Generalization of the Erdős-Gallai Inequality

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Abstract

P. Erdős and T. Gallai gave necessary and sufficient conditions for a sequence of non-negative integers to be graphic. Here,their result is generalized to multigraphs with a specified multiplicity. This both generalizes and provides a new proof of a result in the literature by Chungphaisan [2].

1 Introduction

A sequence π of non-negative integers is said to be *graphic* if it can be realized by a simple graph with degree sequence π. Similarly, π is said to be *multigraphic* if it can be realized by a multigraph where multiple edges between two vertices are allowed. In 1960, P. Erdős and T. Gallai gave necessary and sufficient conditions for a sequence of non-negative integers to be graphic, proving the following theorem.

Theorem 1.1 ([1]) *A sequence $\pi = (d_1, d_2, \ldots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \cdots \geq d_p$ is graphic if and only if*

(1) $\sum_{i=1}^{p} d_i$ is even, and

(2) $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} min\{d_i, k\}$, for every k, $1 \leq k \leq p$.

Several other proofs have been given to this theorem since then. In 1969, F. Harary [4] gave a lengthy direct proof, while in 1973, C. Berge [6] gave a shorter proof using network flows. Later in 1986, S.A. Choudum [7] gave a simple direct proof by induction on the sum of the sequence. Here we provide another proof in a more general setting that uses Tutte's f-factor theorem.

A multigraph is said to have *multiplicity* (or *index*) λ, or said to be λ-*multigraphic* if the maximum number of edges joining each pair of vertices is λ. In 1974, Chungphaisan gave necessary and sufficient conditions for sequences to be at most λ-multigraphic, proving the following result.

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Theorem 1.2 ([2]) *A sequence* $\pi = (d_1, d_2, \ldots, d_p)$ *of non-negative integers with* $d_1 \geq d_2 \geq \cdots \geq d_p$ *is multigraphic with multiplicity at most* λ *if and only if*

(1) $\sum_{i=1}^{p} d_i$ *is even, and*

(2) $\sum_{i=1}^{k} d_i \leq \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\}$, *for every k,* $1 \leq k \leq p$.

But this theorem does not guarantee to have a realization of the sequence with a graph with multiplicity exactly λ. Multiplicity of the realization graph may be λ or less than λ.

In this paper, this result will be modified to generalize it for multigraphs with exact specified multiplicity. Also, another proof for Theorem 1.2 using Tutte's f-factor theorem is given.

Before stating Tutte's f-factor Theorem, let's give some necessary definitions. To assist the reader, Tutte's notation [9] is adopted throughout this paper.

The *valency* of a vertex x in a graph G is the degree of x in G and is denoted by val(G, x). If f is a function from the vertex set $V(G)$ into the set of integers, define another function f' by the rule $f'(x) = \text{val}(G, x) - f(x)$ for each vertex x of G. Given such a function f, an *f-factor* is a spanning subgraph F of G satisfying val$(F, x) = f(x)$ for each vertex x of G.

A *G-triple* is an ordered triple (S, T, U) where $\{S, T, U\}$ partitions $V(G)$. For any subset S of $V(G)$, $f(S) = \sum_{v \in S} f(v)$. For any disjoint subsets S and T of $V(G)$, $m(S, T)$ denotes the number of edges of G joining vertices in S to vertices in T.

If $B = (S, T, U)$ is a G-triple and C is any component of $G[U]$ in G, then define

$$J(B, f, C) = f(C) + m(V(C), T).$$

C is an ODD component if $J(B, f, C)$ is an odd integer. Note that capital letters are used to distinguish it from "odd component", the usual terminology used to represent a component with an odd number of vertices. The number of ODD components of U in G with respect to B and f is denoted by $h(B, f)$. The *deficiency* $\delta(B, f)$ of the G-triple $B = (S, T, U)$ with respect to f is defined as follows:

$$\delta(B, f) = h(B, f) - f(S) - f'(T) + m(S, T).$$

An *f-barrier* of G is a G-triple $B = (S, T, U)$ such that $\delta(B, f) > 0$.

I can now state Tutte's f-factor Theorem.

Theorem 1.3 ([9]) *Given G and f, exactly one of the following statements is true:*

(1) G has an f-factor.

(2) G has an f-barrier.

In other words, if we let f be a vertex-function of a graph G, then G has an f-factor or there exists a G-triple $B = (S, T, U)$ of G with $\delta(B, f) > 0$, but not both.

Now let's give one last definition. A sequence $\pi = (d_1, d_2, \ldots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \cdots \geq d_p$ is *λ-admissible* if they satisfy the conditions of Theorem 1.2, namely,

(1) $\sum_{i=1}^{p} d_i$ is even, and

(2) $\sum_{i=1}^{k} d_i \leq \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\}$, for every k, $1 \leq k \leq p$.

2 Preliminary Results

Lemma 2.1 *A sequence $\pi = (d_1, d_2, \ldots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \cdots \geq d_p$ and even sum is λ-admissible if and only if for all G-triples (S, T, U) we have*

$$m(S,T) \leq d(S) + \lambda(p-1)|T| - d(T)$$

where $G = \lambda K_p$ and $d(S) = \sum_{v_i \in S} d_i$ for $d(v_i) = d_i$.

Proof. Assume π is λ-admissible. Since the vertices in T are not ordered according to their degrees, we have $\sum_{v_j \in T} d_j \leq \sum_{i=1}^{|T|} d_i$, hence

$$\sum_{v_j \in T} d_j \leq \sum_{i=1}^{|T|} d_i \leq \lambda|T|(|T|-1) + \sum_{i=|T|+1}^{p} min\{d_i, \lambda|T|\}$$

by λ-admissibility. It is also clear that

$$\begin{aligned} \lambda|T|(|T|-1) &= \lambda(p-1)|T| - \lambda|S||T| - \lambda|T||U| \\ &= \lambda(p-1)|T| - m(S,T) - \lambda|T||U| \end{aligned}$$

We also have

$$\begin{aligned} \sum_{i=|T|+1}^{p} min\{d_i, \lambda|T|\} &\leq d(S) + \sum_{v_j \in U} min\{d_j, \lambda|T|\} \\ &\leq d(S) + \lambda|U||T| \end{aligned}$$

Hence, we get

$$\begin{aligned} \sum_{j \in T} d_j \leq \sum_{i=1}^{|T|} d_i &\leq \lambda|T|(|T|-1) + \sum_{i=|T|+1}^{p} min\{d_i, \lambda|T|\} \\ &\leq \lambda(p-1)|T| - m(S,T) - \lambda|T||U| + d(S) + \lambda|U||T| \\ &= \lambda(p-1)|T| - m(S,T) + d(S) \end{aligned}$$

proving $m(S,T) \leq d(S) + \lambda(p-1)|T| - d(T)$.

Now assume $m(S,T) \leq d(S) + \lambda(p-1)|T| - d(T)$ holds for all G-triples (S,T,U). Now let $T = \{v_i : 1 \leq i \leq k\}$, $S = \{v_i : d_i \leq \lambda k$ and $i > k\}$, and $U = \{v_i : d_i > \lambda k$ and $i > k\}$.

Then we have

$$\begin{aligned} m(S,T) &\leq d(S) + \lambda(p-1)|T| - d(T) \\ \lambda|S||T| &\leq d(S) + \lambda(|T|-1)|T| + \lambda|S||T| + \lambda|U||T| - d(T) \end{aligned}$$

implying $d(T) \le \lambda k(k-1) + d(S) + \lambda k|U|$. By the choice of S and U, $d(S) = \sum_{v_i \in S} min\{d_i, \lambda k\}$, and $\lambda|U||T| = \sum_{v_i \in U} min\{d_i, \lambda k\}$. So,

$$\begin{aligned} d(T) &\le \lambda k(k-1) + \textstyle\sum_{v_i \in S} min\{d_i, \lambda k\} + \sum_{v_i \in U} min\{d_i, \lambda k\} \\ &= \lambda k(k-1) + \textstyle\sum_{i=k+1}^{p} min\{d_i, \lambda k\} \end{aligned}$$

showing $\sum_{i=1}^{t} d_i \le \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\}$. Thus π is λ-admissible. □

Before stating the main theorem, let's prove two results which are analogous to Havel - Hakimi ([5], [3]) conditions.

Lemma 2.2 *If $\pi = (d_1, d_2, \ldots, d_p)$ with $d_1 \ge d_2 \ge \cdots \ge d_p$ is a degree sequence of a λ-multigraph, we can find a realization of π with the largest multiplicity (which is λ) occurring between the two vertices of the largest degree.*

Proof. Let G be a λ-multigraph with degree sequence $\pi = (d_1, d_2, \ldots, d_p)$, v_1 and v_2 be the vertices of the largest degrees, and let $m(x, y)$ represent the number of edges between the vertices x and y.

Case 1: $m(v_1, v) \ne \lambda$ and $m(v_2, v) \ne \lambda$ for any $v \in V(G)$. Now let v_i and v_j be two vertices in $V(G)$ such that $d_G(v_i) = d_i$, $d_G(v_j) = d_j$, and $m(v_i, v_j) = \lambda$. The first step of the algorithm will result in a graph with $m(v_2, v_i) = \lambda$, and after the second step we'll have a new graph with $m(v_1, v_2) = \lambda$ that still has π as the degree sequence.

Step 1: Let $g(v) = m(v_2, v) - m(v_j, v)$. Obviously, $g(v_i) < 0$ since $m(v_2, v_i) < \lambda$. Also, $g(v_j) + g(v_2) = 0$ and $\sum_{v \in V(G)} g(v) = d_2 - d_j \ge 0$. So,

$$\begin{aligned} \textstyle\sum_{v \in V(G) - \{v_2, v_i, v_j\}} g(v) &= \left(\textstyle\sum_{v \in V(G)} g(v)\right) - g(v_i) - g(v_j) - g(v_2) \\ &= \left(\textstyle\sum_{v \in V(G)} g(v)\right) - g(v_i) \\ &\ge -g(v_i) \end{aligned}$$

Hence, for each $v \in V(G) - \{v_2, v_i, v_j\}$, we can choose an integer $t(v)$ such that $0 \le t(v) \le g(v)$ and $\sum_{v \in V(G) - \{v_2, v_i, v_j\}} t(v) = -g(v_i)$. Now, for each $v \in V(G) - \{v_2, v_i, v_j\}$, we can remove $t(v)$ edges joining v to v_2 and add $t(v)$ edges between v to v_j, then remove $t(v)$ edges joining v_i and v_j and add $t(v)$ edges between v_i to v_2. So, $\sum_{v \in V(G) - \{v_2, v_i, v_j\}} t(v) = -g(v_i)$ edges are added between v_2 and v_i in total. This produces a new graph with the same degree sequence where $m(v_2, v_i) = \lambda$.

Step 2: Replace v_1, v_2, and v_i for v_2, v_i, and v_j respectively in the above algorithm to get $m(v_1, v_2) = \lambda$.

Case 2: Assume that $m(v_1, v_k) = \lambda$ or $m(v_2, v_k) = \lambda$ for some $v_k \in V(G)$. Now we can directly apply Step 2 for $\{v_1, v_2, v_k\}$. □

The following is an obvious corollary of the previous lemma. If $\pi = (d_1, d_2, \ldots, d_p)$ is a sequence of non-negative integers with $d_1 \ge d_2 \ge \cdots \ge d_p$, let $\pi' = (d'_1, d'_2, d'_3 \ldots, d'_p)$ be a rearrangement (if necessary) of $\{d_1 - \lambda, d_2 - \lambda, d_3, \ldots, d_p\}$ so that $d'_1 \ge d'_2 \ge d'_3 \ge \cdots \ge d'_p$.

Corollary 2.3 *If a sequence $\pi = (d_1, d_2, \ldots, d_p)$ of non-negative integers with $d_1 \ge d_2 \ge \cdots \ge d_p$ is λ multigraphic, then π' is at most λ-multigraphic.*

Proof. Assume that $\pi = (d_1, d_2, \dots, d_p)$ with $d_1 \geq d_2 \geq \dots \geq d_p$ is multigraphic with exact multiplicity λ. By the Lemma 2.2, there exists a multigraph G such that $m_G(v_1, v_2) = \lambda$. By removing these λ edges, we have a new graph with degree sequence π'. Since the new graph may or may not have multiplicity λ. π' is at most λ-multigraphic.

□

Note that the converse is not necessarily true unless π is λ-admissible.

3 Main Theorem

Now, let me state the main theorem which is a generalization of the Erdős-Gallai Theorem. This result is not only of interest in its own right, but has also been very useful in the study of maximal sets of Hamilton cycles in multipartite graphs [8].

Theorem 3.1 *A sequence $\pi = (d_1, d_2, \dots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \dots \geq d_p$ is multigraphic with exact multiplicity λ if and only if*

(i) π is λ-admissible, and

(ii) $\pi' = (d'_1, d'_2, d'_3 \dots, d'_p)$ is λ-admissible where π' is a rearrangement (if necessary) of $\{d_1 - \lambda, d_2 - \lambda, d_3, \dots, d_p\}$ so that $d'_1 \geq d'_2 \geq d'_3 \geq \dots \geq d'_p$.

Proof. First assume that the sequence $\pi = (d_1, d_2, \dots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \dots \geq d_p$ is multigraphic with multiplicity λ, and let G be a graph realizing this degree sequence. Then for any set Q of k vertices in G, the total degree of the vertices in Q is equal to the twice the number of edges in Q plus the number of edges between the sets Q and $G - Q$. The maximum number of edges in Q is $\lambda\binom{k}{2}$ and the maximum number of edges between Q and $G - Q$ is $\sum_{i=k+1}^{p} min\{d_i, \lambda k\}$. Hence, $\sum_{i=1}^{k} d_i \leq \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\}$ follows for every k, $1 \leq k \leq p$. Clearly, $\sum_{i=1}^{p} d_i$ is even since it counts each edge twice. So, π is λ-admissible. By Corollary 2.3, π' is at most λ-multigraphic. By the above reasoning and since $\sum_{i=1}^{p} d'_i = \sum_{i=1}^{p} d_i - 2\lambda$ is even, π' is λ-admissible as well.

Now, assume that π and π' are λ-admissible. We need to show that if $G = \lambda K_p$, $H = G - G[u, v]$ has an f-factor with $f(v_i) = d'_i$ for all $v_i \in H$ where u and v are vertices with $f(u) = d_1 - \lambda$ and $f(v) = d_2 - \lambda$. We will use Tutte's f-factor Theorem and show that $\delta(B, f) \leq 0$ for all $B = (S, T, U)$, where $\{S, T, U\}$ is a partition of $V(H)$. By Lemma 2.1, for any G-triples (S, T, U) we have

$$m_G(S, T) \leq f(S) + \lambda(p-1)|T| - f(T) \ (i)$$

$$m_G(S, T) \leq f_0(S) + \lambda(p-1)|T| - f_0(T), \ (ii)$$

where f is described as above and $f_0(v_i) = d_i$ if $v_i \notin \{u, v\}$ and $f_0(u) = d_1$, $f_0(v) = d_2$.

Let $m_H(S, T) = m(S, T)$. Note that when

$$m(S, T) \leq f(S) + f'(T) \qquad (\star)$$

holds, we have $\delta = h(B,f) - f(S) - f'(T) + m(S,T) \leq h(B,f)$. When $G[U]$ has at most one ODD component, it means that $\delta \leq 0$, since $\delta \equiv \sum_{i=1}^{p} d_i \pmod 2$ by [9], and $\sum_{i=1}^{p} d'_i$ is even. So, showing (⋆⋆) is enough to show $\delta \leq 0$ when $U \neq K$ for $K = \{u, v\}$.

Also, $m(S,T) = m_G(S,T) - a\lambda$, where $a = 1$ if $|S \cap K| = |T \cap K| = 1$, and $a = 0$ otherwise.

Case 1: $a = 0$ and $U \neq K$.

If $|S \cap K| = 0$, we need to show

$$\lambda|S||T| \leq f(S) + \lambda(p-1)|T| - \lambda|T \cap K| - f(T).$$

By (ii), we have

$$\begin{aligned} \lambda|S||T| &\leq f_0(S) + \lambda(p-1)|T| - f_0(T) \\ &= f(S) + \lambda(p-1)|T| - \lambda|T \cap K| - f_0(T) + \lambda|T \cap K| \\ &= f(S) + \lambda(p-1)|T| - \lambda|T \cap K| - f(T). \end{aligned}$$

If $|T \cap K| = 0$, we need to show $\lambda|S||T| \leq f(S) + \lambda(p-1)|T| - f(T)$, which actually follows directly from (i).

Thus we have $m(S,T) \leq f(S) + f'(T)$ and $\delta \leq 0$ by (⋆) when $a = 0$.

Case 2: $a = 1$.

We need to show $\lambda|S||T| - \lambda \leq f(S) + \lambda(p-1)|T| - \lambda - f(T)$ which is same as $\lambda|S||T| \leq f(S) + \lambda(p-1)|T| - f(T)$, which is direct from (i).

Case 3: $U = K$.

In this case, $G[U]$ has two components.

By (i), $m(S,T) \leq f(S) + \lambda(p-1)|T| - f(T) = f(S) + f'(T)$. So,

$$\delta = h(B,f) - f(S) - f'(T) + m(S,T) \leq h(B,f).$$

Since $\delta \equiv \sum_{i=1}^{p} d_i \pmod 2$, by [9] either $\delta = 0$ or $\delta = 2$, and $\delta = 2$ when $G[U]$ has two ODD components and $m(S,T) = f(S) + f'(T)$.

Assume $h = 2$, and $m(S_0,T_0) = f(S_0) + f'(T_0)$ for some (S_0,T_0) that partitions $V(H) - K$. Since $h = 2$, both $J(B,f,u) = f(u) + m(u,T_0) = f(u) + \lambda|T_0|$, and $f(v) + \lambda|T_0|$ are odd. This implies $f(u) \equiv f(v) \pmod 2$. Since $a = 0$ in this case, $m(S_0,T_0) = f(S_0) + f'(T_0)$ implies

$$m(S_0,T_0) = f(S_0) + \lambda(p-1)|T_0| - f(T_0) \qquad (\star\star).$$

Now let $T = T_0 \cup \{u\}$, $S = S_0$, and $U = \{v\}$. Since $a = 0$, in Case 1, it is shown that

$$\begin{aligned} m(S,T) &\leq f(S) + f'(T) \\ &= f(S) + \lambda(p-1)|T| - \lambda - f(T) \end{aligned}$$

which implies $\lambda|S_0||T_0+1| \leq f(S_0)+\lambda(p-1)|T_0+1|-\lambda-f(T_0)-f(u)$. By $(\star\star)$, we have $\lambda|S_0||T_0| = f(S_0)+\lambda(p-1)|T_0|-f(T_0)$.Thus,

$$\begin{aligned}
\lambda|S_0||T_0+1| &\leq f(S_0)+\lambda(p-1)|T_0+1|-\lambda \\
&\quad -f(T_0)-f(u) \\
f(S_0)+\lambda(p-1)|T_0|-f(T_0)+\lambda|S_0| &\leq f(S_0)+\lambda(p-1)|T_0+1|-\lambda \\
&\quad -f(T_0)-f(u) \\
f(u) &\leq \lambda(p-|S_0|-2) \\
&= \lambda|T_0|
\end{aligned}$$

Similarly we can show $f(v) \leq \lambda|T_0|$ which implies $f(u)+f(v) = f(K) \leq 2\lambda|T_0|$.

But by $(\star\star)$, we can also write $f(T_0)$ as $f(T_0) = \lambda|T_0|(|T_0-1|)+f(S_0)+2\lambda|T_0|$, which implies $f(K) \geq 2\lambda|T_0|$. So, we must have $f(K) = 2\lambda|T_0|$ with $f(u) = f(v) = \lambda|T_0|$.

So, $J(B,f,u) = f(u)+m(u,T_0) = \lambda|T_0|+\lambda|T_0| = 2\lambda|T_0|$, and $J(B,f,v) = 2\lambda|T_0|$ are both even, implying $h=0$ when $m(S_0,T_0) = f(S_0)+f'(T_0)$.

Thus, when $U = K$ and $h=2$, there is no G-triple (S_0,T_0,U) such that $m(S_0,T_0) = f(S_0)+f'(T_0)$. Hence $\delta \leq 0$ in this last case as well.

In each case, it is shown that $\delta \leq 0$. Therefore, for $G = \lambda K_p$, $H = G-G[u,v]$ has an f-factor where $f(v_i) = d_i'$ for all $v_i \in H$,and $f(u) = d_1-\lambda$, $f(v) = d_2-\lambda$ as required. Clearly, this f-factor is a multigraph with multiplicity at most λ that has π' as degree sequence. In this f-factor, there are no edges between the vertices u and v with $f(u) = d_1-\lambda$ and $f(v) = d_2-\lambda$. So, by adding λ edges between u and v, we obtain a new graph which accepts π as the degree sequence and has exact multiplicity λ. □

4 Concluding Remarks

If we assume only the first condition of Theorem 3.1, namely $\pi = (d_1,d_2,\ldots,d_p)$ is λ-admissible, we can similarly show that the graph $G = \lambda K_p$ has an f-factor with $f(v_i) = d_i$ as follows.

By Lemma 2.1, λ-admissibility implies $m(S,T) \leq f(S)+\lambda(n-1)|T|-f(T)$. And since $\lambda(p-1)|T|-f(T) = f'(T)$ in this case, we have

$$\delta = h(B,f)-f(S)-f'(T)+m(S,T) \leq h(B,f).$$

For $G = \lambda K_p$, $G[U]$ has only one component for any G-triple, so $\delta \leq 1$. But, by [9], $\delta \equiv \sum_{i=1}^{p} d_i \pmod 2$, implying $\delta \leq 0$.

This provides a very short new proof to Theorem 1.2; Chungphaisan's result, and also to the Erdős-Gallai Inequality when we let $\lambda = 1$.

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