A new lower bound on critical graphs with maximum degree of 8 and 9

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Abstract

In this article, we give new lower bounds for the size of edge chromatic critical graphs with maximum degrees of $8, 9$ respectively. Furthermore, it implies that if G is a graph embeddable in a surface S with characteristics $c_S = -1$ or -2, then G is *class one* if maximum degree $\Delta \geq 8$ or 9 respectively.

Key words: edge chromatic number, critical graph.

1 Introduction

A *k-edge-coloring* of a graph G is a function $\phi : E(G) \mapsto \{1, \cdots, k\}$ such that any two adjacent edges receive different colors. The *edge chromatic number*, denoted by $\chi_e(G)$, of a graph G is the smallest integer k such that G has a k-edge-coloring. Vizing's Theorem [13] states that the edge chromatic number of a simple graph G is either Δ or $\Delta+1$, where Δ denotes the maximum vertex degree of G. A graph G is *class one* if $\chi_e(G) = \Delta$ and is *class two* otherwise. A class two graph G is *critical* if $\chi_e(G-e) < \chi_e(G)$ for each edge e of G. A critical graph G is Δ-*critical* if it has maximum

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degree Δ. The following conjecture was proposed by Vizing [14] concerning the sizes of critical graphs.

Conjecture 1.1. *If $G = (V, E)$ is a critical simple graph, then* $|E| \geq \frac{1}{2}(|V|(\Delta - 1) + 3)$.

We list some results of critical graphs with small maximum degrees in the following.

Theorem 1.2. *Let G be a critical graph with maximum degree Δ. Then,*

(1) ([5]) $|E| \geq \frac{\Delta+1}{3}|V|$ *if* $6 \leq \Delta \leq 8$.

(2) ([9]) $|E| \geq \frac{10}{3}|V|$ *if* $\Delta \geq 9$.

(3)([15, 16]) $|E| \geq \begin{cases} \frac{12}{7}|V|, \frac{15}{7}|V|, \frac{87}{35}|V|, \frac{31}{11}|V| & \text{if } \Delta = 4, 5, 6, 7 \text{ respectively.} \\ \frac{2(\Delta+3)}{7}|V| & \text{if } 8 \leq \Delta \leq 17. \end{cases}$

(4) ([7]) $|E| \geq \{\ 4|V|, \frac{17}{4}|V|, \frac{46}{10}|V| \quad \text{if } \Delta = 10, 11, 12 \text{ respectively.}$

In section 3, we will present new lower bounds on the size of Δ-critical graphs: $|E(G)| \geq \frac{13}{4}|V(G)|$ if $\Delta = 8$, and $|E(G)| \geq 3.6|V(G)|$ if $\Delta = 9$. These bounds are better than those in [9] and [18] and are closer to Conjecture 1.1 for $\Delta = 8, 9$. In section 4, by applying these results, we obtain that if G is a simple graph with maximum degree Δ that is embeddable in a surface S of characteristic $c_S = -1$, or -2, then G is *class one* if $\Delta \geq 8$ or 9 respectively.

Let V and E be the vertex set and edge set of a graph G, respectively. Let $|V|$ and $|E|$ be the cardinality of V and E of G, respectively. Two vertices are adjacent to each other if there is an edge of G connecting them. A k-vertex (or, ($\leq k$)-vertex, ($\geq k$)-vertex) is a vertex of degree k (or $\leq k$, $\geq k$, respectively). For a vertex $x \in V(G)$, let $N(x)$ be the set of vertices adjacent to x. A vertex y is a *neighbor* of x if $y \in N(x)$. A k-neighbor y of x is a *neighbor* of x having a degree of k. For $V' \subseteq V(G)$, let $N(V') = \cup_{x \in V'} N(x)$. V_k (or $V_{\geq k}$, $V_{\leq k}$) is the set of k-vertices (or, ($\geq k$)-vertices, ($\leq k$)-vertices, respectively). Let $d(x)$ be the degree of the vertex x. We define $d_k(x)$ (or $d_{\geq k}(x)$, $d_{\leq k}(x)$) as the number of k-vertices (or ($\geq k$)-vertices, ($\leq k$)-vertices, respectively) adjacent to x in G.

2 Adjacency Lemmas

In this section, first we list some useful known adjacency Lemmas of critical graphs, later we give an improved adjacency Lemma. The following one belongs to Vizing [14], and we refer it as VAL.

Lemma 2.1. *(VAL) If G is a Δ-critical graph and xy is an edge of G, then x is adjacent to at least $(\Delta - d(y)+1)$ Δ-vertices other than y, $d(x)+d(y) \geq \Delta + 2$ and every vertex is adjacent to at least two Δ-vertices.*

Zhang ([17]) gave an adjacency lemma on two adjacent vertices whose sum of degrees is $\Delta + 2$, it was obtained independently by Sanders and Zhao [12]. But in this paper, it is referred as Zhang's Adjacency Lemma.

Lemma 2.2. *(Zhang [17]) Let G be critical, $xy \in E(G)$ and $d(x) + d(y) = \Delta + 2$. The following hold:*

(1) each neighbor of x, y is a Δ-vertex; (2) every vertex of $N(N(x, y)) \setminus \{x, y\}$ is of degree at least $\Delta - 1$; and (3) if $d(x), d(y) < \Delta$, then every vertex of $N(N(x, y)) \setminus \{x, y\}$ is a Δ-vertex.

For an edge e of a Δ-critical graph G, $G-e$ has a Δ-edge-coloring. Given two colors j and k, the subgraph of $G - e$ induced by the edges colored either j or k, call it $G(j, k)$, has maximum degree two, and is thus the disjoint union of paths and cycles. A bi-colored (k, j)-path is a component of $G(k, j)$ which is a path. Let $P_{k,j}(x)$ be a bi-colored (k, j)-path starting at x. A vertex v sees color j if v is adjacent to an edge colored by j. Given a vertex v in G that sees j and doesn't see k, *swapping* (k, j) *along* v means switching the colors j and k along the (j, k)-bi-colored path staring at v. If edge xy misses q, swapping xy to a color q means that swapping (j, q)

along x where xy colored by j. Let ϕ be a Δ-edge-coloring of $G - e$. Let $\phi(v)$ be the set of colors appearing on the edges adjacent to the vertex v.

For the purpose of simplicity, in this paper, let G be the Δ-critical graph with $7 \leq \Delta \leq 9$. Following lemmas summarize results in [6-11] which will be used to prove the main theorem.

Lemma 2.3. *[10] Let x be a 3-vertex in G. Then*

(1) there are at least two Δ-vertices y in $N(x)$ with $d_{\leq\Delta-2}(y) \leq 1$, and

(2) x has a neighbor which is adjacent to at least $\Delta - 6$ Δ-vertices z with $d_{\leq\Delta-3}(z) = 0$.

Lemma 2.4. *[10] Let x be a 4-vertex in G.*

(1) If x is adjacent to four Δ-vertices and one of its neighbors is adjacent to three $(\leq \Delta - 2)$-vertices, then each of the remain three neighbors of x is adjacent to only one $(\leq \Delta - 2)$-vertex, which is x;

(2) If x is adjacent to a $(\Delta - 1)$-vertex, then there are at least two Δ-vertices in $N(x)$ which are adjacent to at most two $(\leq \Delta - 2)$-vertices. Moreover, if x is adjacent to two $(\Delta - 1)$-vertices, then each of the two Δ-neighbors is adjacent to exactly one $(\leq \Delta - 2)$-vertex, which is x.

For the sake of convenience of discussion, we denote by $\delta_1(x)$ the minimum degree of vertices adjacent to x. The following three lemmas could be obtained by mimicking proofs of Lemma 2.5, lemma 2.7 in [7]and lemma 2.5 in [6] without restriction of $\Delta \geq 10$. To avoid repetition, we omit the proofs here.

Lemma 2.5. *[7] Let x be a 5-vertex in G and w be a $\delta_1(x)$-neighbor of x.*

(i) If $d(w) = \Delta$ and it is adjacent to four $(\leq \Delta - 3)$-vertices, then, the remain neighbors of x are all Δ-vertices and none of them is adjacent to any $(\leq \Delta - 3)$-vertices other than x.

(ii) $d(w) = \Delta - 1$.

(ii-1) If w is adjacent to two $(\leq \Delta - 2)$ vertices other than x, then the remain four neighbors of x are all Δ-vertices and each of them is adjacent to all $(\geq \Delta - 1)$-vertices other than x.

(ii-2) If w is adjacent to one $(\leq \Delta - 2)$ vertices other than x, then there are three $(\geq \Delta - 1)$-neighbors y of x including at least one Δ-neighbor satisfying the following: if y is a Δ-vertex, then it is adjacent to at most two $(\leq \Delta - 1)$-vertices; if y is a $(\Delta - 1)$-vertex, then it is adjacent to only one $(\leq \Delta - 1)$-vertex which is x.

(iii) $d(w) = \Delta - 2$.

(iii-1) If w is adjacent to one $(\leq \Delta - 2)$-vertex other than x, then all other four neighbors of x are Δ-vertices and each of them is adjacent to $(\geq \Delta - 1)$-vertices other than x.

(iii-2) If w is adjacent to only one $(\leq \Delta - 2)$-vertex which is x, then there are three $(\geq \Delta - 1)$-neighbors of x including at least two Δ-neighbors y satisfying the following: if it is a Δ-vertex, then it is adjacent to at most two $(\leq \Delta - 2)$-vertices; if it is a $(\Delta - 1)$-vertex, then it is adjacent to one $(\leq \Delta - 2)$-vertex which is x.

Lemma 2.6. *[6] Let x be a 5-vertex of a Δ-critical graph G. $|V_\Delta \cap N(x)| = 2$ and $|V_{\Delta-1} \cap N(x)| = 3$, then $N(N(x) \cap V_\Delta) \subseteq V_{\geq \Delta - 2}$.*

Lemma 2.7. *[7] Let x be a 6-vertex in G and w be a $\delta_1(x)$-neighbor of x where $\delta_1(x) = \Delta - 2$, or $\Delta - 1$. Then we have following:*

(i) $d(w) = \Delta - 2$.

(i-1) If w is adjacent to three $(\leq \Delta - 2)$-vertices, then each of the five neighbors of x other than w is Δ-vertex and is adjacent to all $(\geq \Delta - 2)$-vertices other than x.

(i-2) If w is adjacent to two $(\leq \Delta-2)$-vertices, then there are four $(\geq \Delta-1)$-neighbors of x including at least two Δ-neighbors y satisfying: if y is a Δ-vertex, then it is adjacent to at most two $(\leq \Delta-2)$-vertices; if y is a $(\Delta-1)$-vertex, then it is adjacent to one $(\leq \Delta-2)$-vertex which is x.

(i-3) If (i-1) and (i-2) do not happen, then each $\Delta-2$-neighbor of x is adjacent to one $(\leq \Delta-2)$-vertex which is x, and each Δ-neighbor of x is adjacent to at most three $(\leq \Delta-2)$-vertices.

(ii) $d(w) = \Delta - 1$.

(ii-1) If w is adjacent to four $(\leq \Delta-3)$-vertices, then each of the five neighbors of x other than w is Δ-vertex and is adjacent to all $(\geq \Delta-2)$-vertices other than x.

(ii-2) If (ii-1) does not happen, then each $(\Delta-1)$-neighbor of w is adjacent to at most three $(\leq \Delta-3)$-vertices.

Lemma 2.8. *[11] Let x be a j-vertex of a Δ-critical graph which is adjacent to a k-vertex y. if $j < \Delta, k < \Delta$, then x is adjacent to at least $\Delta - k + 1$ vertices z satisfying the following: $z \neq y; z$ is adjacent to at least $2\Delta - j - k$ vertices different from x of degree at least $2\Delta - j - k + 2$; and if z is not adjacent to y, then z is adjacent to at least $2\Delta - j - k + 1$ vertices different from x of degree at least $2\Delta - j - k + 2$.*

Lemma 2.2 (L.Zhang [17]) gives some information on two adjacent vertices of a critical graph whose sum of degrees is $\Delta + 2$. Naturally, we ask that are there any similar results for two adjacent vertices of a critical graph whose sum of degrees is $\Delta + 3$? The following Lemma gives partial answer to the question.

Lemma 2.9. *Let x be a j-vertex of a critical graph G which is adjacent to a vertex w such that $d(x) + d(w) = \Delta + 3$ and $|N(x) \cap V_{\leq \Delta-1}| = 2$. Then* *there are at least $j-2$ Δ-vertices $y \in N(x)$ satisfying: $y \neq w; y$ is adjacent to all vertices of degree at least $\Delta - 1$.*

Figure 1: Δ-edge coloring ϕ of G-xw exhibited at $N(x) \cup N(w)$ in Lemma 2.9.

Proof. Since G is critical, $G-xw$ has an edge Δ-coloring. Each color shows either at x or at w, or G has an edge Δ-coloring. Without loss of generality, the edges incident with x in $G-xw$ are colored $1, \cdots, j-1$, while those incident with w are colored $j-1, \cdots, \Delta$ since $j+k = \Delta+3$ (See Figure 1).

By Lemma 2.2-2.4 in [?], we have following observation, that is, there are $j-2$ vertices z in $N(x)\setminus\{w\}$ with xz colored by a color in $\{1, \cdots, j-2\}$ so that each of them has degree at least $\Delta - 1$ where $j+k = \Delta+3$. On the other hand, by VAL, x is adjacent to at least $j-2$ Δ-vertices. Thus it is sufficient to consider following **two cases** without loss of generality.

Case I: There is a Δ-vertex z in $N(x) \setminus \{w\}$ with xz colored by color $j-1$ and there is a $\leq (\Delta-1)$-vertex y in $N(x) \setminus \{w\}$ with xy colored by a color in $\{1, \cdots, j-2\}$.

Case II: There is a $(\leq \Delta - 1)$-vertex $z \in N(x) \setminus \{w\}$ with xz colored by $j-1$, and there are $j-2$ Δ-vertices y in $N(x) \setminus \{w\}$ with xy colored by colors in $\{1, \cdots, j-2\}$.

Since proof of Case I is not only similar to, but also harder than that of Case II, we are to give the proof of case I only.

We call a swapping (i,j) along a vertex u is a *nice* swapping if the swapping does not affect the colors of edges incident with vertices x and w.

Proof of Case I. Without loss of generality, we assume xy is colored 1. It is sufficient to show that z is adjacent to all vertices of degree at least $\Delta - 1$. We use C to denote the set of colors: $\{1, \cdots, \Delta\}$.

(1) We claim that y may miss one color $j-1$ only.

(1-1) Claim that y sees each color in $\{j, \cdots, \Delta\}$.

It is obvious since each path $P_{1,k}(x)$ must end at w for each $k \in \{j, \cdots, \Delta\}$.

(1-2) Claim that y sees each color in $\{1, \cdots, j-2\}$.

For a color $r \in \{2, \cdots, j-2\}$, if y misses it, then we can do a *nice* swapping (r, Δ) along x. Under current edge coloring, y sees r but misses color Δ which contradicts (1-1). Hence y may miss color $j-1$ only since $d(y) \leq \Delta - 1$.

(2) Consider a neighbor u of z such that zu is colored $r \in \{j, \cdots, \Delta\}$. We claim that u must see each color in $\{1, \cdots, \Delta\}$.

(2-1) u sees $j-1$. Otherwise, we recolor zu, xz with $j-1, r$ respectively. Under current coloring, y must see $j-1$ but r. A contradiction to (1).

(2-2) u sees each color in $\{j, \cdots, \Delta\}$.

Otherwise, assume that u misses $\ell \in \{j, \cdots, \Delta\}$. Here, we use $P_{i,q}(v)_\phi$ to denote (i,q)-bi-colored path starting at v under edge coloring ϕ of $G - xw$. Consider $P_{j-1,\ell}(u)_\phi$. We do a nice swapping $(j-1, \ell)$ along u. Denote new edge coloring of $G - xw$ by ϕ'.

If $P_{j-1,\ell}(u)_\phi$ ends at x, then it doesn't pass through y as y misses $j-1$. Note that colors of edges adjacent to y don't be affected under ϕ'. So, by using same argument as in (1), y must see $j-1$, but misses color ℓ. A contradiction.

If $P_{j-1,\ell}(u)_\phi$ ends at y, note that colors of edges adjacent to x and w haven't been affected under ϕ'. Now y sees $j-1$, but it misses ℓ, a contradiction rises again.

Now we consider the case that $P_{j-1,\ell}(u)_\phi$ doesn't end either at x or at y. Under ϕ', we recolor zu, xz with $j-1, r$ respectively. Denote current edge coloring of $G - xw$ by ϕ''. Please note that colors of edges adjacent to y haven't been affected. Under ϕ'', by using same argument as in (1), y must see $j-1$, but misses color r which causes a contradiction.

(2-3) We claim that u sees each color in $\{1, \cdots, j-2\}$.

Without loss of generality, we assume that u misses color 1. We do a nice swapping $(1, \Delta)$ along x. By mimicking the proof as in (2-1) and (2-2) under current edge coloring of $G - xw$, we have that u must see each color in $C \setminus \{\Delta, 2, \cdots, j-1\}$ which implies that u sees color 1. A contradiction.

(3) Consider a neighbor v of z such that zv is colored by a color $b \in \{1, \cdots, j-2\}$, we are to show that v must see each color in $C \setminus \{j-1\}$.

(3-1) Claim that v sees each color in $\{j, \cdots, \Delta\}$.

Assume that v misses a color $p \in \{j, \cdots, \Delta\}$. Note that v and x are not in the same component of $G(b, p)$, and so as vertex v and w. Hence we do a nice swapping (b, p) along v. Now v sees p. By applying the same argument in (2), we have that v sees each color in C. A contradiction.

(3-2) Claim that v sees each color in $\{1, \cdots, j-2\}$.

Assume that v misses a color $b' \in \{1, \cdots, j-2\}$. We implement a nice swapping (b', Δ) along x. By (2), v sees each color in C, a contradiction. Hence, from (3-1) and (3-2), we obtain that $d(v) \geq \Delta - 1$. Thus, we finish our proof of Case I.

□

3 Main Results

Theorem 3.1. *Let G be a Δ-critical graph with $7 \le \Delta \le 9$. Then $|E(G)| \ge \frac{|V(G)|}{2}q$ where $q = 6.5$ or 7.2 if $\Delta = 8$ or 9 respectively.*

Proof. Suppose to the contrary, the Theorem is not true. Then $\sum_{x \in V}(d(x) - q) < 0$. We are to use charge-discharge method to get contradictions. We call $c(x) = d(x) - q$ the *initial charge* of the vertex x and will assign a new charge to each vertex x according to the following rules.

(R1) Let x be a 2-vertex and $u, v \in N(x)$. x receives $d(y) - q$ from each adjacent Δ-vertex y and each $z \in N(u) \setminus \{x, v\}$ sends $\frac{d(z)-q}{\Delta}$ to x via u and each $z \in N(v) \setminus \{x, u\}$ sends $\frac{d(z)-q}{\Delta}$ to x via v. Note that each Δ-vertex adjacent to both u and v sends $2 \times \frac{d(z)-q}{\Delta}$ to x in total.

(R2) Let x be a 3-vertex. Let $w \in N(x)$ with $d(w) = \Delta - 1$. x receives $d(y) - q$ from each adjacent $(\ge \Delta - 1)$-vertex y, and each Δ-vertex $z \in N(x, w) \setminus \{x, w\}$ sends $\frac{\Delta - q}{\Delta}$ to x via w. Note that Δ-vertices adjacent to w sends at least $(\Delta - 3) \times \frac{\Delta - q}{\Delta}$ to x in total.

(R3) If x is a 6-vertex and $\Delta = 8$, then x receives $\frac{0.5}{4}$ from each adjacent 7-vertex, $\frac{0.5}{3}$ from each adjacent 8-vertex.

If x is a 7-vertex and $\Delta = 9$, then x receives $\frac{0.2}{5}$ from each adjacent 8-vertex, $\frac{0.2}{3}$ from each adjacent 9-vertex.

(R4) Let x be a $(\le \lfloor q \rfloor - 1)$-vertex. Let

$$q_k = \begin{cases} \frac{q - \lfloor q \rfloor}{3} & \text{if } k = \Delta. \\ \frac{0.5}{4} & \text{if } k = 7, \Delta = 8. \\ \frac{0.2}{5} & \text{if } k = 8, \Delta = 9. \end{cases}$$

Note that $c(x) < 0$, x receives $\frac{k - q - s \times q_k}{j}$ from each adjacent k-vertex y for $k \ge \Delta - 1$ where $d_{(\le \lfloor q \rfloor - 1)}(y) = j$ and $d_{\lfloor q \rfloor}(y) = s$.

Let $c'(x)$ be the new charge of each vertex.

(I) Claim that $c'(x) > 0$ if $d(x) = 2$.

Let $u, v \in V_\Delta \cap N(x)$. By Lemma 2.2, each of u, v is adjacent to at least $(\Delta - 2)$ Δ-vertices different from u, v. Therefore, by (R1), $c'(x) \geq c(x) + 2 \times (\Delta - q) + 2 \times (\Delta - 2) \times \frac{\Delta - q}{\Delta} > 0$ where $\Delta = 8, 9$ respectively.

(II) Claim that $c'(x) \geq 0$ if $d(x) + \delta_1(x) = \Delta + 2$.

Let y be a vertex adjacent to x with $d(x) + d(y) = \Delta + 2$. Assume that x is a d-vertex with $3 \leq d \leq \lfloor q \rfloor$. By Zhang's Adjacency Lemma, $|N(x) \cap V_\Delta| = d - 1$ and each vertex in $N(N(x)) \setminus \{x, y\}$ has degree $\geq \Delta - 1$. Considering vertex y may have degree of $\leq q$ and x, y may share some Δ-neighbors, so x receives at least $\frac{\Delta - q}{2}$ from each adjacent Δ-vertex and $max\{d(y) - q, 0\}$ from y. Please note that $\delta_1(x)$ is the minimum degree of vertices adjacent to x. So by (R2),(R3),(R4), and lemma 2.3-2.7,

$$c'(x) \geq \begin{cases} -3.5 + 2 \times 1.5 + 0.5 = 0 & \text{if } d(x) = 3, \delta_1(x) = 7,\ \Delta = 8. \\ (4 - q) + 3 \times (\Delta - q - \frac{q - \lfloor q \rfloor}{3}) > 0 & \text{if } d(x) = 4, \delta_1(x) = \Delta - 2,\ \Delta = 8, 9. \\ (5 - q) + 4 \times \frac{\Delta - q}{2} > 0 & \text{if } d(x) = 5, \delta_1(x) = \Delta - 3,\ \Delta = 8, 9. \\ -0.5 + 5 \times \frac{0.5}{3} > 0 & \text{if } d(x) = 6, \delta_1(x) = 4,\ \Delta = 8. \\ -4.2 + 2 \times 1.8 + 0.8 + 5 \times \frac{1.8}{9} > 0 & \text{if } d(x) = 3, \delta_1(x) = 8,\ \Delta = 9. \\ -1.2 + 5 \times \frac{1.8}{2} > 0 & \text{if } d(x) = 6, \delta_1(x) = 5,\ \Delta = 9. \\ -0.2 + 6 \times \frac{0.2}{3} > 0 & \text{if } d(x) = 7, \delta_1(x) = 4,\ \Delta = 9. \end{cases}$$

Now we assume that x is a $(> \lfloor q \rfloor)$-vertex. By Lemma 2.2 and (R4), x sends out at most $d(x) - q$ to its adjacent vertex y, so $c'(x) \geq 0$.

(III) Claim that $c'(x) \geq 0$ if $d(x) + \delta_1(x) = \Delta + 3$ and $d(x) \leq \lfloor q \rfloor$.

First we consider that of $d(x) = 3$ and $\delta_1(x) = \Delta$. By Lemma 2.3, there are *two* Δ-vertices in $N(x)$, each of them is adjacent to at least $(\Delta - 1)$ $(\geq \Delta - 1)$-vertices. Then x receives at least $2 \times 1.5 + \frac{1.5}{2}$ from adjacent vertices if $\Delta = 8$. So $c'(x) \geq -3.5 + 3.75 > 0$ for $\Delta = 8$. Furthermore, by (R2), x receives at least $2 \times 1.8 + \frac{1.8}{2}$ if $\Delta = 9$. So, $c'(x) \geq -4.2 + 3.6 + 0.9 > 0$ for $\Delta = 9$.

Next we consider a vertex x with $d(x) \geq 4$ and $d(x) + \delta_1(x) = \Delta + 3$. Let y be a vertex adjacent to x with $d(x) + d(y) = \Delta + 3$. Assume that x is a d-vertex with $4 \leq d \leq \lfloor q \rfloor$. By Lemma 2.8[11] and Lemma 2.9, there are at least $d-2$ Δ-vertices in $(N(x) \setminus \{x, y\}$, each of them is adjacent to all vertices of degree $\geq \Delta - 1$. Be aware that vertex y may have degree $\leq q$ and x, y may share some Δ-neighbors, so x receives at least $\frac{\Delta - q}{2}$ from each adjacent Δ-vertex and $max\{d(y) - q, 0\}$ from y. So by Lemma 2.9,2.5 and 2.7, (R3) and (R4), we have

$$c'(x) \geq \begin{cases} (4-q) + 2 \times (\Delta - q) > 0 & \text{if } d(x) = 4, \delta_1(x) = \Delta - 1,\ \Delta = 8, 9. \\ -1.5 + 3 \times \frac{1.5}{2} > 0 & \text{if } d(x) = 5, \delta_1(x) = 6,\ \Delta = 8. \\ -0.5 + 4 \times \frac{0.5}{3} > 0 & \text{if } d(x) = 6, \delta_1(x) = 5,\ \Delta = 8. \\ -2.2 + 3 \times \frac{1.8}{2} + \frac{1.8}{4} > 0 & \text{if } d(x) = 5, \delta_1(x) = 7,\ \Delta = 9. \\ -1.2 + 4 \times \frac{1.8}{2} > 0 & \text{if } d(x) = 6, \delta_1(x) = 6,\ \Delta = 9. \\ -0.2 + 6 \times \frac{0.2}{3} > 0 & \text{if } d(x) = 7, \delta_1(x) = 5,\ \Delta = 9. \end{cases}$$

From now on, by (II) and (III), we consider the cases of $d(x)+\delta_1(x) \geq \Delta+4$, and of $d(x) > \lfloor q \rfloor$ if $d(x) + \delta_1(x) = \Delta + 3$.

(IV) Claim that $c'(x) > 0$ if $d(x) = 4$.

By discussion in previous paragraph, we have that $\delta_1(x) = \Delta$. There are two cases may arise: either there is *one* Δ-vertex $y \in N(x)$ with $d_{\leq \Delta-2}(y) = 3$ and each of rest vertices $z \in N(x)$ has $d_{\leq \Delta-2}(z) = 1$, or there is at least one Δ-vertex $y \in N(x)$ with $d_{\leq \Delta-2}(y) = 1$ and there are at most *three* vertices $z \in N(x)$ such that $d_{\leq \Delta-2}(z) \leq 2$. For former case, by (R4), x receives at least $\frac{\Delta-q}{3} + 3 \times (\Delta - q) > 2.5$ or 3.2 for $\Delta = 8$ or 9 respectively, so $c'(x) > 0$. For later case, by (R4), x receives at least $3 \times \frac{\Delta-q}{2} + (\Delta - q) > 2.5$ or 3.2 from its adjacent vertices for $\Delta = 8$ or 9 respectively. Hence, $c'(x) \geq 0$.

(V) Claim that $c'(x) \geq 0$ if $d(x) = 5$.

If $\delta_1(x) = \Delta - 1$, to avoid repetition, we consider the worst case, that is, x is adjacent to *two* Δ-vertices and *three* $(\Delta - 1)$-vertices. By Lemma 2.5(ii), there are two Δ-vertices in $N(x)$ which incident with all vertices of degree of $\geq \Delta - 2$. Note that each adjacent $(\Delta - 1)$-vertex sends at least $\frac{\Delta-1-q}{3}$ to x. Hence, $c'(x) \geq (5-q) + 2 \times (\Delta - q - 3 \times \frac{(q-\lfloor q \rfloor)}{3}) + 3 \times \frac{q-\lfloor q \rfloor}{3} > 0$ if $\delta_1(x) = \Delta - 1$, $\Delta = 8, 9$.

If $\delta_1(x) = \Delta$, by Lemma 2.5,

$$c'(x) \geq \begin{cases} -1.5 + min\{4 \times 1.5, 3 \times \frac{1.5}{2} + \frac{1.5}{3}, 5 \times \frac{1.5}{3}\} > 0 & \text{if } \Delta = 8. \\ -2.2 + min\{4 \times 1.8, 3 \times \frac{1.8}{2} + \frac{1.8}{3}, 5 \times \frac{1.8}{3}\} > 0 & \text{if } \Delta = 9. \end{cases}$$

(VI) Claim that $c'(x) \geq 0$ if $d(x) = 6$.

If $\delta_1(x) = \Delta - 2$, x is adjacent to at least *three* Δ-vertices. Let w be $\delta_1(x)$-neighbor of x. By Lemma 2.7, we consider following three cases.

(a) $d(w) = \Delta - 2$ with $d_{\leq \Delta-3}(w) = 3$ and each of remain vertices $z \in N(x) \setminus \{w\}$ has $d_{\leq \Delta-2}(z) = 1$. So by (R3), x receives at least $3 \times \frac{0.5}{3} = 0.5$ if $\Delta = 8$, and x receives at least $3 \times 1.8 > 1.2$ if $\Delta = 9$. Hence, $c'(x) \geq 0$.

(b) $d(w) = \Delta - 2$ with $d_{\leq \Delta-2}(w) = 2$ and each of rest vertices $z \in N(x) \setminus \{w\}$ has $d_{\leq \Delta-2}(z) \leq 2$. Then x receives at least $3 \times \frac{0.5}{3} = 0.5$ if $\Delta = 8$, and $3 \times \frac{1.8}{2} > 1.2$ if $\Delta = 9$. Hence, $c'(x) \geq 0$.

(c) Each $(\Delta - 2)$-neighbor w of x has $d_{\leq \Delta-2}(w) = 1$ and each Δ-vertex $z \in N(x)$ has $d_{\leq \Delta-2}(z) \leq 3$. By Lemma 2.7 and (R4), x receives $3 \times \frac{0.5}{3} = 0.5$ if $\Delta = 8$, and x receives at least $3 \times \frac{1.8}{3} > 1.2$ if $\Delta = 9$. Hence, $c'(x) \geq 0$.

If $\delta_1(x) = \Delta - 1$ or Δ, then x is adjacent to either *two* Δ-vertices and *four* $(\Delta - 1)$-vertices, or at least *three* Δ-vertices. By VAL, Lemma 2.7, (R3) and (R4), we have

$$c'(x) \geq \begin{cases} -0.5 + min\{2(\frac{0.5}{3}) + 4(\frac{0.5}{4}), 3(0.5)\} > 0 & \text{if } \delta_1(x) = 7,\ \Delta = 8. \\ -0.5 + 6(\frac{0.5}{3}) > 0 & \text{if } \delta_1(x) = 8,\ \Delta = 8. \\ -1.2 + min\{2(1.8), 2(\frac{1.8}{5}) + 4(\frac{0.8}{2}), 2(\frac{1.8}{5}) + 4(0.8)\} > 0 & \text{if } \delta_1(x) = 8,\ \Delta = 9. \\ 5(\frac{1.8}{5}) > 0 & \text{if } \delta_1(x) = 9,\ \Delta = 9. \end{cases}$$

(VII) Claim $c'(x) \geq 0$ if $d(x) = 7$.

Note that if $\Delta = 9$, x sends noting out but receives charges. Since x is adjacent to at least three 9-vertices, then by ($R3$), x receives at least $3 \times \frac{0.2}{3} = 0.2$ from its adjacent 9-vertices. $c'(x) \geq 0$. Next, we consider $\Delta = 8$. So x may send some charges out. By (II), $\delta_1(x) \geq 4$. By VAL, (R3) and (R4), $c'(x) \geq -1 + (\delta_1(x) - 2) \times \frac{0.5}{\delta_1(x)-2} = 0$ where $\delta_1(x) = 4, 5, 6$ respectively and $\Delta = 8$. Therefore, $c'(x) \geq 0$ if $\delta_1(x) = 4, 5, 6$. Be aware that x sends nothing out if $\delta_1(x) \geq 7$.

(VIII) Claim that $c'(x) \geq 0$ if $d(x) = 8$.

If $\delta_1(x) = 3$, then by (II), we consider $\Delta = 8$ only. Either x is adjacent to *seven* ($\geq$ 7)-vertices and *one* 3-vertex, or is adjacent to *six* Δ-vertices and *two* ($\leq$ 6)-vertices. By (R3) and (R4), x sends at most $\max\{1.5, 2 \times \frac{1.5}{2}\}$ out. Hence, $c'(x) \geq 0$.

If $\delta_1(x) = 4, 5, 6$ or 7, by VAL, (R3) and (R4), x sends out at most

$$\begin{cases} 1.5 & \text{if } \delta_1(x) = 4, 5,\ \Delta = 8. \\ 3 \times \frac{0.5}{3} = 0.5 & \text{if } \delta_1(x) = 6,\ \Delta = 8. \\ 1.8 & \text{if } \delta_1(x) = 4, 5, 6,\ \Delta = 9. \\ 3 \times \frac{0.2}{3} = 0.2 & \text{if } \delta_1(x) = 7,\ \Delta = 9. \\ 0 & \text{if } \delta_1(x) = 7, 8,\ \Delta = 8 \text{ and } \delta_1(x) = 8,\ \Delta = 9. \end{cases}$$

Hence $c'(x) \geq 0$.

(IX) Claim that $c'(x) \geq 0$ if $d(x) = 9$.

Be aware that $\Delta = 9$ only. If $3 \leq \delta_1(x) \leq 7$, there are at least $(\Delta - \delta_1(x) + 1)$ Δ-vertices in $N(x)$ by VAL. Let n^* =number of ($\leq$ 6)-vertices in $N(x)$, n_7 =number of 7-vertices in $N(x)$. By (R3) and (R4), x sends out at most

$$\begin{cases} max_{n_7 \leq 3}\{n^* \times \frac{1.8 - n_7 \frac{0.2}{3}}{n^*}\} = 1.8 & \text{if } \delta_1(x) = 3, 4, 5, 6, 7,\ \Delta = 9. \\ 0 & \text{if } \delta_1(x) = 8,\ \Delta = 9. \end{cases}$$

Hence $c'(x) \geq 0$.

From (I)-(IX), $c'(x) \geq 0$ and therefore, $\sum_{x \in V(G)} c'(x) \geq 0$. Since the discharge rules only move charge around and do not change the sum, we have $0 \leq \sum_{x \in V(G)} c'(x) = \sum_{x \in V(G)} c(x) < 0$. This contradiction completes the proof.

4 Class one graphs with $c_S = -1, -2$.

Theorem 4.1. *Let G be a simple graph that is embeddable in a surface S of characteristic $c_S = -1$, or -2, then G is class one if $\Delta \geq 8$, or 9 respectively.*

Before we proceed our proof of the Theorem, we need following results on critical graphs with small orders.

Lemma 4.2. *(Beineke and Fiorini [1],Brinkmann and Steffen [2, 3, 4])*
(i) There are no critical graphs of even order up to 14;
(ii) there are only two critical graphs of order 11, both of which are 3-critical;
(iii) Petersen graph minus a vertex is the only non-trivial critical graph on up to 10 vertices, which is 3-critical;
(IV) There are only three critical graphs of order 13, which are 3-critical.

Proof of Theorem 4.1. By Theorem 3.1 and Theorem 1.2, we only need to prove it when $\Delta = 8, 9$ respectively. Let V and F be vertex set and face set of G respectively. Suppose to the contrary, let G be the smallest counterexample with respect to edges. Then G is Δ-critical where $\Delta = 8, 9$ respectively. By Eulor's Formula, we have

$$\begin{cases} \sum_{x \in V}(d(x) - 6) + \sum_{f \in F}(d(f) - 3) = 6 & \text{if } c_S = -1,\ \Delta = 8. \\ \sum_{x \in V}(d(x) - 6) + \sum_{f \in F}(d(f) - 3) = 12 & \text{if } c_S = -2,\ \Delta = 9. \end{cases}$$

By Theorem 3.1, we have

$$\begin{cases} 0.5 \times |V| \leq 6 & \text{if } c_S = -1,\ \Delta = 8. \\ 1.2 \times |V| \leq 12 & \text{if } c_S = -2,\ \Delta = 9. \end{cases}$$

Hence, $|V| \leq 12$ or $|V| \leq 10$ for $\Delta = 8$ or 9 respectively. By Lemma 4.2, we have contradictons.

Remark: The theorem 4.1 was proved in [8]. But the new lower bounds in this paper imply the results in [8].

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