Chromatic Uniqueness of Complete Bipartite Graphs With Certain Edges Deleted

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ABSTRACT

For integers p, q, s with $p \geq q \geq 3$ and $1 \leq s \leq q-1$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2−connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. In this paper, we prove that for any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, if $9 \leq s \leq q-1$ and $\Delta(G') = s-3$ where $G' = K_{p,q} - G$, then G is chromatically unique.

Keywords: Chromatic Polynomial; Chromatically unique; Chromatically equivalent.

1. Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be *chromatically equivalent* (or simply χ−*equivalent*), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ−*unique*) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ−*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint*, or simply χ−*disjoint*.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2−connected) bipartite graphs which

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can be obtained from $K_{p,q}$ by deleting a set of s edges. For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E,\ x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$. Let $\Delta(G')$ denote the maximum degree of G'.

Dong et al. [1] have shown that any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, is chromatically unique if one of the following conditions holds.

(i) $5 \leq s \leq q-1$ and $\Delta(G') = s-1$, or

(ii) $7 \leq s \leq q-1$ and $\Delta(G') = s-2$.

In this paper, we give a similar result by examining the chromatic uniqueness of $G \in \mathcal{K}_2^{-s}(p,q)$, where $9 \leq s \leq q-1$ and $\Delta(G') = s-3$. This result was obtained by using the same approach introduced by Dong et al. in [1].

2. Preliminary results and notation

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [3]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma A. *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

Partition $\mathcal{K}^{-s}(p,q)$ into the following subsets:

$$\mathcal{D}_i(p,q,s) = \left\{ G \in \mathcal{K}^{-s}(p,q) \;\middle|\; \Delta(G') = i \right\}, \quad i = 1, 2, \ldots, s.$$

Throughout this paper, we fix the following conditions for p, q and s:

$$p \geq q \geq 3 \qquad \text{and} \qquad 1 \leq s \leq q-1.$$

The following result was obtained in [1].

Theorem B. *Let $p \geq q \geq 3$ and $1 \leq s \leq q-1$.*

(i) $\mathcal{D}_1(p,q,s)$ is χ-closed.

(ii) $\cup_{2 \leq i \leq (s+3)/2} \mathcal{D}_i(p,q,s)$ is χ-closed for $s \geq 2$.

(iii) $\mathcal{D}_i(p,q,s)$ *is* χ*−closed for each* i *with* $\lceil (s+3)/2 \rceil \le i \le min\{s, q-2\}$.

(iv) $\mathcal{D}_{q-1}(p,q,s) \cap \mathcal{K}_2^{-s}(p,q)$ *is* χ*−closed for* $s = q-1$.

For a bipartite graph $G = (A, B; E)$, let $\mathcal{I}(G)$ be the set of independent sets in G and

$$\Omega(G) = \{ Q \in \mathcal{I}(G) \mid Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \}.$$

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{1}$$

Lemma C (Dong et al. [2]). *For* $G \in \mathcal{K}^{-s}(p,q)$,

$$\alpha'(G,3) = |\Omega(G)| \ge 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4−independent partitions $\{ A_1, A_2, A_3, A_4 \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned}(2^{|A|-1} - 1)(2^{|B|-1} - 1) &+ \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \\ &= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|} + 3^{|B|}) - 2.\end{aligned}$$

Define

$$\alpha'(G,4) = \alpha(G,4) - \left\{ (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|} + 3^{|B|}) - 2 \right\}.$$

Observe that for $G, H \in \mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4) = \alpha(H,4) \quad \text{if and only if} \quad \alpha'(G,4) = \alpha'(H,4).$$

The following lemmas will be used to prove our main result.

Lemma D (Dong et al. [2]). *For* $G = (A, B; E) \in \mathcal{K}^{-s}(p,q)$ *with* $|A| = p$ *and* $|B| = q$,

$$\begin{aligned}\alpha'(G,4) = &\sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) \\ &+ |\{ \{Q_1, Q_2\} \mid Q_1, Q_2 \in \Omega(G),\ Q_1 \cap Q_2 = \emptyset \}|.\end{aligned}$$

Lemma E (Dong et al. [2]). *For a bipartite graph* $G = (A, B; E)$, *if* uvw *is a path in* G' *with* $d_{G'}(u) = 1$ *and* $d_{G'}(v) = 2$, *then for any* $k \ge 2$,

$$\alpha(G,k) = \alpha(G + uv, k) + \alpha(G - \{u,v\}, k-1) + \alpha(G - \{u,v,w\}, k-1).$$

By using lemma E, we obtain the following lemma.

Lemma F. *For a bipartite graph $G = (A, B; E)$, if uvw, uvy and wvy are three paths in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 3$, then for any $k \geq 2$,*

$$\begin{aligned}\alpha(G,k) &= \alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1)+ \\ &\quad \alpha(G-\{u,v,y\},k-1)+\alpha(G-\{u,v,w,y\},k-1).\end{aligned}$$

Proof. Since $P(G,\lambda) = P(G+uv,\lambda) + P(G\cdot uv,\lambda)$, we have

$$\alpha(G,k) = \alpha(G+uv,k) + \alpha(G\cdot uv,k).$$

Let x be the vertex in $G\cdot uv$ produced by identifying u and v, and z the vertex in $G\cdot uv\cdot xw$ produced by identifying x and w. Notice that x is adjacent to all vertices in $V(G\cdot uv) - \{x,w,y\}$ and z is adjacent to all vertices in $V(G(uv\cdot xw) - \{z,y\}$. Thus

$$\begin{aligned} G\cdot uv + xw + xy &= K_1 + (G-\{u,v\}), \\ (G\cdot uv + xw)\cdot xy &= K_1 + (G-\{u,v,y\}), \\ G\cdot uv\cdot xw + zy &= K_1 + (G-\{u,v,w\}) \quad \text{and} \\ G\cdot uv\cdot xw\cdot zy &= K_1 + (G-\{u,v,w,y\}). \end{aligned}$$

We also observe that for any graph H, $\alpha(K_1+H,k) = \alpha(H,k-1)$, since $P(K_1+H,\lambda) = \lambda P(H,\lambda-1)$. Hence

$$\begin{aligned}\alpha(G\cdot uv,k) &= \alpha(G\cdot uv+xw,k)+\alpha(G\cdot uv\cdot xw,k) \\ &= \alpha(G\cdot uv+xw+xy,k)+\alpha((G\cdot uv+xw)\cdot xy,k)+ \\ &\quad \alpha(G\cdot uv\cdot xw+zy,k)+\alpha(G\cdot uv\cdot xw\cdot zy,k) \\ &= \alpha(K_1+(G-\{u,v\}),k)+\alpha(K_1+(G-\{u,v,y\}),k)+ \\ &\quad \alpha(K_1+(G-\{u,v,w\}),k)+\alpha(K_1+(G-\{u,v,w,y\}),k) \\ &= \alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,y\},k-1)+ \\ &\quad \alpha(G-\{u,v,w\},k-1)+\alpha(G-\{u,v,w,y\},k-1).\end{aligned}$$

The lemma is then obtained. □

3. Main result

In [2], Dong et al. proved that every 2−connected graph in $\mathcal{D}_s(p,q,s)$ is χ−unique. Then, Dong et al. in [1] also proved that G is χ−unique for every $G \in \mathcal{D}_{s-1}(p,q,s)$, where $s \geq 5$, and that G is χ−unique for every $G \in \mathcal{D}_{s-2}(p,q,s)$, where $s \geq 7$. In this section, we shall prove that for each graph G in $\mathcal{D}_{s-3}(p,q,s)$, where $s \geq 9$, G is χ−unique. We first have the following lemma.

Lemma 1. *For any $G \in \mathcal{D}_{s-3}(p,q,s)$, where $s \geq 9$, G' is one of the 20 graphs in Figure 1.*

Our main result is Theorem 1.

Theorem 1. *For any $G \in \mathcal{K}_2^{-s}(p,q)$, with $p \geq q \geq s+1 \geq 10$, if $\Delta(G') = s-3$, then G is χ−unique.*

Proof. Since $s \geq 9$, $(s+3)/2 \leq s-3$ and thus by Theorem B(iii), $\mathcal{D}_{s-3}(p,q,s)$ is χ−closed. By Lemma 1, if $G \in \mathcal{D}_{s-3}(p,q,s)$, then G' is one of the graphs in Figure 1. Thus $\mathcal{D}_{s-3}(p,q,s)$ contain 48 graphs, which are named as V_1, V_2, ..., V_{48}. We show the graphs V_1, V_2, and V_{21} to V_{24} in Table 1. The reader may refer to [6] for the complete listing of the graphs. We then group these graphs according to their values of $\alpha'(V_i,3)$ which can be calculated by using Lemma C.

$$\begin{aligned}
\mathcal{T}_1 &= \{ V_1, V_2 \} \\
\mathcal{T}_2 &= \{ V_3, V_4 \} \\
\mathcal{T}_3 &= \{ V_5, V_6, V_7, V_8 \} \\
\mathcal{T}_4 &= \{ V_9, V_{10}, V_{11}, V_{12}, V_{13}, V_{14}, V_{15}, V_{16} \} \\
\mathcal{T}_5 &= \{ V_{17}, V_{18}, V_{19}, V_{20}, V_{21}, V_{22}, V_{23}, V_{24} \} \\
\mathcal{T}_6 &= \{ V_{25}, V_{26}, V_{27}, V_{28}, V_{29}, V_{30} \} \\
\mathcal{T}_7 &= \{ V_{31}, V_{32}, V_{33}, V_{34}, V_{35}, V_{36}, V_{37}, V_{38}, V_{39}, V_{40} \} \\
\mathcal{T}_8 &= \{ V_{41}, V_{42}, V_{43}, V_{44}, V_{45}, V_{46} \} \\
\mathcal{T}_9 &= \{ V_{47}, V_{48} \}.
\end{aligned}$$

Observe that for any i, j with $1 \leq i < j \leq 9$, $\alpha'(V_{i_1},3) > \alpha'(V_{j_1},3)$ if $V_{i_1} \in \mathcal{T}_i$ and $V_{j_1} \in \mathcal{T}_j$. Thus by Lemmas A and 1, $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 9)$ are χ−disjoint and since $\mathcal{D}_{s-3}(p,q,s)$ is χ−closed, each $\mathcal{T}_i$ $(1 \leq i \leq 9)$ is χ−closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are χ−unique, it suffices to show that for any two graphs, V_{i_1}, $V_{i_2} \in \mathcal{T}_i$, if $V_{i_1} \not\cong V_{i_2}$, then either $\alpha'(V_{i_1},4) \neq \alpha'(V_{i_2},4)$ or $\alpha(V_{i_1},5) \neq \alpha(V_{i_2},5)$. The values of $\alpha'(V_i,4)$ can be obtained by using Lemma D.

We shall establish several inequalities of the form $\alpha'(V_i,4) < \alpha'(V_j,4)$ for some i, j. Since the method used to obtain these inequalities is standard, long and rather repetitive, we shall not discuss all of them here. In the following we shall only show two examples of detail comparisons. In the first example, we compare $\alpha'(V_1,4)$ and $\alpha'(V_2,4)$ when $p > q$, and in the second example, we compute $\alpha(V_{21},5) - \alpha(V_{23},5)$ when $p = q$. The reader may refer to [4] for all the other detail comparisons.

(1) V_1 and V_2 when $p > q$

$$\alpha'(V_1,4) - \alpha'(V_2,4)$$

$$= \left[\left(\sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-i-1} + 2^{q-2} - 2) \right) + 7 \cdot 2^{p-1} + 7 \cdot 2^{q-2} + 2^{q-5} + \right.$$

$$\left. 7 \cdot 2^{s-4} - 35 \right] - \left[\left(\sum_{i=1}^{s-3} \binom{s-3}{i} (2^{q-i-1} + 2^{p-2} - 2) \right) + 7 \cdot 2^{p-2} + \right.$$

$$\left. 7 \cdot 2^{q-1} + 2^{p-5} + 7 \cdot 2^{s-4} - 35 \right]$$

$$= \left[\sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1}) \right] + 7(2^{p-1} - 2^{q-1}) -$$

$$7(2^{p-2} - 2^{q-2}) - (2^{p-5} - 2^{q-5})$$

$$< -7 \binom{s-3}{4} (2^{p-5} - 2^{q-5}) + 7 \cdot 2^4 (2^{p-5} - 2^{q-5}) -$$

$$7 \cdot 2^3 (2^{p-5} - 2^{q-5}) - (2^{p-5} - 2^{q-5})$$

$$< (2^{p-5} - 2^{q-5}) \left[-7 \binom{s-3}{4} + 7 \cdot 16 - 7 \cdot 8 - 1 \right]$$

$$= (2^{p-5} - 2^{q-5}) \left[-7 \binom{s-3}{4} + 55 \right]$$

$$\leq (2^{p-5} - 2^{q-5})(-50) \qquad \left[\text{since} \quad \binom{s-3}{4} \geq \binom{6}{4} = 15 \right]$$

$$< 0.$$

(2) V_{21} and V_{23} when $p = q$

When $p = q$, $\alpha'(V_{21}, 4) - \alpha'(V_{23}, 4) = 0$. Since $\alpha'(V_{21}, 4) = \alpha'(V_{23}, 4)$, we need to calculate $\alpha(V_{21}, 5) - \alpha(V_{23}, 5)$. Using Lemma F, we have

$$\alpha(V_{21}, 5) - \alpha(V_{23}, 5)$$

$$= \left[\alpha(V_{21} + a_1 b_1, 5) + \alpha(V_{21} - \{a_1, b_1\}, 4) + \alpha(V_{21} - \{a_1, b_1, c_1\}, 4) + \right.$$

$$\left. \alpha(V_{21} - \{a_1, b_1, d_1\}, 4) + \alpha(V_{21} - \{a_1, b_1, c_1, d_1\}, 4) \right] -$$

$$\left[\alpha(V_{23} + a_2b_2, 5) + \alpha(V_{23} - \{a_2, b_2\}, 4) + \alpha(V_{23} - \{a_2, b_2, c_2\}, 4) + \right.$$

$$\left. \alpha(V_{23} - \{a_2, b_2, d_2\}, 4) + \alpha(V_{23} - \{a_2, b_2, c_2, d_2\}, 4) \right]$$

$$= \Big(\alpha(V_{21} - \{a_1, b_1, c_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, c_2\}, 4) \Big) +$$

$$\Big(\alpha(V_{21} - \{a_1, b_1, d_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, d_2\}, 4) \Big) +$$

$$\Big(\alpha(V_{21} - \{a_1, b_1, c_1, d_1\}, 4) - \alpha(V_{23} - \{a_2, b_2, c_2d_2\}, 4) \Big)$$

since $V_{21} + a_1b_1 \cong V_{23} + a_2b_2$ and $V_{21} - \{a_1, b_1\} \cong V_{23} - \{a_2, b_2\}$

$$= \Big(\alpha'(V_{21} - \{a_1, b_1, c_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, c_2\}, 4) \Big) +$$

$$\Big(\alpha'(V_{21} - \{a_1, b_1, d_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, d_2\}, 4) \Big) +$$

$$\Big(\alpha'(V_{21} - \{a_1, b_1, c_1, d_1\}, 4) - \alpha'(V_{23} - \{a_2, b_2, c_2d_2\}, 4) \Big)$$

$$= \left(\sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-3-i} + 2^{q-3} - 2) - \sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-2-i} + 2^{q-4} - 2) \right) +$$

$$\left(\sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-3-i} + 2^{q-3} - 2) - \sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-2-i} + 2^{q-4} - 2) \right) \cdot$$

$$\left(\sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-4-i} + 2^{q-3} - 2) - \sum_{i=1}^{s-3} \binom{s-3}{i} (2^{p-2-i} + 2^{q-5} - 2) \right)$$

$$= 2\left(\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{q-4} - 2^{p-3-i}) + \sum_{i=1}^{s-3} \binom{s-3}{i}(3 \cdot 2^{q-5} - 3 \cdot 2^{p-4-i}) \right)$$

$$= 7\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{q-5} - 2^{p-4-i})$$

$$= 7\left\{ \binom{s-3}{1}(2^{q-5} - 2^{p-5}) + \binom{s-3}{2}(2^{q-5} - 2^{p-6}) + \right.$$

$$\left. \binom{s-3}{3}(2^{q-5} - 2^{p-7}) + \cdots + \binom{s-3}{s-3}(2^{q-5} - 2^{p-s-1}) \right\}$$

$$= 7\left\{ \binom{s-3}{1}(2^{p-5} - 2^{p-5}) + \binom{s-3}{2}(2^{p-5} - 2^{p-6}) + \right.$$

$$\left. \binom{s-3}{3}(2^{p-5} - 2^{p-7}) + \cdots + \binom{s-3}{s-3}(2^{p-5} - 2^{p-s-1}) \right\} \quad \text{since } p = q$$

$$> 0 \quad \text{since } s \geq 9$$

Similarly, we can show that for any two graphs, V_{i_1}, $V_{i_2} \in \mathcal{T}_i$, if $V_{i_1} \not\cong V_{i_2}$, then either $\alpha'(V_{i_1}, 4) \neq \alpha'(V_{i_2}, 4)$ or $\alpha(V_{i_1}, 5) \neq \alpha(V_{i_2}, 5)$ (see [4]). Hence the proof of the theorem is complete. □

Using the same method, we obtained the following extention of Theorem 1.

Theorem 2. (Roslan and Peng [5]) *For any $G \in \mathcal{K}_2^{-s}(p, q)$, with $p \geq q \geq s + 1 \geq 12$, if $\Delta(G') = s - 4$, then G is χ-unique.*

We end this paper with the following conjecture which is true for $t = 3$, 4, 5 and 6.

Conjecture. *For any $G \in \mathcal{K}_2^{-s}(p, q)$ with $p \geq q \geq s + 1 \geq 2t$ ($t = 3, 4, 5, \ldots$), if $\Delta(G') = s - t + 2$, then G is χ-unique.*

Acknowledgments. The authors would like to thank the referee for his helpful comments.

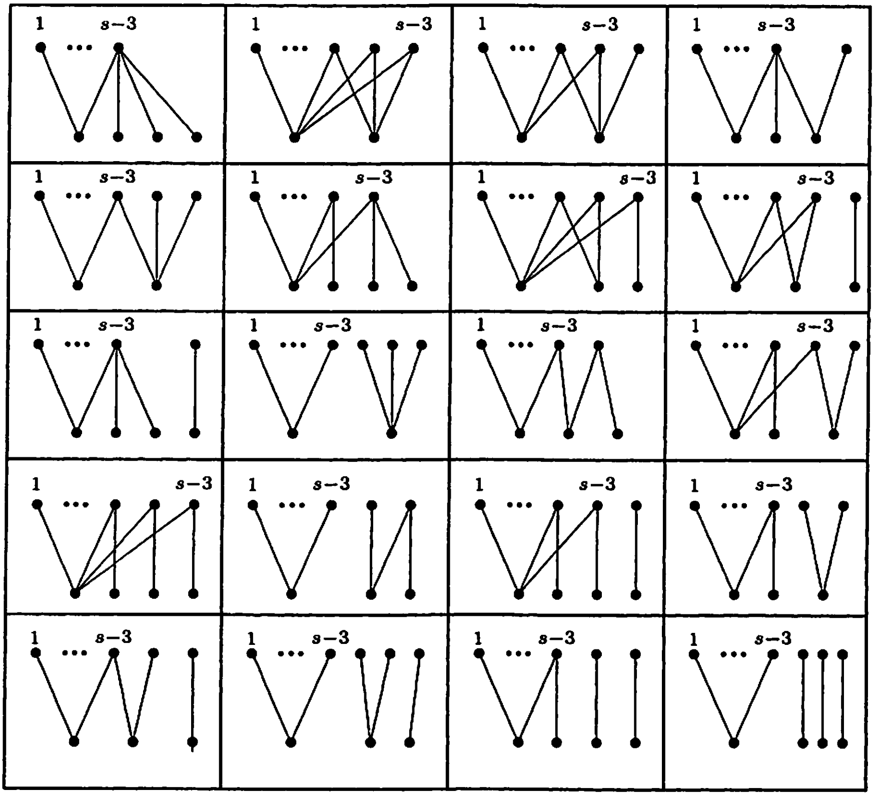


FIGURE 1

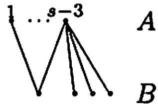
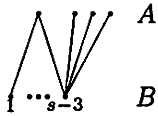
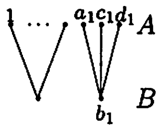
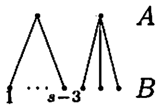
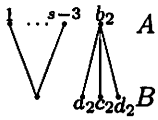
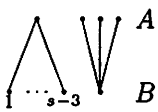
Graphs V_i	Graphs V_i' ($V_i' = K_{p,q} - V_i$) $\lvert A\rvert = p, \lvert B\rvert = q$	$\alpha'(V_i, 3)$	$\alpha'(V_i, 4)$
V_1		$2^{s-3} + 13$	$\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{p-i-1} + 2^{q-2} - 2) + 7 \cdot 2^{p-1} + 7 \cdot 2^{q-2} + 2^{q-5} + 7 \cdot 2^{s-4} - 35$
V_2		$2^{s-3} + 13$	$\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{q-i-1} + 2^{p-2} - 2) + 7 \cdot 2^{p-2} + 7 \cdot 2^{q-1} + 2^{p-5} + 7 \cdot 2^{s-4} - 35$
V_{21}		$2^{s-3} + 6$	$\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{p-i-1} + 2^{q-2} - 2) + 2^{p} + 3 \cdot 2^{p-4} + 7 \cdot 2^{q-2} + 7 \cdot 2^{s-3} - 21$
V_{22}		$2^{s-3} + 6$	$\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{q-i-1} + 2^{p-2} - 2) + 7 \cdot 2^{p-2} + 2^{q} + 3 \cdot 2^{q-4} + 7 \cdot 2^{s-3} - 21$
V_{23}		$2^{s-3} + 6$	$\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{p-i-1} + 2^{q-2} - 2) + 7 \cdot 2^{p-2} + 2^{q} + 3 \cdot 2^{q-4} + 7 \cdot 2^{s-3} - 21$
V_{24}		$2^{s-3} + 6$	$\sum_{i=1}^{s-3} \binom{s-3}{i}(2^{q-i-1} + 2^{p-2} - 2) + 2^{p} + 3 \cdot 2^{p-4} + 7 \cdot 2^{q-2} + 7 \cdot 2^{s-3} - 21$

TABLE 1

References

1 F.M. Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, An attempt to classify bipartite graphs by chromatic polynomials. *Discrete Math.* 222 (2000), 73–88.

2 F.M.Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, Sharp bounds for the numbers of 3-partitions and the chromaticity of bipartite graphs, *J. Graph Theory* 37 (2001), 48–77.

3 R.C. Read, W.T. Tutte, Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), *Selected Topics in Graph Theory III, Academic Press*, New York, 1988, pp. 15–42.

4 H. Roslan and Y.H. Peng, Chromatic Uniqueness of Complete Bipartite Graphs With Certain Edges Deleted. *Research Report* **2/2003**, *Institute for Mathematical Research*, Universiti Putra Malaysia (2003).

5 H. Roslan and Y.H. Peng, Chromatic Uniqueness of Complete Bipartite Graphs With Certain Edges Deleted, II. *Research Report* **9/2003**, *Institute for Mathematical Research*, Universiti Putra Malaysia (2003).

6. `http://www.fsas.upm.edu.my/~yhpeng/publish/table(s-3).pdf`