Nilpotent Adjacency Matrices and Random Graphs

René Schott*, George Stacey Staples†

Abstract

While powers of the adjacency matrix of a finite graph reveal information about walks on the graph, they fail to distinguish closed walks from cycles. Using elements of an appropriate commutative, nilpotent-generated algebra, a "new" adjacency matrix Λ can be associated with a random graph on n vertices. Letting X_k denote the number of k-cycles occurring in a random graph, this algebra together with a probability mapping allow $\mathbb{E}(X_k)$ to be recovered in terms of $\operatorname{tr} \Lambda^k$. Higher moments of X_k can also be computed, and conditions are given for the existence of higher moments in growing sequences of random graphs by considering infinite-dimensional algebras. The algebras used can be embedded in algebras of fermion creation and annihilation operators, thereby establishing connections with quantum computing and quantum probability theory. In the framework of quantum probability, the nilpotent adjacency matrix of a finite graph is a quantum random variable whose m^{th} moment corresponds to the m-cycles contained in the graph.

AMS subject classification: 05C38, 05C80, 60B99, 81P68
Key words: cycles, Hamiltonian, enumeration, random graphs, quantum computing

1 Introduction

The reader is referred to [8] for graph theory beyond the essential notation and terminology found here. A *graph* $G = (V, E)$ is a collection of vertices V and a set E of unordered pairs of vertices called *edges*. A *directed graph* is a graph whose edges are *ordered pairs* of vertices. Two vertices $v_i, v_j \in V$ are *adjacent* if there exists an edge $e = (v_i, v_j) \in E$. A graph is *finite* if

*IECN and LORIA, Université Henri Poincaré-Nancy 1, 54506 Vandoeuvre-lès-Nancy, France, email: schott@loria.fr

†Department of Mathematics and Statistics, Southern Illinois University Edwardsville, Edwardsville, IL 62026-1653, email: sstaple@siue.edu

V and E are finite sets, that is, if $|V|$ and $|E|$ are finite numbers. A *loop* in a graph is an edge of the form (v, v). A graph is said to be *simple* if it contains no loops and no *multiple edges*; i.e., no pair of adjacent vertices shares more than one edge.

A *k-walk* $\{v_0, \ldots, v_k\}$ in a graph G is a sequence of vertices in G with *initial vertex* v_0 and *terminal vertex* v_k such that there exists an edge $(v_j, v_{j+1}) \in E$ for each $0 \leq j \leq k-1$. A k-walk contains k edges. A *k-path* is a k-walk in which no vertex appears more than once. A *closed k-walk* is a k-walk whose initial vertex is also its terminal vertex. A *k-cycle* is a closed k-path with $v_0 = v_k$. A *Hamiltonian cycle* is an n-cycle in a graph on n vertices; i.e., it contains V. A *k-trail* is a k-walk in which no edge appears more than once. A *k-circuit* is a closed k-walk. An *Euler circuit* is a circuit encompassing every edge in E exactly once.

When working with a finite graph G on n vertices, one often utilizes the *adjacency matrix* A associated with G. If the vertices are labeled $\{1, \ldots, n\}$, A is defined by

$$A_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

A simple but useful result of this definition, which can also be generalized to directed graphs, is given here without proof.

Proposition 1.1. *Let G be a graph on n vertices with associated adjacency matrix A. Then for any positive integer k, the $(i,j)^{th}$ entry of A^k is the number of k-walks $i \to j$. In particular, the entries along the main diagonal of A^k are the numbers of closed k-walks in G.*

What the usual adjacency matrix fails to provide, however, is a method of counting self-avoiding walks and cycles in G. This problem is overcome by constructing a nilpotent adjacency matrix.

All graphs in this work are assumed to contain no multiple edges and no loops. Graphs may be directed or undirected.

The methods employed here are original with the authors. The technique involves mapping combinatorial structures into algebras where self-intersections are "sieved out" by multiplication. Then the remaining structures, representing cycles and paths, are recovered by projection.

Other algebraic-probabilistic approaches to graph theory include the works of Hashimoto, Hora, and Obata [2] and Obata [4]. Overlaps between quantum probability and graph theory have also been discussed by Lehner [3].

1.1 Algebraic preliminaries

Let V be a finite set, and let $\mathcal{N}_V$ denote the abelian algebra generated by the collection $\{\zeta_i\}$, $(i \in V)$ along with the scalar $1 = \zeta_\emptyset$ subject to the following multiplication rules:

$$\zeta_i \zeta_j = \zeta_j \zeta_i \quad \text{for } i \neq j, \text{ and} \tag{1.2}$$

$$\zeta_i^{\,2} = 0 \quad \text{for } i \in V. \tag{1.3}$$

A general element $\alpha \in \mathcal{N}_V$ can be expanded as

$$\alpha = \sum_{\underline{i} \in \mathcal{P}(V)} \alpha_{\underline{i}} \zeta_{\underline{i}}, \tag{1.4}$$

where $\mathcal{P}(V)$ is the power set of V used as a multi-indexing set, $\alpha_{\underline{i}} \in \mathbb{R}$, and $\zeta_{\underline{i}} = \prod_{\iota \in \underline{i}} \zeta_\iota$.

For a fixed finite set E, let $\mathcal{I}_E$ denote the abelian algebra generated by the collection $\{\gamma_i\}$ $(i \in E)$ along with the scalar $1 = \gamma_\emptyset$ subject to the following multiplication rules:

$$\gamma_i \gamma_j = \gamma_j \gamma_i \quad \text{for } i \neq j, \text{ and} \tag{1.5}$$

$$\gamma_i^{\,2} = \gamma_i \quad \text{for } i \in E. \tag{1.6}$$

It is evident that a general element $\beta \in \mathcal{I}_E$ can also be expanded as in (1.4).

The *inner-product* is defined by

$$\langle u, v \rangle = \left\langle \sum_{\underline{i} \in \mathcal{P}(V)} u_{\underline{i}} \zeta_{\underline{i}}, \sum_{\underline{j} \in \mathcal{P}(V)} v_{\underline{j}} \zeta_{\underline{j}} \right\rangle = \sum_{\underline{i} \in \mathcal{P}(V)} u_{\underline{i}} v_{\underline{i}}. \tag{1.7}$$

Hence, arbitrary $u \in \mathcal{N}_V$ has the canonical decomposition

$$u = \sum_{\underline{i} \in \mathcal{P}(V)} \langle u, \zeta_{\underline{i}} \rangle \zeta_{\underline{i}}. \tag{1.8}$$

Finally, define the double angle bracket to mean the sum of all scalar coefficients. That is, for $u \in \mathcal{N}_V$,

$$\langle\langle u \rangle\rangle = \sum_{\underline{i} \in \mathcal{P}(V)} u_{\underline{i}}. \tag{1.9}$$

1.2 Nilpotent Adjacency Matrices

Definition 1.2. Define the *nilpotent adjacency matrix* associated with G by

$$\Lambda_{ij} = \begin{cases} \zeta_j, & \text{if } (v_i, v_j) \in E(G) \\ 0, & \text{otherwise.} \end{cases} \tag{1.10}$$

Observe that $\Lambda \in \mathrm{Mat}_n(\mathcal{N}_V)$, the algebra of $n \times n$ matrices with entries in the abelian nilpotent-generated algebra $\mathcal{N}_V$.

Proposition 1.3. *Let Λ be the nilpotent adjacency matrix of a graph G on n vertices. For any $m \geq 1$, summing the coefficients of $(\Lambda^m)_{ii}$ yields the number of m-cycles based at v_i occurring in G.*

Proof. Proof is by induction on m. When $m = 1$, the proposition is true by definition of Λ.

Now assuming the proposition holds for m and considering the case $m+1$,

$$\left(\Lambda^{m+1}\right)_{ii} = \left(\Lambda^m \times \Lambda\right)_{ii} = \sum_{\ell=1}^{n} \left(\Lambda^m\right)_{i\ell} \Lambda_{\ell i}. \tag{1.11}$$

Considering a general term of the sum,

$$\left(\Lambda^m\right)_{i\ell} = \sum_{m\text{-paths } w_m : v_i \to v_\ell} w_m, \text{ and} \tag{1.12}$$

$$\Lambda_{\ell i} = \begin{cases} 1\text{-path } w_1 : v_\ell \to v_i, & \text{if } (v_\ell, v_i) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

It should then be clear that terms of the product

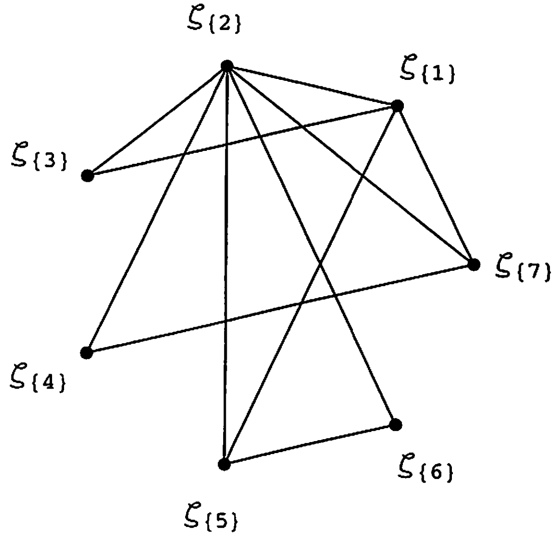
$$\left(\Lambda^m\right)_{i\ell} \Lambda_{\ell i} \tag{1.14}$$

are nonzero if and only if they correspond to $m+1$-paths $v_i \to v_\ell \to v_i$. Summing over all vertices v_ℓ gives the sum of all $m+1$-cycles based at v_i. □

Because Λ has entries in $\mathcal{N}_V$, Λ^k is identically the zero matrix for all $k > n$. As a result, $(I - t\Lambda)^{-1}$ exists as the finite sum $\sum_{k=0}^{n} t^k \Lambda^k$ for real parameter t, and $\mathrm{tr}\, \Lambda^k$ is recovered as the $\mathcal{N}_V$-valued coefficient of t^k in the power series expansion of $\mathrm{tr}(I - t\Lambda)^{-1}$.

Example 1.4. The 5-cycles contained in the randomly generated graph in Figure 1.1 are recovered by examining the trace of Λ^5. Dividing by five compensates for the five choices of base point, and dividing by two compensates for possible orientations.

In[50]:= **NilpotentLabeledPlotGraph[A]**



In[58]:= **NilpotentAdjacencyMatrix[A] // MatrixForm**

Out[58]//MatrixForm=

$$\begin{pmatrix} 0 & \zeta_{\{2\}} & \zeta_{\{3\}} & 0 & \zeta_{\{5\}} & 0 & \zeta_{\{7\}} \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{\{2\}} & 0 & 0 & 0 & 0 & \zeta_{\{7\}} \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & 0 & 0 & \zeta_{\{6\}} & 0 \\ 0 & \zeta_{\{2\}} & 0 & 0 & \zeta_{\{5\}} & 0 & 0 \\ \zeta_{\{1\}} & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & 0 & 0 & 0 \end{pmatrix}$$

In[55]:= **Simplify[Tr[ClMatrixPower[M, 5]] / 2 / 5]**

Out[55]= $\zeta_{\{1,2,3,4,7\}} + \zeta_{\{1,2,3,5,6\}} + \zeta_{\{1,2,4,5,7\}} + \zeta_{\{1,2,5,6,7\}}$

In[56]:= **ScalarSum[%]**

Out[56]= 4

Figure 1.1: A randomly generated graph on 7 vertices.

A nilpotent adjacency matrix for random graphs can also be defined by attaching edge existence probabilities to the nilpotent generators of $\mathcal{N}_V$. Using this approach, $\mathbb{E}(X_k)$ is recovered from the trace of Λ^k [7].

In the number of algebra multiplications required, cycle enumeration is reduced to matrix multiplication. Hence, the time complexity of enumerating a graph's k-cycles requires no more than $O(kn^3)$ algebra multiplications. Several NP-complete problems are moved into class P in this context[6].

However, computing higher moments of X_k requires computing probabilities $\mathbb{P}(X_k = \ell)$ for $\ell \geq 0$, and the abelian nilpotent-generated algebra $\mathcal{N}_V$ is not sufficient for this purpose. In order to compute higher moments, it is necessary to define a nilpotent adjacency matrix with entries in $\mathcal{I}_{V\times V}\otimes\mathcal{N}_V$.

2 Cycles in random graphs

Consider a random graph $G = (V, E)$ on n vertices, $V = \{v_1, \ldots, v_n\}$. Let $2 \leq k \leq n$, and let $\omega \in \{1, 2\}$ be defined by

$$\omega = \begin{cases} 1 & \text{if } G \text{ is directed or } k = 2 \\ 2 & \text{otherwise.} \end{cases} \tag{2.1}$$

For each ordered pair $(v_i, v_j) \in V \times V$, define the probability of existence of edge (v_i, v_j) in the graph G by

$$p_{ij} = \mathbb{P}\{(v_i, v_j) \in E\}. \tag{2.2}$$

Defining the random variable X_k as the number of k-cycles occurring in the graph, the goal is to compute $\mathbb{E}(X_k)$ as well as the variance and the higher moments.

Definition 2.1. Labeling the vertices with nilpotents and the edges with idempotents, the *edge-labeled nilpotent adjacency matrix of* $G = (V, E)$ is defined by

$$\Lambda_{ij} = \begin{cases} \gamma_{(i,j)}\,\zeta_j & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

for $i, j \in V$. It is clear that $\Lambda \in \mathrm{Mat}_n(\mathcal{I}_{V\times V}\otimes\mathcal{N}_V)$, the algebra of $n \times n$ matrices with entries in $\mathcal{I}_{V\times V}\otimes\mathcal{N}_V$.

Definition 2.2. Let $u \in \mathcal{I}_{V\times V}\otimes\mathcal{N}_V$ and define

$$\varphi = \sum_{\underline{i}\in\mathcal{P}(V\times V)} \left(\prod_{\iota\in\underline{i}} p_\iota\right) \gamma_{\underline{i}} \in \mathcal{I}_{V\times V}. \tag{2.4}$$

The φ-*evaluation* of u is then defined as the linear functional

$$\langle\cdot\rangle_\varphi : \mathcal{I}_{V\times V}\otimes\mathcal{N}_V \to \mathbb{R},$$

$$\langle u\rangle_\varphi = \langle \sum_{\substack{\underline{i}\in\mathcal{P}(V\times V)\\ \underline{j}\in\mathcal{P}(V)}} u_{\underline{i}\,\underline{j}}\,\gamma_{\underline{i}}\,\zeta_{\underline{j}}, \varphi\rangle = \sum_{\substack{\underline{i}\in\mathcal{P}(V\times V)\\ \underline{j}\in\mathcal{P}(V)}} u_{\underline{i}\,\underline{j}}\,\varphi_{\underline{i}}, \tag{2.5}$$

where $\varphi_{\underline{i}}$ denotes the product $\prod_{\iota\in\underline{i}} p_\iota$.

If $u = u_{\underline{i}\,\underline{j}}\,\gamma_{\underline{i}}\,\zeta_{\underline{j}}$ for some $\underline{i}\in\mathcal{P}(V\times V)$, $\underline{j}\in\mathcal{P}(V)$ where $|\underline{i}| = k$, and $|\underline{j}| = \ell$, then u is referred to as a $k\otimes\ell$-*vector*.

When $k \geq 3$, $\operatorname{tr}\Lambda^k$ will give ωk copies of each k-cycle in G. In the particular case $k = 2$, only two copies will be obtained because only one orientation is possible. Let

$$\tau_k = \frac{1}{\omega k}\operatorname{tr}\Lambda^k. \tag{2.6}$$

Because the graph contains no multiple edges and no loops, $\tau_1 = 0$ and all values of k are hereby assumed to be greater than or equal to 2. Because the edge probabilities are independent, the φ-evaluation of each $k\otimes k$-vector is the probability of existence of a k-cycle in G. Then, τ_k represents the collection of all $k\otimes k$-vectors associated with the edges and vertices belonging to the k-cycles of nonzero probability in G.

Further,

$$\mathbb{E}(X_k) = \sum_{k\text{-cycles}} \mathbb{P}\{(U_i)\} = \langle\tau_k\rangle_\varphi, \tag{2.7}$$

where U_i denotes the event that the i^{th} k-cycle exists, X_k is the number of k-cycles in G, and $\langle\tau_k\rangle_\varphi$ denotes the φ-evaluation of τ_k.

Now define the map

$$\vartheta : \mathcal{I}_{V\times V}\otimes\mathcal{N}_V \to \mathcal{I}_{V\times V}\otimes\mathcal{N}_{\mathcal{P}(V\times V)}$$

by linear extension of

$$\vartheta\left(\gamma_{\underline{\ell}}\,\zeta_{\underline{j}}\right) = \gamma_{\underline{\ell}}\,\zeta_{f(\underline{\ell})}, \tag{2.8}$$

where $\underline{\ell}\in\mathcal{P}(V\times V)$ is a fixed multi-index, $\underline{j}\in\mathcal{P}(V)$ is an arbitrary multi-index, and $f : \mathcal{P}(V\times V) \to [2^{|V\times V|}]$ is an *enumeration* of the power set of $V\times V$. Each subset of $V\times V$ is now associated with one nilpotent generator of the $2^{|V\times V|}$-dimensional algebra $\mathcal{N}_{\mathcal{P}(V\times V)}$. The associated vertex sets are discarded.

With the proper tools in place, the strategy is as follows:

- Considering the k^{th} power of the edge-labeled nilpotent adjacency matrix reveals the collection of k-cycles within the graph. The trace is an element of the algebra. These cycles are then associated with nilpotent generators in a higher-dimensional algebra.

- Taking the ℓ^{th} power of the resulting element recovers all *distinct ℓ-tuples* of k-cycles because each k-cycle is associated with a distinct nilpotent element.

- Because the edges of the graph are associated with idempotent elements, the probability evaluation can be applied to edge sets representing multiple k-cycles regardless of multiplicity. That is, a single edge may be part of numerous distinct cycles, but its probability need only be considered once when computing the probability of existence for a given collection of cycles.

Remark 2.3. Throughout the remainder of the paper, the quantity $\binom{n}{k}\frac{(k-1)}{\omega}$ refers to the maximum number of k-cycles possible in a graph on n vertices.

Theorem 2.4. *Let Λ be the edge-labeled nilpotent adjacency matrix of a random graph $G = (V, E)$. For fixed positive integer $k \leq |V|$, let X_k denote the number of k-cycles in G, and let $\hat{\tau}_k = \vartheta\left(\frac{1}{\omega k}\mathrm{tr}(\Lambda^k)\right)$. Then,*

$$\mathbb{P}\{X_k = \ell\} = \left\langle (-1)^\ell 2e^{-\hat{\tau}_k} - 2\sum_{j=0}^{\ell-1}\frac{(-1)^{j+\ell}}{j!}\hat{\tau}_k{}^j - \frac{\hat{\tau}_k{}^\ell}{\ell!}\right\rangle_\varphi . \tag{2.9}$$

Proof. Given $\hat{\tau}_k = \vartheta(\tau_k)$, denote by $\hat{\tau}_k{}^{(i)}$ the $k \otimes k$-vector associated with the i^{th} k-cycle enumerated in G. Utilizing idempotency of the edges and nilpotency of the vertices and expanding $\hat{\tau}_k$ in terms of the k-cycles it represents, namely

$$\hat{\tau}_k = \sum_{i=1}^{\binom{n}{k}\frac{(k-1)!}{\omega}} \hat{\tau}_k^{(i)},$$

one can see that $\sum_{i<j} \hat{\tau}_k^{(i)}\hat{\tau}_k^{(j)}$ gives the collection of $s \otimes t$-vectors associated with edge- and vertex-sets of *pairs* of k-cycles, where s and t are positive integers satisfying $k < s, t \leq 2k$. Because distinct cycles are associated with distinct nilpotent generators, a straightforward inductive argument shows that for any positive integer j,

$$\hat{\tau}_k{}^j = j! \sum_{i_1<i_2<\cdots<i_j} \hat{\tau}_k^{(i_1)} \cdots \hat{\tau}_k^{(i_j)}. \tag{2.10}$$

Because the φ-evaluation determines the probability of distinct subsets of edges in G, it is further evident that the probability that G contains a particular ℓ-tuple of k-cycles is given by

$$\mathbb{P}(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_\ell}) = \left\langle \hat{\tau}_k^{(i_1)} \cdots \hat{\tau}_k^{(i_\ell)} \right\rangle_\varphi . \tag{2.11}$$

Now the probability that G contains one or more k-cycles is found by the principle of inclusion-exclusion:

$$\begin{aligned}
\mathbb{P}(U_1 \cup \cdots \cup U_{\binom{n}{k}\frac{(k-1)!}{\omega}}) &= \sum_i \mathbb{P}(U_i) - \sum_{i_1<i_2} \mathbb{P}(U_{i_1} \cap U_{i_2}) \\
+ \sum_{i_1<i_2<i_3} \mathbb{P}(U_{i_1} \cap U_{i_2} \cap U_{i_3}) - &\cdots + (-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}-1]} \mathbb{P}(U_{i_1} \cap \cdots \cap U_{i_{\binom{n}{k}\frac{(k-1)!}{\omega}}}) \\
= \left\langle \sum_i \hat{\tau}_k^{(i)} \right\rangle_\varphi - \left\langle \sum_{i<j} \hat{\tau}_k^{(i)} \hat{\tau}_k^{(j)} \right\rangle_\varphi &+ \left\langle \sum_{i<j<\ell} \hat{\tau}_k^{(i)} \hat{\tau}_k^{(j)} \hat{\tau}_k^{(\ell)} \right\rangle_\varphi - \cdots \\
&+ (-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}-1]} \left\langle \hat{\tau}_k^{(1)} \cdots \hat{\tau}_k^{(\binom{n}{k}\frac{(k-1)!}{\omega})} \right\rangle_\varphi .
\end{aligned} \tag{2.12}$$

This is simplified by applying (2.10) as follows:

$$\begin{aligned}
\mathbb{P}(U_1 \cup \cdots \cup U_{\binom{n}{k}\frac{(k-1)!}{\omega}}) &= \langle \hat{\tau}_k \rangle_\varphi - \frac{1}{2} \left\langle \hat{\tau}_k{}^2 \right\rangle_\varphi + \frac{1}{3!} \left\langle \hat{\tau}_k{}^3 \right\rangle_\varphi - \cdots \\
&\cdots + \frac{(-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}-1]}}{\left(\binom{n}{k}\frac{(k-1)!}{\omega}\right)!} \left\langle \hat{\tau}_k{}^{\binom{n}{k}\frac{(k-1)!}{\omega}} \right\rangle_\varphi .
\end{aligned} \tag{2.13}$$

Note that in the remainder of the proof, superscripts on $\hat{\tau}_k$ are exponents.

Then, by linearity of φ-evaluation and nilpotency,

$$\mathbb{P}(U_1 \cup \cdots \cup U_{\binom{n}{k}\frac{(k-1)!}{\omega}}) = 1 - \left\langle e^{-\hat{\tau}_k} \right\rangle_\varphi . \tag{2.14}$$

Similarly, the probability that G contains *two* or more k-cycles is computed by inclusion-exclusion. That is,

$$\begin{aligned}
\mathbb{P}(X_k \geq 2) &= \sum_{i_1<i_2} \mathbb{P}(U_{i_1} \cap U_{i_2}) - \sum_{i_1<i_2<i_3} \mathbb{P}(U_{i_1} \cap U_{i_2} \cap U_{i_3}) \\
&+ \cdots + (-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}]} \mathbb{P}(U_{i_1} \cap \cdots \cap U_{i_{\binom{n}{k}\frac{(k-1)!}{\omega}}}) \\
&= \langle \hat{\tau}_k \rangle_\varphi - \frac{1}{2} \left\langle \hat{\tau}_k{}^2 \right\rangle_\varphi + \frac{1}{3!} \left\langle \hat{\tau}_k{}^3 \right\rangle_\varphi - \cdots \\
&\cdots + \frac{(-1)^{[\binom{n}{k}\frac{(k-1)!}{\omega}]}}{\left(\binom{n}{k}\frac{(k-1)!}{\omega}\right)!} \left\langle \hat{\tau}_k{}^{\binom{n}{k}\frac{(k-1)!}{\omega}} \right\rangle_\varphi ,
\end{aligned} \tag{2.15}$$

which is simplified to become

$$\mathbb{P}(X_k \geq 2) = \left\langle e^{-\hat{\tau}_k} + \hat{\tau}_k - 1\right\rangle_\varphi . \tag{2.16}$$

It is now evident that if X_k denotes the number of k-cycles appearing in G, the probability that G contains ℓ or more k-cycles is equal to the probability one or more ℓ-tuples of k-cycles exist in G. Using inclusion-exclusion and induction, the following result is obtained:

$$\mathbb{P}\{X_k(n) \geq \ell\} = \left\langle (-1)^\ell e^{-\hat{\tau}_k} + \sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell-1}}{j!} {\hat{\tau}_k}^{\,j} \right\rangle_\varphi . \tag{2.17}$$

Therefore,

$$\mathbb{P}\{X_k = \ell\} = \mathbb{P}\{X_k \geq \ell\} - \mathbb{P}\{X_k \geq \ell + 1\} =$$

$$\left\langle (-1)^\ell e^{-\hat{\tau}_k} + \sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell-1}}{j!} {\hat{\tau}_k}^{\,j} \right\rangle_\varphi - \left\langle (-1)^{\ell+1} e^{-\hat{\tau}_k} + \sum_{j=0}^{\ell} \frac{(-1)^{j+\ell}}{j!} {\hat{\tau}_k}^{\,j} \right\rangle_\varphi$$

$$= \left\langle (-1)^\ell 2e^{-\hat{\tau}_k} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} {\hat{\tau}_k}^{\,j} - \frac{{\hat{\tau}_k}^{\,\ell}}{\ell!} \right\rangle_\varphi \tag{2.18}$$

□

Corollary 2.5. *Let $m > 1$ be fixed, and let G be a random graph on n vertices with associated edge-labeled nilpotent adjacency matrix Λ. Then for $k \leq n$,*

$$\mathbb{E}\left({X_k}^m\right) = \sum_{\ell=1}^{\infty} \ell^m \left\langle (-1)^\ell 2e^{-\hat{\tau}_k} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} {\hat{\tau}_k}^{\,j} - \frac{{\hat{\tau}_k}^{\,\ell}}{\ell!} \right\rangle_\varphi . \tag{2.19}$$

The variance of X_k is then given by

$$\mathrm{var} X_k = \sum_{\ell=1}^{\infty} \left(\ell^2 \left\langle (-1)^\ell 2e^{-\hat{\tau}_k} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} {\hat{\tau}_k}^{\,j} - \frac{{\hat{\tau}_k}^{\,\ell}}{\ell!} \right\rangle_\varphi \right) - {\langle \hat{\tau}_k \rangle_\varphi}^2 . \tag{2.20}$$

For a finite random graph, the asymptotic behavior of $\lim_{m\to\infty} \mathbb{E}({X_k}^m)$ is characterized in the next proposition.

Proposition 2.6. *Let $G = (V, E)$ be a finite random graph. Fix $k \geq 2$, and define the following quantities:*

$$c_\ell = \mathbb{P}\{X_k = \ell\} \tag{2.21}$$
$$\lambda = \max\{\ell : c_\ell \neq 0\}. \tag{2.22}$$

Then,

$$\lim_{m\to\infty} \frac{\mathbb{E}(X_k{}^m)}{\lambda^m} = \left\langle (-1)^\lambda 2e^{-\hat{\tau}_k} - 2\sum_{j=0}^{\lambda-1} \frac{(-1)^{j+\lambda}}{j!}\hat{\tau}_k{}^j - \frac{\hat{\tau}_k{}^\lambda}{\lambda!} \right\rangle_\varphi . \tag{2.23}$$

Proof. By definition of the m^{th} moment of X_k,

$$\mathbb{E}(X_k{}^m) = \sum_\ell \ell^m \, \mathbb{P}\{X_k = \ell\} = \sum_\ell \ell^m c_\ell . \tag{2.24}$$

Let λ denote the maximum value of ℓ such that $c_\ell \neq 0$. Then,

$$\frac{\mathbb{E}(X_k{}^m)}{\lambda^m} = \sum_\ell \frac{\ell^m}{\lambda^m} c_\ell = c_\lambda + \sum_{\ell<\lambda} \frac{\ell^m}{\lambda^m} c_\ell . \tag{2.25}$$

Observing that λ is finite, $\frac{\ell}{\lambda} < 1$, and $0 \leq c_\ell \leq 1$, the proof is complete. □

3 Convergence of Moments

Let $\mathcal{G} = (G_n)$ denote a sequence of random graphs with vertex sets V_n and edge sets E_n. For each $n > 0$, let G_n denote a random graph on n vertices having no more than $\frac{2}{\omega}\binom{n}{2}$ edges, with probabilities of existence $\{p_1, \ldots, p_{\frac{2}{\omega}\binom{n}{2}}\}$. Further assume that each G_n is a subgraph of G_{n+1}. In other words,

$$v_i \in V_n \Rightarrow v_i \in V_{n+1} \text{ , and}$$
$$(v_i, v_j) \in E_n \Rightarrow (v_i, v_j) \in E_{n+1}.$$

For each $n > 0$, the edge-labeled nilpotent adjacency matrix of G_n has entries in $\mathcal{I}_{V_n \times V_n} \otimes \mathcal{N}_{V_n}$. Each algebra is therefore canonically embedded in the infinite-dimensional algebra $\mathcal{I} \otimes \mathcal{N}$, defined by

$$\mathcal{I} \otimes \mathcal{N} = \bigoplus_{n=1}^{\infty} (\mathcal{I}_{V_n \times V_n} \otimes \mathcal{N}_{V_n}) . \tag{3.1}$$

The φ-evaluation is extended naturally to $\mathcal{I} \otimes \mathcal{N}$. The goal is to determine a condition on the matrices $\{\Lambda_n\}$ so that $\lim_{n\to\infty} \mathbb{E}(X_k(n)^m) < \infty$. That is, a condition is sought to ensure that

$$\lim_{n\to\infty} \sum_{\ell=1}^{\infty} \ell^m \left\langle (-1)^\ell 2e^{-\hat{\tau}_k(n)} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} \hat{\tau}_k(n)^j - \frac{\hat{\tau}_k(n)^\ell}{\ell!} \right\rangle_\varphi < \infty. \tag{3.2}$$

For the case $m = 1$,

$$\lim_{n\to\infty} \mathbb{E}(X_k(n)) = \lim_{n\to\infty} \left\langle \frac{1}{\omega k} \mathrm{tr}\left({\Lambda_n}^k\right) \right\rangle_\varphi, \tag{3.3}$$

provided the limit exists. The next theorem gives a sufficient condition for convergence of higher moments.

Theorem 3.1. *Let $(G_n) = (V_n, E_n)$ be an increasing sequence of random graphs such that G_n is a subgraph of G_{n+1} for all n. Let integers $k, m \geq 2$ be fixed. For each $n \in \mathbb{N}$, let Λ_n denote the edge-labeled nilpotent adjacency matrix of G_n, and define $\hat{\tau}_k(n) = \vartheta\left(\frac{1}{\omega k}\mathrm{tr}\left({\Lambda_n}^k\right)\right)$. Suppose that there exist positive real numbers L and N such that for all $\ell > L$, the following inequality is satisfied for all $n > N$:*

$$\left\langle (-1)^\ell e^{-\hat{\tau}_k(n)} + \sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell-1}}{j!} \hat{\tau}_k(n)^j \right\rangle_\varphi < \frac{1}{\ell^{m+1+\varepsilon}}. \tag{3.4}$$

Then $\lim_{n\to\infty} \mathbb{E}(X_k(n)^m)$ exists.

Proof. Let k be fixed. Observe that for fixed n and ℓ,

$$\mathbb{P}(X_k(n) = \ell) \leq \mathbb{P}(X_k(n) \geq \ell). \tag{3.5}$$

Hence,

$$\left\langle (-1)^\ell 2e^{-\hat{\tau}_k(n)} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} \hat{\tau}_k(n)^j - \frac{\hat{\tau}_k(n)^\ell}{\ell!} \right\rangle_\varphi \leq \left\langle (-1)^\ell e^{-\hat{\tau}_k(n)} + \sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell-1}}{j!} \hat{\tau}_k(n)^j \right\rangle_\varphi. \tag{3.6}$$

Now,

$$\mathbb{E}(X_k(n)^m) = \sum_{\ell=1}^{L} \ell^m \left\langle (-1)^\ell 2e^{-\hat{\tau}_k(n)} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} \hat{\tau}_k(n)^j - \frac{\hat{\tau}_k(n)^\ell}{\ell!} \right\rangle_\varphi$$

$$+ \sum_{\ell=L+1}^{\infty} \ell^m \left\langle (-1)^\ell 2e^{-\hat{\tau}_k(n)} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} \hat{\tau}_k(n)^j - \frac{\hat{\tau}_k(n)^\ell}{\ell!} \right\rangle_\varphi . \quad (3.7)$$

Assuming the conditions in the statement of the theorem and applying inequality (3.6) yield the following inequality, valid for all $n > N$:

$$\sum_{\ell=L+1}^{\infty} \ell^m \left\langle (-1)^\ell 2e^{-\hat{\tau}_k(n)} - 2\sum_{j=0}^{\ell-1} \frac{(-1)^{j+\ell}}{j!} \hat{\tau}_k(n)^j - \frac{\hat{\tau}_k(n)^\ell}{\ell!} \right\rangle_\varphi$$

$$\leq \sum_{\ell=L+1}^{\infty} \frac{\ell^m}{\ell^{m+1+\varepsilon}} = \sum_{\ell=L+1}^{\infty} \frac{1}{\ell^{1+\varepsilon}}. \quad (3.8)$$

□

4 Links to Quantum Computing

The algebra $\mathcal{N}_V$ can be constructed within the group algebra of $\mathbb{Z}_2{}^n$. The idempotent-generated algebra $\mathcal{I}_{V\times V}$ can be realized using mutually orthogonal hyperplane projections in $\mathbb{R}^{|V\times V|}$. The realization chosen here unites both algebras within a context familiar to physicists and quantum probabilists.

For fixed $n > 0$, the n-particle *fermion algebra* is defined as the associative algebra generated by the collection $\{f_i, f_i{}^+\}$, where $1 \leq i \leq n$, satisfying the following:

$$\{f_i{}^+, f_j\} = \delta_{ij} \quad (4.1)$$

$$\{f_i, f_j\} = \{f_i{}^+, f_j{}^+\} = 0. \quad (4.2)$$

Here, $\{a, b\} = ab + ba$ is the anti-commutator, and δ_{ij} is the Kronecker delta function. For each $1 \leq i \leq n$, $f_i{}^+$ denotes the i^{th} fermion creation operator, while f_i denotes the i^{th} fermion annihilation operator.

When V is a set with n elements, the algebra $\mathcal{N}_V$ is constructed within the $2n$-particle fermion algebra by writing

$$\zeta_i = f_i{}^+ f_{n+i}^+. \quad (4.3)$$

The algebra $\mathcal{I}_{V\times V}$ is constructed within a $2n(n-1)$-particle fermion algebra. Fix $n > 0$ and consider elements of the form

$$\gamma_i = \frac{1}{2}\left(1 + \left(\frac{f_i + f_i{}^+}{2}\right)\left(\frac{f_{n^2-n+i} + f^+_{n^2-n+i}}{2}\right)\right). \tag{4.4}$$

Direct calculation shows

$$\gamma_i{}^2 = \left(\frac{1 + \left(\frac{f_i+f_i{}^+}{2}\right)\left(\frac{f_{n^2-n+i}+f^+_{n^2-n+i}}{2}\right)}{2}\right)^2$$

$$= \frac{1}{4} + \frac{1}{2}\left(\frac{f_i + f_i{}^+}{2}\right)\left(\frac{f_{n^2-n+i} + f^+_{n^2-n+i}}{2}\right)$$

$$+\frac{1}{2}\left(\frac{f_i + f_i{}^+}{2}\right)\left(\frac{f_{n^2-n+i} + f^+_{n^2-n+i}}{2}\right)\left(\frac{f_i + f_i{}^+}{2}\right)\left(\frac{f_{n^2-n+i} + f^+_{n^2-n+i}}{2}\right)$$

$$= \frac{2\left(\frac{f_i+f_i{}^+}{2}\right)\left(\frac{f_{n^2-n+i}+f^+_{n^2-n+i}}{2}\right) + 2}{4} = \gamma_i. \tag{4.5}$$

Because each γ_i is written using a pair $(i, n^2 - n + i)$ of fermion creation/annihilation operator pairs and because these pairs are disjoint for $i \neq j$, direct calculation also shows that $\gamma_i\,\gamma_j = \gamma_j\,\gamma_i$ for $i \neq j$.

Letting $\mathcal{F}$ denote the infinite-dimensional fermion algebra,

$$\mathcal{I} \otimes \mathcal{N} \subset \mathcal{F} \otimes \mathcal{F}.$$

The edge-labeled nilpotent adjacency matrix associated with a finite graph can itself be considered a quantum random variable whose m^{th} moment corresponds to the number of m-cycles occurring in the graph [5]. Considering sequences of such quantum random variables associated with ascending sequences of random graphs is a topic for further research.

Acknowledgment. The authors owe a debt of gratitude to the referee for a number of valuable suggestions.

References

[1] G. Dobinski, Summierung der Reihe für m = 1, 2, 3, 4, 5, ... *Grunert Archiv (Arch. Math. Phys.)* 61 (1877), 333-336.

[2] Y. Hashimoto, A. Hora, N. Obata, Central limit theorems for large graphs: method of quantum decomposition, J. Math. Phys., 44 (2003), 71-88.

[3] F. Lehner, Cumulants in noncommutative probability theory III, Creation and annihilation operators on Fock spaces, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (2005), 407-437.

[4] N. Obata, Quantum probabilistic approach to spectral analysis of star graphs, Interdisciplinary Information Sciences, **10** (2004), 41-52.

[5] R. Schott, G.S. Staples, Clifford algebras, random graphs, and quantum random variables, presented at 27th International Conference on Quantum Probability, Nottingham, England, July, 2006.

[6] R. Schott, G.S. Staples, Reductions in computational complexity using Clifford algebras, Preprint, 2006. `http://www.siue.edu/~sstaple/index_files/NPtoP.pdf`.

[7] G.S. Staples, A new adjacency matrix for finite graphs, submitted to Proceedings of the 7th International Conference on Clifford Algebras and Their Applications, Toulouse, France, May, 2005.

[8] D. West, *Introduction to Graph Theory, Second Ed.*, Prentice Hall, Upper Saddle River, 2001.