A new bound on maximum genus of simple graphs*

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Abstract: Let G be a connected simple graph with girth g and minimal degree $\delta \geq 3$. If G is not up-embeddable, then, when G is 1-edge connected,

$$\gamma_M(G) \geq \frac{D_1(\delta,g)-2}{2D_1(\delta,g)-1}\beta(G) + \frac{D_1(\delta,g)+1}{2D_1(\delta,g)-1};$$

when G is $k(k=2,3)$-edge connected,

$$\gamma_M(G) \geq \frac{D_k(\delta,g)-1}{2D_k(\delta,g)}\beta(G) + \frac{D_k(\delta,g)+1}{2D_k(\delta,g)}.$$

$D_k(\delta,g)(k=1,2,3)$ are increasing functions on δ and g.
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1 Introduction

Graph $G=(V(G),E(G))$ considered in this paper are all simple, undirected and connected. For graphical notations without explanation, see [1].

The *maximum genus*, $\gamma_M(G)$, of a connected graph G is the largest integer k such that there exists a cellular embedding of G in the orientable surface with genus k. Recall that any cellular embedding of G has at least one region. By the Euler polyhedral equation, the maximum genus

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$\gamma_M(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the cycle rank or Betti number of G. A graph G is *up-embeddable* if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$ exactly.

For a spanning tree T in G, $\xi(G,T)$ is the number of components of $G \backslash E(T)$ with odd number of edges. $\xi(G) = \min_T \xi(G,T)$ is called the *Betti deficiency number* of G, where the minimum is taken over all spanning trees T of G. There are two equivalent characterizations on the maximum genus of a graph, due to Xuong [12], Liu [7], and Nebesky [10], respectively. The following first theorem gives a formula on $\gamma_M(G)$ by means of $\xi(G)$ and $\beta(G)$.

Theorem A(Xuong [12], Liu [7]) *Let G be a graph, then $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$; and graph G is up-embeddable if and only if $\xi(G) \leq 1$.*

Let A be an edge subset of $E(G)$. $c(G \setminus A)$ denotes the number of components of $G \setminus A$, and $b(G \setminus A)$ denotes the number of components of $G \setminus A$ with odd Betti number. Let F be a subgraph of G, $E(F,G)$ is the set of edges which has one end in F and the other is not in F. For any set X, $|X|$ denotes the cardinality of X. In 1981, Nebesky [10] obtained a combinatorial expression of $\xi(G)$ in terms of an edge set.

Theorem B(Nebesky [10]) *Let G be a graph, then*

$$\xi(G) = \max_{A \subseteq E(G)} \{c(G \backslash A) + b(G \backslash A) - |A| - 1\}.$$

Based on Theorem B, Liu and Huang [8] provided some characterizations on not up-embeddable graphs.

Theorem C(Liu and Huang [8]) *Let G be a graph, if G is not up-embeddable, i.e., $\xi(G) \geq 2$, then there exists an edge subset $A \subseteq E(G)$ satisfying the following properties:*

(1) $c(G \backslash A) = b(G \backslash A) \geq 2$;

(2) *for any component F of $G \backslash A$, F is an induced subgraph of G;*

(3) $\xi(G) = 2c(G \backslash A) - |A| - 1$.

The study on maximum genus was inaugurated by Nordhaus, Stewart and White [11]. From then on, various classes of graphs have been proved up-embeddable. A formerly known result in paper[12] states that every 4-edge connected graph is up-embeddable. For the vertex-(or edge-) connectivity ≤ 3, there exists many graphs(see paper[6]) which are not up-embedabble. In this paper, we obtained a new lower bound on maximum genus of simple graphs with edge-connectivity ≤ 3 in terms of the girth and minimal degree, which complements the results in paper[3] and improves the results in paper[4] and [5].

2 The main results

The *degree* $d_G(v)$ of a vertex v is the number of edges incident with v. The *distance* between two vertices u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. For an edge $e = xy$ and a vertex v in G, define $d_G(e,v) = \min\{d_G(x,v), d_G(y,v)\}$ to be the *distance* between the edge $e = xy$ and vertex v in G. Especially, $d_G(uv,u) = d_G(uv,v) = d_G(u,u) = 0$. For any vertex or edge x in G, the $i(i \geq 0)$*neighbor set* of x in G is $N_i(x) = \{v \mid d_G(x,v) = i, v \in V(G)\}$.

A $\delta(\delta \geq 3)$-regular simple graph of girth g with the least possible number of vertices is called a (δ, g)-cage. By considering the vertices whose distance from a given vertex (or an edge) is at most $\lfloor (g-1)/2 \rfloor$, we can obtain the lower bound on the number of vertices of a (δ, g)-cage, which is called *Moore bound* $M(\delta, g)$ (see [2], pp.180):

$$M(\delta,g) = \begin{cases} 1+\delta+\cdots+\delta(\delta-1)^{r-1} = \dfrac{\delta(\delta-1)^r - 2}{\delta - 2}, & g = 2r+1, \\ 2(1+(\delta-1)+\cdots+(\delta-1)^{r-1}) = \dfrac{2(\delta-1)^r - 2}{\delta-2}, & g = 2r. \end{cases}$$

Clearly, a graph with girth g and minimal degree δ has at least $M(\delta, g)$ vertices.

Let H be a connected induced subgraph of G, the vertex $v \in V(H)$ is called a *touching vertex* of H when v is the end of some edges in $E(H,G)$.

Lemma 1 *Let G be a graph with girth g and minimal degree $\delta \geq 3$. H is a connected induced subgraph of G with $\beta(H) \geq 1$.*

(1) *If $|E(H,G)| \leq 2$, then $|V(H)| \geq M(\delta, g)$;*

(2) *If $|E(H,G)| = 3$, then $|V(H)| \geq M(\delta,g) - 1$.*

Proof See the proof of Lemma 2.1 in the paper[9]. □

Let G be a graph with girth g and minimal degree $\delta \geq 3$, $r = \lfloor \frac{g}{2} \rfloor$. If G is not up-embeddable, there exists an edge set $A \subseteq E(G)$ satisfies Theorem C. Define $C(G \setminus A)$ to be the set of components of $G \setminus A$, and

$$\begin{aligned} B_4 &= \{F \mid |E(F,G)| \geq 4, F \in C(G\setminus A)\}; \\ B_i &= \{F \mid |E(F,G)| = i, F \in C(G\setminus A)\}, \quad i = 1,2,3; \\ \beta_i &= \min\{\beta(F) \mid F \in B_i\}, \quad i = 1,2,3,4. \end{aligned}$$

Clearly,

$$c(G\setminus A) = |B_1| + |B_2| + |B_3| + |B_4|. \quad (1)$$

By Theorem C, any component $F \in C(G \setminus A)$ is an induced subgraph of G, then

$$2|E(F)| = \sum_{v \in V(F)} d_G(v) - |E(F,G)| \geq \delta |V(F)| - |E(F,G)|.$$

Hence, the Betti number of F is

$$\begin{aligned}\beta(F) &= |E(F)| - |V(F)| + 1 \\ &\geq \frac{\delta|V(F)| - |E(F,G)|}{2} - |V(F)| + 1 \\ &= \frac{(\delta-2)|V(F)| - |E(F,G)|}{2} + 1.\end{aligned}$$

From Lemma 1, when $F \in B_1 \cup B_2$, $|V(F)| \geq M(\delta, g)$; when $F \in B_3$, $|V(F)| \geq M(\delta, g) - 1$. Thus, when $g = 2r$,

$$\begin{aligned}\beta_1 &\geq \frac{(\delta-2)M(\delta,g) - 1}{2} + 1 = (\delta-1)^r - \frac{1}{2}, \\ \beta_2 &\geq \frac{(\delta-2)M(\delta,g) - 2}{2} + 1 = (\delta-1)^r - 1, \\ \beta_3 &\geq \frac{(\delta-2)\cdot(M(\delta,g)-1) - 3}{2} + 1 = (\delta-1)^r - \frac{\delta}{2} - \frac{1}{2};\end{aligned}$$

when $g = 2r + 1$,

$$\begin{aligned}\beta_1 &\geq \frac{(\delta-2)M(\delta,g) - 1}{2} + 1 = \frac{\delta(\delta-1)^r}{2} - \frac{1}{2}, \\ \beta_2 &\geq \frac{(\delta-2)M(\delta,g) - 2}{2} + 1 = \frac{\delta(\delta-1)^r - 2}{2}, \\ \beta_3 &\geq \frac{(\delta-2)(M(\delta,g)-1) - 3}{2} + 1 = \frac{\delta(\delta-1)^r}{2} - \frac{\delta}{2} - \frac{1}{2}.\end{aligned}$$

By Theorem C, β_1, β_2 and β_3 are odd, so we define the following functions on δ, g:

$$D_1(\delta,g) = \begin{cases} 2\lfloor\frac{(\delta-1)^r}{2}\rfloor + 1, & \text{when } g = 2r, \\ 2\lfloor\frac{\delta(\delta-1)^r}{4}\rfloor + 1, & \text{when } g = 2r+1; \end{cases}$$

$$D_2(\delta,g) = \begin{cases} 2\lfloor\frac{(\delta-1)^r - 1}{2}\rfloor + 1, & \text{when } g = 2r, \\ 2\lfloor\frac{\delta(\delta-1)^r - 2}{4}\rfloor + 1, & \text{when } g = 2r+1. \end{cases}$$

$$D_3(\delta,g) = \begin{cases} 2(\delta-1)^r - \delta, & \text{when } g = 2r, \\ \delta(\delta-1)^r - \delta, & \text{when } g = 2r+1. \end{cases}$$

Obviously, $D_1(\delta,g) \geq D_2(\delta,g) \geq 3$ and

$$\beta_1 \geq D_1(\delta,g),\ \beta_2 \geq D_2(\delta,g),\ 2\beta_3 + 1 \geq D_3(\delta,g). \qquad (2)$$

For each edge $e \in A$, the end vertices of e must belong to two distinct components of $G \setminus A$, since any component $F \in C(G\setminus A)$ is an induced subgraph of G. On the other hand, the edge $e \in E(F,G)$ must belong to A. Thus,

$$|A| = \frac{1}{2} \sum_{F \in C(G\setminus A)} |E(F,G)| \geq 2|B_4| + \frac{3}{2}|B_3| + |B_2| + \frac{1}{2}|B_1|. \quad (3)$$

In addition,

$$|E(G)| = \sum_{F \in C(G\setminus A)} |E(F)| + |A|, \quad |V(G)| = \sum_{F \in C(G\setminus A)} |V(F)|. \quad (4)$$

Combining equations(1)-(4), we obtain

$$\begin{aligned}
\beta(G) &= |E(G)| - |V(G)| + 1 \\
&= \sum_{F \in C(G\setminus A)} |E(F)| + |A| - \sum_{F \in C(G\setminus A)} |V(F)| + 1 \\
&= \sum_{F \in C(G\setminus A)} |\beta(F)| + |A| - c(G \setminus A) + 1 \\
&\geq \beta_4|B_4| + \beta_3|B_3| + \beta_2|B_2| + \beta_1|B_1| + |A| - c(G \setminus A) + 1 \\
&\geq (\beta_4 + 1)|B_4| + \frac{2\beta_3 + 1}{2}|B_3| + \beta_2|B_2| + \frac{2\beta_1 - 1}{2}|B_1| + 1 \\
&\geq \frac{2\beta_3 + 1}{2}|B_3| + \beta_2|B_2| + \frac{2\beta_1 - 1}{2}|B_1| + 1 \\
&\geq \frac{D_3(\delta,g)}{2}|B_3| + D_2(\delta,g)|B_2| + \frac{2D_1(\delta,g) - 1}{2}|B_1| + 1.
\end{aligned}$$

When G is 1-edge connected, by simple calculations, $D_3(\delta,g) \geq D_2(\delta,g) \geq \frac{1}{3}(2D_1(\delta,g) - 1)$, thus

$$\beta(G) \geq \frac{2D_1(\delta,g) - 1}{3}(\frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1|) + 1,$$

this implies

$$\frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| \leq \frac{3\beta(G) - 3}{2D_1(\delta,g) - 1}. \quad (5)$$

When G is 2-edge connected, $|B_1| = 0$, thus

$$\beta(G) \geq D_2(\delta,g)(\frac{1}{2}|B_3| + |B_2|) + 1,$$

this implies

$$\frac{1}{2}|B_3| + |B_2| \leq \frac{\beta(G) - 1}{D_2(\delta,g)}. \quad (6)$$

When G is 3-edge connected, $|B_1| = |B_2| = 0$, thus

$$\frac{1}{2}|B_3| \leq \frac{\beta(G) - 1}{D_3(\delta, g)}. \tag{7}$$

From Theorem C and equations (1), (3),

$$\begin{aligned}
\xi(G) &= 2c(G \setminus A) - |A| - 1 \\
&\leq 2(|B_4| + |B_3| + |B_2| + |B_1|) - 2|B_4| - \frac{3}{2}|B_3| - |B_2| - \frac{1}{2}|B_1| - 1 \\
&= \frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| - 1.
\end{aligned}$$

When G is 1-edge connected, by equation (5),

$$\xi(G) \leq \frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| - 1 \leq \frac{3\beta(G) - 2(D_1(\delta, g) + 1)}{2D_1(\delta, g) - 1}; \tag{8}$$

When G is 2-edge connected, by equation (6),

$$\xi(G) \leq \frac{1}{2}|B_3| + |B_2| - 1 \leq \frac{\beta(G) - (D_2(\delta, g) + 1)}{D_2(\delta, g)}. \tag{9}$$

When G is 3-edge connected, by equation (7),

$$\xi(G) \leq \frac{1}{2}|B_3| - 1 \leq \frac{\beta(G) - (D_3(\delta, g) + 1)}{D_3(\delta, g)}. \tag{10}$$

Now, by Theorem A, equations(8), (9) and (10), the following theorems are obtained.

Theorem 1 *Let G be a graph with girth g and minimal degree $\delta \geq 3$. If G is not up-embeddable, then*

$$\gamma_M(G) \geq \frac{D_1(\delta, g) - 2}{2D_1(\delta, g) - 1}\beta(G) + \frac{D_1(\delta, g) + 1}{2D_1(\delta, g) - 1}.$$

Proof By Theorem A and equation (8),

$$\begin{aligned}
\gamma_M(G) &= \frac{\beta(G) - \xi(G)}{2} \geq \frac{1}{2} \cdot (\beta(G) - \frac{3\beta(G) - 2(D_1(\delta, g) + 1)}{2D_1(\delta, g) - 1}) \\
&= \frac{D_1(\delta, g) - 2}{2D_1(\delta, g) - 1}\beta(G) + \frac{D_1(\delta, g) + 1}{2D_1(\delta, g) - 1}.
\end{aligned}$$

□

Theorem 2 *Let G be a $k(k=2,3)$-edge connected graph with girth g and minimal degree $\delta \geq 3$. If G is not up-embeddable, then*

$$\gamma_M(G) \geq \frac{D_k(\delta,g)-1}{2D_k(\delta,g)}\beta(G) + \frac{D_k(\delta,g)+1}{2D_k(\delta,g)}.$$

Proof By Theorem A , (11)and (12), the proof is the same with Theorem 1. []

Corollary 1 *Let G be a graph with girth g and minimal degree $\delta \geq 3$.*

(1) *When G is $k(k=2,3)$-edge connected and $\beta(G) \leq 3D_k(\delta,g)$, then G is up-embeddable;*

(2) *When G is 1-edge connected and $\beta(G) < 2D_1(\delta,g)$, then G is up-embeddable.*

Proof When G is $k(k=2,3)$-edge connected and is not up-embeddable, by Theorem 2,

$$\gamma_M(G) \geq \frac{D_k(\delta,g)-1}{2D_k(\delta,g)}\beta(G) + \frac{D_k(\delta,g)+1}{2D_k(\delta,g)} = \frac{\beta(G)}{2} - \frac{\beta(G)-D_k(\delta,g)-1}{2D_k(\delta,g)}.$$

As $\beta(G) \leq 3D_k(\delta,g)$, then

$$\frac{\beta(G)-D_k(\delta,g)-1}{2D_k(\delta,g)} < 1,$$

namely $\gamma_M(G) > \frac{\beta(G)}{2} - 1$. This means G is up-embeddable. When G is 1-edge connected, the proof is the same. []

Let G be a graph with minimal degree $\delta \geq 3$ and girth g, $r = \lfloor \frac{g}{2} \rfloor$. By simple calculation, $D_1(3,4)=5$ and $D_1(4,3)=7$; $D_1(3,g) \geq 2^r+1$, $D_2(3,g) \geq 2^r-1$, $D_3(3,g) \geq 2^{r+1}-3$; $D_1(\delta,3) \geq \frac{1}{2}\delta(\delta-1)$, $D_2(\delta,3) \geq \frac{1}{2}(\delta+1)(\delta-2)$ and $D_3(\delta,3) \geq \delta(\delta-2)$. Furthermore, the lower bound on $\gamma_M(G)$ in Theorem 1 and 2 are increasing functions on $D_k(\delta,g)(k=1,2,3)$ respectively; $D_k(\delta,g)(k=1,2,3)$ all are increasing functions on δ and g. Thus, the following corollaries are direct results from Theorem 1 and 2.

Corollary 2 *Let G be a graph with girth g and minimal degree $\delta \geq 3$. If G is not up-embeddable, then,*

(1) *when $g \geq 4$, $\gamma_M(G) \geq \frac{1}{3}\beta(G) + \frac{2}{3}$;*

(2) *when $\delta \geq 4$, $\gamma_M(G) \geq \frac{5}{13}\beta(G) + \frac{8}{13}$.*

Corollary 3 *Let G be a graph with girth g and minimal degree ≥ 3, $r = \lfloor \frac{g}{2} \rfloor$. If G is not up-embeddable, then,*

(1) *when G is 1-edge connected, $\gamma_M(G) \geq \frac{2^r-1}{2^{r+1}+1}\beta(G) + \frac{2^r+2}{2^{r+1}+1}$;*

(2) *when G is 2-edge connected, $\gamma_M(G) \geq \frac{2^{r-1}-1}{2^r-1}\beta(G) + \frac{2^{r-1}}{2^r-1}$;*

(3) *when G is 3-edge connected, $\gamma_M(G) \geq \frac{2^r-2}{2^{r+1}-3}\beta(G) + \frac{2^r-1}{2^{r+1}-3}$.*

Corollary 4 *Let G be a graph with minimal degree $\delta \geq 3$. If G is not up-embeddable, then,*

(1) *when* G *is* 1-*edge connected,* $\gamma_M(G) \geq \frac{\delta^2-\delta-4}{2(\delta^2-\delta-1)}\beta(G) + \frac{\delta^2-\delta+2}{2(\delta^2-\delta-1)}$;

(2) *when* G *is* 2-*edge connected,* $\gamma_M(G) \geq \frac{\delta^2-\delta-4}{2(\delta+1)(\delta-2)}\beta(G) + \frac{\delta(\delta-1)}{2(\delta+1)(\delta-2)}$;

(3) *when* G *is* 3-*edge connected,* $\gamma_M(G) \geq \frac{\delta^2-2\delta-1}{2\delta(\delta-2)}\beta(G) + \frac{(\delta-1)^2}{2\delta(\delta-2)}$.

Through simple comparison, the Corollary 2 complement the results in paper[3], the Corollary 3 and 4 improve the lower bound on $\gamma_M(G)$ of graph G with the same connectivity in papers[4] and [5].

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