On kernels by monochromatic paths in D-join

by

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ABSTRACT: In [2] it was introduced the concept of the kernel by monochromatic paths, which generalize concept of kernel. In this paper we prove the necessary and sufficient conditions for the existence of kernels by monochromatic paths in the D-join of digraphs. We also give sufficient condition for D-join to be monochromatic kernel perfect. The existence of generalized kernel (in distance sense) in D-join were studied in [5]. Moreover we calculate the total number of kernels by monochromatic paths in this product.

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1 Introduction

For concepts not defined here, see [1]. Let D be a finite, directed graph (for short: a digraph) where $V(D)$ is the set of vertices and $A(D)$ is the set of arcs of D. By a path from a vertex x_i to a vertex x_n, $n \geq 2$, we mean a sequence of vertices $x_1, ..., x_n$ and arcs $(x_i, x_{i+1}) \in A(D)$, for $i = 1, ..., n-1$ and for simplicity we denote it by $x_1...x_n$. A circuit is a path with $x_1 = x_n$. A digraph D is said to be an edge m-coloured digraph if its arcs are coloured with m colours. A path (or a circuit) is called monochromatic if all of its arcs are coloured alike. By $C_D(x_i)$ we denote the family of all monochromatic circuits in D containing the vertex x_i.
A set $J \subset V(D)$ is said to be a kernel by monochromatic paths of the m-coloured digraph D if it satisfies the two following properties:
1. J is independent by monochromatic paths i.e. for any two different vertices $x, y \in J$ there is no monochromatic paths between them and
2. J is dominating by monochromatic paths i.e. for each $x \in (V(D) \setminus J)$ there exists monochromatic paths from x to y, for some $y \in J$.
If the set J satisfies condition in (1) or in (2) then we call it independent by monochromatic paths in D or dominating by monochromatic paths in D, respectively. In addition a subset containing only one vertex and the empty set also is meant as independent by monochromatic paths. If a dominating by monochromatic paths set of D has exactly one vertex, then

we will call it as monochromatic dominating vertex of D. By $NImp(D)$, $NDmp(D)$, $NKmp(D)$ we will denote the numbers of all: independent by monochromatic paths sets of D, dominating by monochromatic paths sets of D and kernels by monochromatic paths of D, respectively.

The concept of kernels by monochromatic paths generalize kernels in classical sense. Sufficient conditions for the existence of kernels by monochromatic paths in m-coloured digraphs also have been investigated by several authors, see by example [2],[4],[6].

Let D be an edge coloured digraph with $V(D) = \{x_1, ..., x_n\}$, $n \geq 2$, and $\alpha = (D_i)_{i\in\{1,...,n\}}$ be a sequence of vertex disjoint edge coloured digraphs on $V(D_i) = \{y_1^i, ..., y_{p_i}^i\}$, $p_i \geq 1$, $i = 1, ..., n$. Then D-join of the digraph D and the sequence α is a digraph $\sigma(\alpha, D)$ such that $V(\sigma(\alpha, D)) = \bigcup_{i=1}^{n} (\{x_i\} \times V(D_i))$ and $A(\sigma(\alpha, D)) = \{((x_s, y_j^s), (x_q, y_t^q))$ coloured m; ($x_s = x_q$ and $(y_j^s, y_t^s) \in A(D_s)$ coloured m) or ($(x_s, x_q) \in A(D)$ coloured m)$\}$. By D_i^c we mean a copy of the digraph D_i in $\sigma(\alpha, D)$.

It may be noted that if all digraphs from the sequence α have the same vertex set, then from the D-join we obtain the generalized lexicographic product of the digraph D and the sequence of the digraphs D_i i.e. $\sigma(\alpha, D) = D[D_1, ..., D_n]$. If all digraphs from sequence α are isomorphic to the same digraph H, then from the D-join we obtain the composition $D[H]$ of the digraphs D and H. The existence of kernels and (k, l)-kernels (i.e. generalizations of kernels in distance sense) in $D[D_1, ..., D_n]$ and in the D-join was studied in [5], [7].

The existence of kernels by monochromatic paths in D-join, where D is a digraph without monochromatic circuits were studied in [3].

In this paper we generalize result from [3] for an arbitrary edge coloured digraph D. Moreover we give the total number of the kernels by monochromatic paths, independent sets by monochromatic paths and dominating sets by monochromatic paths in D-join.

2 The existence of kernels by monochromatic paths in D-join

From the definition of $\sigma(\alpha, D)$ the following Proposition is obvious:

Proposition 1 *Let $(x_i, y_p^i), (x_j, y_q^j) \in V(\sigma(\alpha, D))$ be two different vertices. A path $(x_i, y_p^i)...(x_j, y_q^j)$ is monochromatic in $\sigma(\alpha, D)$ if and only if*
(a) for $i \neq j$ there exists a monochromatic path $x_i...x_j$ in D or
(b) for $i = j$ there exists a monochromatic path $y_p^i...y_q^i$ in D_i^c or $C_D(x_i) \neq \emptyset$.

Theorem 1 *Let D be an edge coloured digraph and $\alpha = (D_i)_{i\in\{1,\dots,n\}}$ be a family of edge coloured vertex disjoint digraphs. A subset $S^* \subset V(\sigma(\alpha, D))$ is independent by monochromatic paths of $\sigma(\alpha, D)$ if and only if $S \subset V(D)$ is independent by monochromatic paths such that $S^* = \bigcup_{i\in\mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$, $S_i \subseteq V(D_i^c)$ and*
(a) S_i is independent by monochromatic paths of D_i^c if $C_D(x_i) = \emptyset$ or
(b) S_i is 1-element set containing an arbitrary vertex from $V(D_i^c)$, otherwise
for every $i \in \mathcal{I}$.

P R O O F: I. Let S^* be independent by monochromatic paths of the D-join $\sigma(\alpha, D)$. Denote $S = \{x_i \in V(D); S^* \cap V(D_i^c) \neq \emptyset\}$. At first we shall prove that S is an independent by monochromatic paths set of D. We proceed by contradiction, suppose that S is not independent by monochromatic paths. This means that there exist $x_i, x_j \in S$ such that there is a monochromatic path $x_i...x_j$ in digraph D. The Proposition 1(a) implies that for each vertices $(x_i, y_p^i) \in V(D_i^c)$ and $(x_j, y_q^j) \in V(D_j^c)$, where $1 \leq p \leq p_i$ and $1 \leq q \leq p_j$ a path $(x_i, y_p^i)...(x_j, y_q^j)$ is monochromatic in $\sigma(\alpha, D)$. By the definition of the set S we have that $S^* \cap V(D_i^c) \neq \emptyset$ and $S^* \cap V(D_j^c) \neq \emptyset$, so there exists a monochromatic path between vertices from S^*, contradiction with independence by monochromatic paths of S^*. The definition of the set S implies that $S^* = \bigcup_{i\in\mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$. We shall prove that $S_i = S^* \cap V(D_i^c)$ is independent by monochromatic paths if $C_D(x_i) = \emptyset$ or S_i contains exactly one arbitrary vertex from $V(D_i^c)$, otherwise. We distinguish two possible cases:

I.1. Let $C_D(x_i) = \emptyset$.

To prove that S_i is independent by monochromatic paths assume on contrary that there exist $(x_i, y_p^i), (x_i, y_q^i) \in S_i$ and a monochromatic path between them in D_i^c. Because $S_i = S^* \cap V(D_i^c)$, so $(x_i, y_p^i), (x_i, y_q^i) \in S^*$, a contradiction that S^* is an independent by monochromatic paths set in D-join $\sigma(\alpha, D)$.

I.2. Let $C_D(x_i) \neq \emptyset$.

Then by Proposition 1(b) for arbitrary two vertices $(x_i, y_p^i), (x_i, y_q^i) \in V(D_i^c)$ there exists a monochromatic path $(x_i, y_p^i)...(x_i, y_q^i)$ in $\sigma(\alpha, D)$. Hence S_i is 1-element set containing an arbitrary vertex from $V(D_i^c)$.

From the above cases we obtain that S_i is independent by monochromatic paths of D_i^c if there no exist in D monochromatic circuit containing the vertex x_i or S_i contains exactly one vertex from $V(D_i^c)$, otherwise.

II. Let $S \subset V(D)$ be an independent by monochromatic paths set of D and let S_i be as in the statement of the Theorem. We will prove that

$S^* = \bigcup_{i \in \mathcal{I}} S_i$ is an independent by monochromatic paths set of D-join. Let $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ be two different vertices. Consider the following cases:

II.1. Let $(x_i, y_p^i) \in S_i$ and $(x_j, y_q^j) \in S_j$, where $i \neq j$.

By the definition of the set S it follows that $x_i, x_j \in S$. Because S is independent by monochromatic paths of D so by the definition of $\sigma(\alpha, D)$ immediately follows that there is no a monochromatic path between vertices (x_i, y_p^i) and (x_j, y_q^j) in D-join.

II.2. Let $(x_i, y_p^i), (x_j, y_q^j) \in S_i$, where $p \neq q$.

Because S_i contains at least two vertices, so by assumption the set S_i is independent by monochromatic paths of D_i^c and $C_D(x_i) = \emptyset$. Assume on the contrary that S_i is not independent by monochromatic paths in $\sigma(\alpha, D)$. Hence there exists a monochromatic path from (x_i, y_p^i) to (x_i, y_q^i) in $\sigma(\alpha, D)$. This means that there is a monochromatic path $(x_i, y_p^i)...(x_i, y_q^i)$ in $\sigma(\alpha, D)$ such that at least one inner vertex of this path not belong to $V(D_i^c)$. This implies that there exists in D a monochromatic circuit containing the vertex x_i, contradiction with $C_D(x_i) = \emptyset$.

Taking two above cases into considerations we obtain that for two distinct $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ there is no exist a monochromatic path between them, hence S^* is independent by monochromatic paths of $\sigma(\alpha, D)$.

Thus the Theorem is proved. □

Theorem 2 *Let D be an edge coloured digraph and $\alpha = (D_i)_{i \in \{1,\dots,n\}}$ be a family of edge coloured vertex disjoint digraphs. A subset $Q^* \subseteq V(\sigma(\alpha, D))$ is dominating by monochromatic paths of $\sigma(\alpha, D)$ if and only if $Q \subseteq V(D)$ is dominating by monochromatic paths such that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$, $Q_i \subseteq V(D_i^c)$ and*
(a) Q_i is dominating by monochromatic paths of D_i^c if $C_D(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there is no exist in D a monochromatic paths $x_i...x_j$
or
(b) Q_i is an arbitrary nonempty subset of $V(D_i^c)$, otherwise
for every $i \in \mathcal{I}$.

P R O O F: I. Let Q^* be a dominating by monochromatic paths of the D-join $\sigma(\alpha, D)$. Denote $Q = \{x_i \in V(D); Q^* \cap V(D_i^c) \neq \emptyset\}$. At first we shall prove that Q is a dominating by monochromatic paths set of D. Let $x_j \notin Q$. By the definition of the set Q we have that for each $1 \leq r \leq p_j$ holds $(x_j, y_r^j) \notin Q^*$. Because Q^* is dominating by monochromatic paths so there exists $(x_i, y_s^i) \in Q^*$, where $i \neq j$ such that a path $(x_j, y_r^j)...(x_i, y_s^j)$ is monochromatic in D-join. Evidently $x_i \in Q$ and from the Proposition 1(a) it follows that there exists a monochromatic path $x_j...x_i$ in D. Hence Q is dominating by monochromatic paths. The definition of the set Q implies that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$. Consider the possible cases:

I.1. Let $C_D(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there is no exist in D a monochromatic path $x_i ... x_j$.

Because Q^* is dominating by monochromatic paths in $\sigma(\alpha, D)$ so using our assumptions and Proposition 1(b) immediately follows that $Q_i = Q^* \cap V(D_i^c)$ is a dominating by monochromatic paths of D_i^c.

I.2. Assume that Case I.1 does not hold.

We shall prove that Q_i is an arbitrary nonempty subset of $V(D_i^c)$. If there exists in D monochromatic circuit containing the vertex x_i, then by Proposition 1(b) for an arbitrary two vertices $(x_i, y_p^i), (x_i, y_q^i) \in V(D_i^c)$ a path $(x_i, y_p^i) ... (x_i, y_q^i)$ is monochromatic in $\sigma(\alpha, D)$. If there exists $j \in \mathcal{I}$ and $j \neq i$ such that there is in D monochromatic path $x_i ... x_j$, then for an arbitrary vertex $(x_i, y_p^i) \in V(D_i^c)$ there exists in $\sigma(\alpha, D)$ a monochromatic path $(x_i, y_p^i) ... (x_j, y_q^j)$, where $(x_j, y_q^j) \in Q^*$. Consequently $Q_i = Q^* \cap V(D_i^c)$ is an arbitrary nonempty subset of $V(D_i^c)$.

II. Let $Q \subseteq V(D)$ be a dominating by monochromatic paths set of the digraph D and Q_i, $i \in \mathcal{I}$, be as in the statement of the Theorem. We shall prove that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$ is dominating by monochromatic paths of $\sigma(\alpha, D)$. Let $(x_j, y_p^j) \notin Q^*$ and distinguish two possible cases:

II.1. Let $(x_j, y_p^j) \notin Q^*$ and $j \notin \mathcal{I}$.

Then by the definition of the set Q, $x_j \notin Q$. Because Q is dominating by monochromatic paths of D, so there exists $i \in \mathcal{I}$ such that $x_i \in Q$ and $x_j ... x_i$ is monochromatic in D. Hence there exists $1 \leq q \leq p_i$ such that $(x_i, y_q^i) \in Q^*$. Consequently by Proposition 1(a) there exists a monochromatic path $(x_j, y_p^j) ... (x_i, y_q^i)$ in D-join $\sigma(\alpha, D)$. Hence Q^* is dominating by monochromatic paths in this case.

II.2. Let $(x_j, y_p^j) \notin Q^*$ and $j \in \mathcal{I}$.

If Q_j is dominating by monochromatic paths of D_i^c, then there is a monochromatic path from (x_j, y_p^j) to Q_j in $\sigma(\alpha, D)$. If Q_j is an arbitrary nonempty subset of $V(D_i^c)$, then by the assumption of Q_j we have that $C_D(x_j) \neq \emptyset$ or there exists $t \in \mathcal{I}$ and $t \neq j$ such that there is in D a monochromatic paths $x_j ... x_t$. Using the Proposition 1(b) we obtain that there exists in D-join a monochromatic paths from (x_j, y_p^j) to Q_j or there is a monochromatic path from (x_j, y_p^j) to Q_t, respectively. Hence Q^* is dominating by monochromatic paths of $\sigma(\alpha, D)$.

Thus the Theorem is proved. □

Theorem 3 *Let D be an edge coloured digraph and $\alpha = (D_i)_{i\in\{1,\dots,n\}}$ be a family of edge coloured vertex disjoint digraphs. A subset $J^* \subset V(\sigma(\alpha, D))$ is a kernel by monochromatic paths of $\sigma(\alpha, D)$ if and only if there exists a kernel by monochromatic paths $J \subset V(D)$ such that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i; x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and*
(a) J_i is kernel by monochromatic paths of D_i^c if $C_D(x_i) = \emptyset$ or
(b) J_i is 1-element set containing an arbitrary vertex from $V(D_i^c)$, otherwise for every $i \in \mathcal{I}$.

P R O O F: I. Let J^* be a kernel by monochromatic paths of the D-join $\sigma(\alpha, D)$. Let $J = \{x_i \in V(D); J^* \cap V(D_i^c) \neq \emptyset\}$. From Theorem 1 and Theorem 2 immediately follows that J is a kernel by monochromatic paths of D. The definition of the set J implies that $J^* = \bigcup_{i\in\mathcal{I}} J_i$, where $\mathcal{I} = \{i; x_i \in J\}$. We distinguish two possible cases:

I.1. Let $C_D(x_i) = \emptyset$

This means that there is no exist in D a monochromatic circuit containing the vertex x_i. Hence by Proposition 1(b) for any arbitrary two vertices belonging to $V(D_i^c)$ there is no exist in $\sigma(\alpha, D)$ a monochromatic path containing the vertex from $V(D_j^c)$ where $j \neq i$. We shall show that $J_i = J^* \cap V(D_i^c)$ is a kernel by monochromatic paths of D_i^c. By Theorem 1(a) immediately follows that J_i is independent by monochromatic paths in $\sigma(\alpha, D)$. Now we will prove that J_i is dominating by monochromatic paths of D_i^c. Because J is independent by monochromatic paths of D hence for each $j \in \mathcal{I}$ and $j \neq i$ there is no exist a monochromatic paths $(x_i, y_p^i)\dots(x_j, y_q^j)$ for all $p = 1, \dots, p_i$ and $q = 1, \dots, q_j$. Moreover by assumption $C_D(x_i) = \emptyset$, so by Theorem 2(a) we obtain that J_i is dominating by monochromatic paths of D_i^c. All this together implies that J_i is a kernel by monochromatic paths of D_i^c.

I.2. Let $C_D(x_i) \neq \emptyset$

Then using the Theorem 1(b) and Theorem 2(b) immediately follows that J_i is 1-element set containing an arbitrary vertex from $V(D_i^c)$.

II. Let $J \subset V(D)$ be a kernel by monochromatic paths of the digraph D. Let $\mathcal{I} = \{i; x_i \in J\}$ and J_i be as in the statements of the Theorem. We shall prove that $J^* = \bigcup_{i\in\mathcal{I}} J_i$ is a kernel by monochromatic paths of $\sigma(\alpha, D)$. Evidently by Theorem 1 the set J^* is independent by monochromatic paths of D_i^c. We will show that J^* is dominating by monochromatic paths. Let $(x_j, y_p^j) \notin J^*$. Consider two possible cases:

II.1. Let $(x_j, y_p^j) \notin J^*$ and $j \notin \mathcal{I}$.

Then by the definition of the set J, $x_j \notin J$. Because J is dominating by monochromatic paths of D, so there exists $i \in \mathcal{I}$ such that $x_i \in J$ and

$x_j...x_i$ is monochromatic in D. Hence there exists $1 \leq q \leq p_i$ such that $(x_i, y_q^i) \in J^*$. Consequently the Proposition 1(a) implies the existence a monochromatic path $(x_j, y_p^j)...(x_i, y_q^i)$ in $\sigma(\alpha, D)$. Hence J^* is dominating by monochromatic paths in this case.

II.2. Let $(x_j, y_p^j) \notin J^*$ and $j \in \mathcal{I}$.

If J_j is kernel by monochromatic paths of D_i^c, then there is in D_i^c a monochromatic paths from (x_j, y_p^j) to J_j and this implies the existence of a monochromatic path from (x_j, y_p^j) to J^*. If $J_j = \{(x_j, y_q^j)\}$, where (x_j, y_q^j) is an arbitrary vertex of D_i^c, then by the assumption of the Theorem $C_D(x_j) \neq \emptyset$. Hence by Proposition 1(b) there exists a monochromatic path $(x_j, y_p^j)...(x_j, y_q^j)$ in $\sigma(\alpha, D)$.

All this together give that J^* is an dominating by monochromatic paths set of D-join $\sigma(\alpha, D)$.

Thus the Theorem is proved. □

3 Monochromatic kernel perfectness of the D-join

Let D be an edge coloured digraph. A digraph D whose every induced subdigraph has a kernel by monochromatic paths is called monochromatic kernel perfect digraph.

From the definition of the $\sigma(\alpha, D)$ the following Proposition is obvious:

Proposition 2 *Every induced subdigraph of $\sigma(\alpha, D)$ is*
(a) a digraph of the form $\sigma(\widetilde{\alpha}, \widetilde{D})$, where $\widetilde{D}$ is a subdigraph of D with $V(\widetilde{D}) = \{x_t; t \in \widetilde{\mathcal{I}}\}$, $|\widetilde{\mathcal{I}}| > 1$, $\widetilde{\mathcal{I}} \subseteq \{1, ..., n\}$ and $\widetilde{\alpha}$ is a family of induced subdigraphs of D_t, where $t \in \widetilde{\mathcal{I}}$ or
(b) an induced subdigraph of D_i for some $1 \leq i \leq n$ or
(c) the union of the digraphs from (a) and (b).

From the definition of the monochromatic kernel perfect digraph and by the Proposition 2 immediately follows:
If $\sigma(\alpha, D)$ is monochromatic kernel perfect, then D and D_i, $i = 1, ..., n$ are monochromatic kernel perfect.

Theorem 4 *Let D be an edge coloured digraph and let $\alpha = (D_i)_{i \in \{1,...,n\}}$ be a family of edge coloured digraphs. Let D be a monochromatic kernel perfect digraph and let D_i be a monochromatic kernel perfect if $C_D(x_i) = \emptyset$ or every subdigraph of D_i has an monochromatic dominating vertex, otherwise. Then $\sigma(\alpha, D)$ is monochromatic kernel perfect.*

P R O O F: Assume that D and D_i, $i = 1, ..., n$ are as in the statements of the Theorem. We shall show that $\sigma(\alpha, D)$ is a monochromatic kernel perfect digraph. From Proposition 2 it follows that we need only to prove that $\sigma(\alpha, D)$ has a kernel by monochromatic paths. By the Theorem 3 and from our assumptions there exists a kernel by monochromatic paths $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a kernel by monochromatic paths of D-join, where $\mathcal{I} = \{i; x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and J_i is a kernel by monochromatic paths of D_i^c if $\mathcal{C}_D(x_i) = \emptyset$ or J_i is 1-element set containing a monochromatic dominating vertex of D_i^c, otherwise.

Thus the Theorem is proved. □

Corollary 1 *If D is a digraph without monochromatic circuits, then $\sigma(\alpha, D)$ is monochromatic kernel perfect if and only if D and D_i, $i = 1, ..., n$, are monochromatic kernel perfect.*

4 The total number of kernels by monochromatic paths in the D-join

In this section we calculate the number of all independent by monochromatic paths sets, dominating by monochromatic paths sets and kernels by monochromatic paths in the D-join $\sigma(\alpha, D)$.

Theorem 5 *Let D be an edge coloured digraph on n vertices, $n \geq 2$ and $\alpha = (D_i)_{i \in \{1,...,n\}}$ be a sequence of edge coloured vertex disjoint digraphs $(D_i)_{i \in \{1,...,n\}}$ on p_i vertices, $p_i \geq 1$. Let $\mathcal{S} = \{S_1, ..., S_j\}$, $j \geq 1$ be a family of all nonempty independent by monochromatic paths sets of the digraph D and let $\mathcal{S} \ni S_r = \{x_i; i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subset \{1, ..., n\}$. Then* $NImp(\sigma(\alpha, D)) = 1 + \sum_{r=1}^{j} \prod_{i \in \mathcal{I}_r} \psi(D_i)$, *where*

$$\psi(D_i) = \begin{cases} NImp(D_i) - 1 & if \;\; \mathcal{C}_D(x_i) = \emptyset \\ p_i & otherwise. \end{cases}$$

P R O O F: Let D be an edge coloured digraph on n-vertices, $n \geq 2$. The Theorem 1 implies that to obtain an independent by monochromatic paths set of $\sigma(\alpha, D)$ first we have to choose an independent by monochromatic paths set of D. Let $\mathcal{S} = \{S_1, ..., S_j\}$, $j \geq 1$ be a family of all nonempty independent by monochromatic paths sets of the digraph D. Assume that $\mathcal{S} \ni S_r = \{x_i; i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subset \{1, ..., n\}$. Next by Theorem 1 in each of the D_i^c, $i \in \mathcal{I}_r$ we have to choose an nonempty independent by monochromatic paths set if $\mathcal{C}_D(x_i) = \emptyset$ or we choose an arbitrary vertex from $V(D_i^c)$, otherwise. Evidently we can do it on $NImp(D_i) - 1$ or p_i ways, respectively. Hence from fundamental combinatorial statements we have $\sum_{r=1}^{j}\prod_{i\in\mathcal{I}_r}\psi(D_i)$ sets, where $\psi(D_i)=\begin{cases} NImp(D_i)-1 & if\ \ C_D(x_i)=\emptyset \\ p_i & otherwise. \end{cases}$

Moreover, the empty set also is an independent by monochromatic paths set of $\sigma(\alpha,D)$. Consequently $NImp(\sigma(\alpha,D))=1+\sum_{r=1}^{j}\prod_{i\in\mathcal{I}_r}\psi(D_i)$.

Thus the Theorem is proved. □

Using the same method we can prove:

Theorem 6 *Let D be an edge coloured digraph on n vertices, $n\geq 2$ and $\alpha=(D_i)_{i\in\{1,...,n\}}$ be a sequence of edge coloured vertex disjoint digraphs $(D_i)_{i\in\{1,...,n\}}$ on p_i vertices, $p_i\geq 1$. Let $\mathcal{Q}=\{Q_1,...,Q_j\}$, $j\geq 1$ be a family of all dominating by monochromatic paths sets of the digraph D and let $\mathcal{Q}\ni Q_r=\{x_i; i\in\mathcal{I}_r\}$, where $\mathcal{I}_r\subseteq\{1,...,n\}$. Then $NDmp(\sigma(\alpha,D))=\sum_{r=1}^{j}\prod_{i\in\mathcal{I}_r}\phi(D_i)$, where*

$$\phi(D_i)=\begin{cases} NDmp(D_i) & if\ C_D(x_i)=\emptyset \text{ and for each } j\in\mathcal{I} \text{ and } j\neq i \text{ there} \\ & \text{is no exist in } D \text{ monchromatic path } x_i...x_j \\ 2^{p_i}-1 & otherwise. \end{cases}$$

Theorem 7 *Let D be an edge coloured digraph on n vertices, $n\geq 2$ and $\alpha=(D_i)_{i\in\{1,...,n\}}$ be a sequence of edge coloured vertex disjoint digraphs $(D_i)_{i\in\{1,...,n\}}$ on p_i vertices, $p_i\geq 1$. Let $\mathcal{J}=\{J_1,...,J_j\}$, $j\geq 1$ be a family of all kernels by monochromatic paths of the digraph D and let $\mathcal{J}\ni J_r=\{x_i; i\in\mathcal{I}_r\}$, where $\mathcal{I}_r\subset\{1,...,n\}$. Then $NKmp(\sigma(\alpha,D))=\sum_{r=1}^{j}\prod_{i\in\mathcal{I}_r}\xi(D_i)$, where*

$$\xi(D_i)=\begin{cases} NKmp(D_i) & if\ \ C_D(x_i)=\emptyset \\ p_i & otherwise. \end{cases}$$

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