_i, d_i\}$. As $d(x_p, S_i) \geq 3$, by symmetry, we may assume that $\{a_i, b_i, c_i\} \subseteq N(x_p, S_i)$ or $\{c_i, b_i, d_i\} \subseteq N(x_p, S_i)$. In both cases, we can choose $S_i' = G[\{x_p, a_i, b_i, c_i\}]$ and $P' = P - x_p + d_i y$ such that S_i' and P' are disjoint. This completes the proof for Claim 3. □

By Claim 3, $P = x_1 \ldots x_f$ is a Hamiltonian path in F. Subject to this fact, we choose $S_1, \ldots, S_k$ and P such that

$$\sum_{i=1}^{k} |E(S_i)| \text{ is maximized.} \tag{3}$$

Claim 4. For each $1 \leq i \leq k$, $d(x_1, S_i) + d(x_f, S_i) \leq 6$.

Proof Otherwise, suppose $d(x_1, S_1) + d(x_f, S_1) \geq 7$. Then we may assume that $d(x_1, S_1) = 4$ and $d(x_f, S_1) \geq 3$. By symmetry role of c_1 and d_1, we may assume that $x_f d_1 \in E(G)$. Then $G[V(S_1 \cup P)]$ can be partitioned into a triangle $a_1 b_1 c_1 a_1$ and a cycle $C' = x_f d_1 x_1 \ldots x_f$ such that they are disjoint, a contradiction. This completes the proof. □

Clearly, $x_1 x_f \notin E(G)$. Applying Lemma 2.3 to P, $d(x_1, P) + d(x_f, P) \leq f - 1$. Then $d(x_1, H) + d(x_f, H) \geq n + k - (f - 1) = 5k + 1$. This implies that there exists $S_i \in H$, say S_1, such that $d(x_1, S_1) + d(x_f, S_1) \geq 6$. By Claim 4, $d(x_1, S_1) + d(x_f, S_1) = 6$. Without loss of generality, we assume that $d(x_1, S_1) \geq d(x_f, S_1)$ throughout the rest of this paper. Then $d(x_1, S_1) \geq 3$ and $d(x_f, S_1) \geq 2$.

Claim 5. If $d(x_1, S_1) = 4$, then $S_1 \cong D$ and $N(x_f, S_1) = \{a_1, b_1\}$.

Proof If $x_f d_1 \in E(G)$, then replace S_1 and P with $G[\{x_1, a_1, b_1, c_1\}]$ and $P - x_1 + d_1$, respectively, we see that $S_1 \cong K_4$ by the choice of (3). Suppose $c_1 x_f \in E(G)$, then $G[V(S_1 \cup P)]$ can be partitioned into a triangle $a_1 b_1 d_1 a_1$ and a cycle $C' = x_f c_1 x_1 \ldots x_f$ such that they are disjoint, a contradiction. Therefore, $x_f c_1 \notin E(G)$ and $S_1 \cong D$. As $d(x_f, S_1) \geq 2$, it follows that $N(x_f, S_1) = \{a_1, b_1\}$. □

We derive the following claim to demonstrate the basic structure of the graph G.

Claim 6. $d(x_1, S_1) = 3$ and $d(x_f, S_1) = 3$.

Proof It suffices to prove $d(x_1, S_1) \leq 3$. Otherwise, suppose $d(x_1, S_1) = 4$. By Claim 5, $S_1 \cong D$ and $N(x_f, S_1) = \{a_1, b_1\}$. Now denote $P_1 = c_1 x_1 \ldots x_{f-1}$ and $G_1 = G[V(S_1 \cup P)]$. Since $a_1 b_1 d_1 a_1$ is a triangle, we obtain $d(c_1, P_1) + d(x_{f-1}, P_1) \leq f - 1$ by applying Lemma 2.3 and the assumption that Theorem 1.4 is false. Note that $x_{f-1} c_1 \notin E(G)$ and $x_{f-1} d_1 \notin E(G)$. As $S_1 \cong D$, we obtain $d(x_{f-1}, G_1) + d(c_1, G_1) \leq f + 4$. Consequently, it follows that

$$e(\{c_1, x_{f-1}, x_1, x_f\}, H - S_1) \geq 2(n + k) - (2f + 9) = 10(k - 1) + 1. \quad (4)$$

Then there exists $S_i \in H - S_1$, say S_2, such that $e(\{c_1, x_{f-1}, x_1, x_f\}, S_2) \geq 11$.

In view of the existence of S_1 and $x_f a_1 d_1 b_1 x_f$, we may assume that $e(\{x_1, x_f\}, S_2) \geq e(\{c_1, x_{f-1}\}, S_2)$. Then $e(\{x_1, x_f\}, S_2) \geq 6$. By Claim 4, we obtain $e(\{x_1, x_f\}, S_2) = 6$ and so $e(\{c_1, x_{f-1}\}, S_2) \geq 5$.

Suppose that $d(x_1, S_2) = 4$. By Claim 5, it turns out that $S_2 \cong D$, $N(x_f, S_2) = \{a_2, b_2\}$. If $x_{f-1} c_2 \in E(G)$, then $G[V(G_1 \cup S_2)]$ can be partitioned into S_1, $S_2 - c_2 + x_f$ and $x_1 c_2 x_{f-1} \ldots x_1$, a contradiction. Hence, $x_{f-1} c_2 \notin E(G)$. By the symmetric role of c_2 and d_2, $x_{f-1} d_2 \notin E(G)$. As $e(\{c_1, x_{f-1}\}, S_2) \geq 5$, we may assume that $x_{f-1} a_2 \in E(G)$ by symmetry. Then at lest one of c_2 and d_2 is not neighbor of c_1, otherwise, $G[V(G_1 \cup S_2)]$ can be partitioned into $c_1 c_2 b_2 d_2 c_1$, $x_f b_1 d_1 a_1 x_f$ and $x_1 a_2 x_{f-1} \ldots x_1$, a contradiction. Without loss of generality, say $c_1 d_2 \notin E(G)$. Then it follows that $N(c_1, S_2) = \{a_2, b_2, c_2\}$ and $N(x_{f-1}, S_2) = \{a_2, b_2\}$. However, we see that $G[V(G_1 \cup S_2)]$ can be partitioned into $c_1 c_2 b_2 c_1$, $S_1 - c_1 + x_f$ and $x_1 d_2 a_2 x_{f-1} \ldots x_1$ such that $c_1 c_2 b_2 c_1$ is a triangle, a contradiction again. Consequently, $d(x_1, S_2) \leq 3$ and so $d(x_f, S_2) \leq 3$ by symmetry. Then it follows from $e(\{x_1, x_f\}, S_2) = 6$ that $d(x_f, S_2) = d(x_1, S_2) = 3$.

Suppose $N(x_f, S_2) \neq N(x_1, S_2)$. Note that $N(x_f, S_2) \cap N(x_1, S_2) \neq \emptyset$, then $S_2 \cong D$ by our assumption that Theorem 1.4 is false. We divide the proof into two cases by symmetry.

Case a. $N(x_1, S_2) = \{a_2, b_2, d_2\}$.

By our assumption, $x_f c_2 \in E(G)$. However, if we replace S_2 and P with $S_2 - c_2 + x_1 \cong K_4$ and $P - x_1 + c_2$, respectively, then $S_2 \cong K_4$ by our choice (3), which contradicts the fact that $S_2 \cong D$.

Case b. $N(x_1, S_2) = \{d_2, b_2, c_2\}$.

By our assumption, $x_f a_2 \in E(G)$. We have $x_f c_2 \in E(G)$, for otherwise, $\{x_f b_2, x_f d_2\} \subseteq E(G)$. If we replace S_2 and P with the subgraph $G[\{x_f, a_2, b_2, d_2\}]$ and $P - x_f + c_2$, respectively, we obtain $S_2 \cong K_4$ by (3), which contradicts the fact $S_2 \cong D$ again. Consequently, $x_1 c_2 x_f \ldots x_1$, $S_2 - c_2$ and S_1 is a partition of $G[V(G_1 \cup S_2)]$.

Hence, $N(x_f, S_2) = N(x_1, S_2)$. Since $S_2 \cong D$, we may assume that $d_2 \in N(x_1, S_2) \cap N(x_f, S_2)$ by the symmetry role of d_2 and c_2. Then, $x_1 d_2 x_f \ldots x_1$, $S_2 - d_2$ and S_1 is a desired partition of $G[V(G_1 \cup S_2)]$. This completes the proof for Claim 6. □

Claim 7. $N(x_1, S_1) = N(x_f, S_1)$.

Proof By contradiction, suppose $N(x_f, S_1) \neq N(x_1, S_1)$. By Claim 6, $d(x_1, S_1) = d(x_f, S_1) = 3$. If $N(x_1, S_1) = \{a_1, b_1, c_1\}$, then we have $x_f d_1 \in E(G)$. Clearly, $G[\{a_1, b_1, c_1, x_1\}] \supseteq S_1' \cong K_4$. If we replace S_1 and P with S_1' and $P - x_1 + d_1$, respectively, we obtain $S_1 \cong K_4$ by (3). As $d(x_f, S_1) = 3$, we may assume that $b_1 x_f \in E(G)$. Consequently, G_1 can be partitioned into two disjoint cycles $S_1 - b_1$ and $x_1 b_1 x_f \ldots x_1$. Therefore, by symmetry, it suffices to consider the case $\{d_1, b_1, c_1\} = N(x_1, S_1)$. Then by our assumption, $x_f a_1 \in E(G)$. We must have $x_f d_1 \notin E(G)$, for otherwise, $S_1 - d_1$ and $x_1 d_1 x_f \ldots x_1$ is a partition of G_1. Consequently, $V(S_1) - \{d_1\} = N(x_f, S_1)$. It is easy to see that $S_1 \cong D$. However, if we replace S_1 and P with $S_1 - d_1 + x_f$ and $P - x_f + d_1$, respectively, we see that $S_2 \cong K_4$ by (3), a contradiction. □

Now we are in the position to complete Theorem 1.4. According to Claim 7, we know $N(x_1, S_1) = N(x_f, S_1)$. Therefore, we may assume that $d_1 \in N(x_1, S_1) \cap N(x_f, S_1)$ by Claim 6. Consequently, $S_1 - d_1$ and $x_1 d_1 x_f \ldots x_1$ is a desired partition of G_1, and so G contains a 2-factor with $k + 1$ disjoint cycles $S_1 - d_1, S_2, \ldots, S_k$ and $x_1 d_1 x_f \ldots x_1$, a final contradiction. This completes the proof of Theorem 1.4.

Acknowledgments

The authors would like to thank the referees for their extremely careful reading of the paper, and their detailed corrections and helpful suggestions.

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