Hamilton – Connectivity of Claw – Free Graphs with Bounded Dilworth Numbers

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Abstract

Let u and v be two vertices in a graph G. We say vertex u dominates vertex v if $N(v) \subseteq N(u) \cup \{u\}$. If u dominates v or v dominates u, then u and v are comparable. The Dilworth number of a graph G, denoted $Dil(G)$, is the largest number of pairwise incomparable vertices in the graph G. A graph G is called claw – free if G has no induced subgraph isomorphic to $K_{1,3}$. It is shown that if G is a k ($k \geq 3$) – connected claw – free graph with $Dil(G) \leq 2k - 5$, then G is Hamilton – connected and a Hamilton path between every two vertices in G can be found in polynomial time.

Keywords: Hamilton – Connectivity, Claw – Free Graphs, Dilworth Numbers.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. For terminology, notation and concepts not defined here see [2]. If v is a vertex in a graph, the closed neighbor $N[v]$ of v is defined as $N(v) \cup \{ v \}$. If $S \subseteq V(G)$, then $N(S)$ denotes the neighbors of S, that is, the set of all vertices in G adjacent to at least one vertex in S. A path that contains every vertex of a graph G is called a Hamilton path of G. A cycle that contains every vertex of a graph G is called a Hamilton cycle of

*This work is partially supported by the research and grant support program at University of South Carolina Aiken.

G. A graph G is called Hamiltonian if it has a Hamilton cycle. A graph G is called Hamilton – connected if every two vertices in G are connected by a Hamilton path. A graph G is called claw – free if G has no induced subgraph isomorphic to $K_{1,3}$.

The definition of the Dilworth number of a graph can be found in [5] (also see [3]). Let u and v be two vertices in a graph G. We say vertex u dominates vertex v if $N(v) \subseteq N[u]$. If u dominates v or v dominates u, then u and v are comparable. The Dilworth number of a graph G, denoted $Dil(G)$, is the largest number of pairwise incomparable vertices in the graph G.

Recently, Li [7] obtained the following result on the Hamiltonicity of claw – free graphs with bounded Dilworth numbers: Let G be a k $(k \geq 2)$ – connected claw – free graph. If $Dil(G) \leq 2k - 1$, then G is Hamiltonian and a Hamiltonian cycle in G can be found in polynomial time.

This paper deals with the Hamilton – connectivity of claw – free graphs with bounded Dilworth numbers. In particular, we prove the following theorem.

Theorem 1. *Let G be a k $(k \geq 3)$ – connected claw – free graph. If $Dil(G) \leq 2k - 5$, then G is Hamilton – connected and a Hamiltonian path between every two vertices in G can be found in polynomial time.*

Notice that Hu, Tian, and Wei [6] proved that every 8 – connected claw – free graph is Hamilton – connected. So if we don't consider finding a Hamiltonian path between every two vertices, the condition $Dil(G) \leq 2k - 5$ can be dropped in Theorem 1 when $k \geq 8$.

Recall that Bertossi [1] proved that determining if line graphs have Hamilton paths is NP – complete. Therefore finding Hamilton paths in line graphs is a hard problem. It is well known that every line graph is claw – free. Thus finding Hamilton paths in claw – free graphs is hard. Hence finding Hamilton paths between all pairs of two vertices in claw – free graphs is also hard. So if we currently want to design polynomial time algorithms for finding Hamilton paths between all pairs of two vertices in claw – free graphs, we need extra constraints on this family of graphs. Such possible constraints are presented in Theorem 1 above.

We need the following additional notations in the reminder of this paper. If P is a path of G, let $\overrightarrow{P}$ denote the path P with a given orientation. If vertices u, v are on P and u precedes v along the direction of P, then we use $\overrightarrow{P}[u,v]$ to denote the consecutive vertices on P from u to v in the direction specified by $\overrightarrow{P}$. The same set of vertices, in reverse order, is denoted by $\overleftarrow{P}[v,u]$. Both $\overrightarrow{P}[u,v]$ and $\overleftarrow{P}[v,u]$ are considered as paths and vertex sets. $|\overrightarrow{P}[u,v]|$ and $|\overleftarrow{P}[v,u]|$ are used to denote the number of vertices in the paths $\overrightarrow{P}[u,v]$ and $\overleftarrow{P}[v,u]$, respectively. If u is on P, then the predecessor, successor, next predecessor, and next successor of u along the orientation of P are denoted by u^-, u^+, u^{--}, and u^{++}, respectively. If H is a connected component of a graph G and u and v are two vertices in H, let uHv denote a shortest path between u and v in H which can be found in polynomial time by using the Dijkstra's algorithm (see Chapter 24 in [4]).

2. The Proof of Theorem 1

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1 and u and v any two vertices in G. Using Dijkstra's algorithm in G, we can find a shortest path $P[u,v] := x_0x_1...x_r$ between u and v in G, where $u = x_0$ and $v = x_r$. If $r = |V(G)| - 1$, then it is finished. We now assume that $r \leq |V(G)| - 2$ and an orientation for $P[u,v]$ is defined as from u to v. Next we will show that a path between u and v of length at least $r+1$ in G can be constructed in polynomial time.

Firstly, if $r = 1$, we check if $(N(u) - \{v\}) \cap (N(v) - \{u\})$ is empty. If $(N(u) - \{v\}) \cap (N(v) - \{u\}) \neq \emptyset$, choose a vertex, say w, in $(N(u) - \{v\}) \cap (N(v) - \{u\})$, then uwv is a path between u and v of length at least $r+1$ in G. If $(N(u) - \{v\}) \cap (N(v) - \{u\}) = \emptyset$, choose a vertex $a \in (N(u) - \{v\})$ and a vertex $b \in (N(v) - \{u\})$. Since G is k $(k \geq 3)$ – connected, $G[V(G) - \{u,v\}]$ is connected. Applying Dijkstra's algorithm in $G[V(G) - \{u,v\}]$, we can find a shortest path, say $P[a,b]$, between a and b in $G[V(G) - \{u,v\}]$. Then $uP[a,b]v$ is a path between u and v of length at least $r+1$ in G.

Now consider the case that $r \geq 2$, Using depth – first search algorithm [9] in the graph $G[V(G) - V(P[u,v])]$, we can find a connected component, say H, in $G[V(G) - V(P[u,v])]$. More details on applying depth – first search algorithm to find a connected component in a graph can be found in Algorithm 8.3 on Page 330 in [8]. This step can be completed in $O(|V(G)| + |V(E)|)$ time.

Find all the neighbors of $V(H)$ on $V(P[u,v])$. we assume that $N(V(H)) \cap V(P[u,v]) = \{a_1, a_2, ..., a_l\}$ such that $h_ia_i \in E$, where $h_i \in V(H)$ for each i, $1 \leq i \leq l$, and a_1, a_2, ..., a_l are labeled in the order of the orientation of $P[u,v]$. Notice that if we apply Dijkstra's algorithm on H, we can find a shortest path between every two vertices h_i and h_j, where $1 \leq i \neq j \leq l$.

Since G is k ($k \geq 3$) – connected, $l \geq k \geq 3$. If $r = 2$ or 3, we can easily construct a path between u and v of length at least $r+1$ in G. Now we assume that $l \geq 4$. For each i, $2 \leq i \leq l-1$, if $h_i a_i^- \in E$ or $h_i a_i^+ \in E$, we can easily construct a path between u and v of length at least $r+1$ in G. Now we assume that $h_i a_i^- \notin E$ and $h_i a_i^+ \notin E$. Since G is claw – free, $a_i^- a_i^+ \in E$. Similarly, if $a_1 \neq x_0$, we can construct a path between u and v of length at least $r+1$ in G or $a_1^- a_1^+ \in E$ and if $a_l \neq x_r$, we can construct a path between u and v of length at least $r+1$ in G or $a_l^- a_l^+ \in E$.

Clearly, If $|\overrightarrow{P}[a_i, a_{i+1}]| \leq 4$ for some i, $2 \leq i \leq l-2$, we can easily construct a path between u and v of length at least $r+1$ in G. Moreover, if $|\overrightarrow{P}[a_i, a_{i+1}]| \leq 3$, where $i = 0$ or $l-1$, we can also easily construct a path between u and v of length at least $r+1$ in G. From now on, we assume that $|\overrightarrow{P}[a_i, a_{i+1}]| \geq 5$ for each i, $2 \leq i \leq l-2$, and $|\overrightarrow{P}[a_i, a_{i+1}]| \geq 4$, where $i = 0$ or $l-1$.

Notice first that if $a_i a_{i+1}^- \in E$ for some i, $2 \leq i \leq l-1$, we can construct a path

$$\overrightarrow{P}[u, a_i^-]\overrightarrow{P}[a_i^+, a_{i+1}^-]a_i h_i H h_{i+1} \overrightarrow{P}[a_{i+1}, v]$$

between u and v of length at least $r+1$ in G. Hence we can assume that $a_i a_{i+1}^- \notin E$ for each i, $2 \leq i \leq l-1$. Traverse along the segment of $\overleftarrow{P}[a_{i+1}^-, a_i^+]$ on P and find the first vertex b_i such that $b_i \in N(a_i)$ for each i, $2 \leq i \leq l-1$. Then $|\overrightarrow{P}[b_i^+, a_{i+1}^-]| \geq 1$ for each i, $2 \leq i \leq l-1$.

Since $b_i^+ \notin N(a_i)$, $b_i^+ \in N(b_i)$ and $h_i \notin N(b_i)$ (since $N(h_i) \cap V(P[u,v]) \subseteq N(V(H)) \cap V(P[u,v])$), $h_i \in N(a_i)$ for each i, $1 \leq i \leq l-1$, a_i and b_i are incomparable for each i, $2 \leq i \leq l-1$.

Now consider the set $D = \{a_2, b_2, a_3, b_3, ..., a_{l-1}, b_{l-1}\}$. Since $|D| = 2(l-2) \geq 2(k-2)$ and $Dil(G) \leq 2k-5$, there exist two distinct comparable vertices in D. Find two distinct vertices x and y in D such that x and y are comparable. Clearly, this step can be completed in polynomial time. Since a_i and b_i are incomparable for each i, $2 \leq i \leq l-1$, we have following possible cases.

Case 1. $x = a_i$ and $y = a_j$ for some i and j, $2 \leq i \neq j \leq l-1$.

If $1 \leq i < j \leq l-1$, we have $a_j^+ \in N(a_j) \subseteq N[a_i]$ or $a_i^+ \in N(a_i) \subseteq N[a_j]$ and G has a path

$$\overrightarrow{P}[u, a_i^-]\overrightarrow{P}[a_i^+, a_j]h_j H h_i a_i \overrightarrow{P}[a_j^+, v]$$

between u and v of length at least $r+1$ in G or a path

$$\overrightarrow{P}[u, a_i]h_iHh_ja_j\overrightarrow{P}[a_i^+, a_j^-]\overrightarrow{P}[a_j^+, v]$$

between u and v of length at least $r+1$ in G. If $1 \le j < i \le l-1$, we can smilarly construct a path between u and v of length at least $r+1$ in G.

Case 2. $x = a_i$ and $y = b_j$ for some i and j, $2 \le i \ne j \le l-1$.

If $1 \le i < j \le l-1$, since $h_i \in N(a_i)$ and $h_i \notin N[b_j]$ (since $N(h_i) \cap V(P[u,v]) \subseteq N(V(H)) \cap V(P[u,v])$), we have $b_j^+ \in N(b_j) \subseteq N[a_i]$ and can construct a path

$$\overrightarrow{P}[u, a_i^-]\overrightarrow{P}[a_i^+, a_j^-]\overrightarrow{P}[a_j^+, b_j]a_jh_jHh_ia_i\overrightarrow{P}[b_j^+, v]$$

between u and v of length at least $r+1$ in G. If $1 \le j < i \le l-1$, we can smilarly construct a path between u and v of length at least $r+1$ in G.

Case 3. $x = b_i$ and $y = b_j$ for some i and j, $2 \le i \ne j \le l-1$.

Then $N(b_i) \subseteq N[b_j]$ or $N(b_j) \subseteq N[b_i]$. We first consider the case that $N(b_i) \subseteq N[b_j]$. Then $b_i^+ \in N(b_i) \subseteq N[b_j]$.

Now consider the subcase that $1 \le i < j \le l-1$. Since $G[b_j, b_i^+, a_j, b_j^+]$ is not isomorphic to a claw and $a_jb_j^+ \notin E$, $b_i^+a_j \in E$ or $b_i^+b_j^+ \in E$. If $b_i^+a_j \in E$, we can construct a path

$$\overrightarrow{P}[u, a_i^-]\overrightarrow{P}[a_i^+, b_i]a_ih_iHh_ja_j\overrightarrow{P}[b_i^+, a_j^-]\overrightarrow{P}[a_j^+, v]$$

between u and v of lenght at least $r+1$ in G. If $b_i^+b_j^+ \in E$, we can construct a path

$$\overrightarrow{P}[u, a_i^-]\overrightarrow{P}[a_i^+, b_i]a_ih_iHh_ja_j\overleftarrow{P}[b_j, a_j^+]\overleftarrow{P}[a_j^-, b_i^+]\overrightarrow{P}[b_j^+, v]$$

between u and v of lenght at least $r+1$ in G. For the subcase that $1 \le j < i \le l-1$, we can smilarly construct a path between u and v of length at least $r+1$ in G.

Similarly, we can construct a path between u and v of length at least $r+1$ in G if $N(b_j) \subseteq N[b_i]$ is true.

Obviously, the above algorithmic procedures of enlarging the path $P[u,v]$ between u and v of length r in G to a path between u and v of length at least $r+1$ in G can be fulfilled in polynomial time.

Apply the similar procedures as above to the newly constructed path between u and v of length at least $r+1$ in G, we can construct a path between u and v of length at least $r+2$ in G. Repeat this process, we can construct a Hamilton path between u and v in G. Notice that we can repeat the processes at most $|V(G)|$ times, therefore we can construct a Hamilton path between u and v in G in polynomial time. QED

3. Acknowledgements

The author would like to thank the referee for his or her suggestions and comments.

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