Spectral radius of graphs with given diameter *

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Abstract

In this paper, we show that among all connected graphs of order n with diameter D, the graph G^* has maximal spectral radius, where G^* is obtained from $K_{n-D} \bigvee \overline{K_2}$ by attaching two paths of order l_1 and l_2 to the two vertices u, v in $\overline{K_2}$, respectively, and $l_1+l_2 = D-2$, $|l_1 - l_2| \leq 1$.

Key words: Graph; Diameter; Spectral radius
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1 Introduction

In this paper, we consider only simple connected graphs. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *adjacency matrix of* G is $A(G) = (a_{ij})$ where $a_{ij} = 1$ if two vertices i and j are adjacent in G and 0 otherwise. The characteristic polynomial of G is just $det(xI - A(G))$. The *eigenvalues* of G are the eigenvalues of its adjacency matrix $A(G)$. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G and denoted by $\rho(G)$. Since $A(G)$ is symmetric nonnegative and irreducible, from the well-known Perron-Frobenius theorem, there is a unique unit positive eigenvector corresponding to $\rho(G)$, we call this vector the *Perron vector.* Note that when edges are added to G, the spectral radius of G increases strictly. We refer the reader to [5] [6] for more details in spectral graph theory.

For two vertices u and v of a connected graph G, the *distance* between u and v, denoted by $d(u, v)$, is the length of a shortest path joining u and v in G. The *diameter* of G, denoted by $D(G)$ (or just D for short), is the maximum distance over all pairs of vertices in G. The diameter is one of the graph invariants that are not only theoretical interests but also

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have many practical applications such as in communication networks. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union $G_1 \cup G_2$ is defined to be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \bigvee G_2$ of G_1 and G_2 is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. Let $\overline{K_t}$ be the empty graph on t vertices. When we say *contracting an edge* $e = uv \in G$, we mean first to delete the edge then coincide the two vertices u, v. For other notations in graph theory, we follow [2].

Suppose G is a k-regular graph on n vertices, λ is the second largest eigenvalue (in absolute value) of G. Alon and Milman [1] showed that $D(G) < 2[\sqrt{2k/(k-\lambda)} \log_2 n]$. Chung [3] further improved this bound to get $D(G) \leq [\log(n-1)/\log(k/\lambda)]$. There are also many other bounds on $D(G)$ by using the so called "Laplacian eigenvalues" of G, see for example [4], [7], [8], [11]. But sharpness of these bounds is not determined. In [9], Fallat and Kirkland determined the tree of order n with maximal algebraic connectivity and given diameter.

In this paper, we study the set $\mathcal{G}$ of connected graphs of order n with given diameter D, and determine the extremal graph that has the maximal spectral radius.

2 Lemmas and results

If $D = 1$, then we can easily get that K_n has the maximal spectral radius, so in the following discussions, we always assume $D \geq 2$. First, we need some lemmas.

Lemma 2.1 *[12] Let u, v be two vertices of the connected graph G and d_v be the degree of v, suppose $v_1, v_2, \cdots, v_s \in N(v) \setminus N(u) (1 \leq s \leq d_v)$, where $v_1, v_2, \cdots, v_s$ are different from u, let $X = (x_1, x_2, \cdots, x_n)$ be the Perron vector of $A(G)$, where x_i corresponds to $v_i, (1 \leq i \leq n)$. Let H be the graph obtained from G by deleting the edges (v, v_i) and adding the edges $(u, v_i), 1 \leq i \leq s$. If $x_u \geq x_v$, then $\rho(G) < \rho(H)$.*

Now, we consider the graph G_{uv} obtained from the connected graph G by subdividing the edge uv, that is, by replacing uv with edges uw, vw where w is an additional vertex. We call the following two types of paths *internal paths* : (a) a sequence of vertices $v_0, v_1, \cdots v_{k+1} (k \geq 2)$ where $v_0, v_1, \cdots v_k$ are distinct, $v_{k+1} = v_0$ of degree at least 3, $d_{v_i} = 2$ for $i = 1, \cdots, k$, and v_{i-1} and v_i $(i = 1, \cdots, k+1)$ are adjacent. (b) A sequence of distinct vertices $v_0, v_1, \cdots v_{k+1} (k \geq 0)$ such that v_{i-1} and v_i $(i = 1, \cdots, k+1)$ are adjacent, $d_{v_0} \geq 3$, $d_{v_{k+1}} \geq 3$ and $d_{v_i} = 2$ whenever $1 \leq i \leq k$.

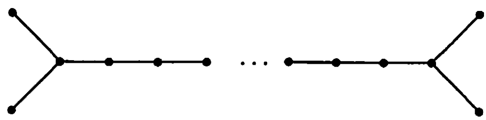

Fig 1. The tree W_n, $(n \geq 7)$

Lemma 2.2 *[6][10] If uv lies on an internal path of the connected graph G and $G \neq W_n$, then $\rho(G_{uv}) < \rho(G)$.*

Let u, v be two vertices in $\overline{K_2}$. Suppose $n - 2D + 1 > 0$, G_1 is obtained from $K_{n-2D+1} \bigvee \overline{K_2}$ by first adding an edge uv, then subdividing the edge uv by inserting $2D - 3$ vertices. G_2 is obtained from $K_{n-D} \bigvee \overline{K_2}$ by attaching two paths of order l_1 and l_2 at u and v respectively, $l_1 + l_2 = D - 2$. As shown in Fig.2. It is easy to see that G_1 and G_2 have diameters D.

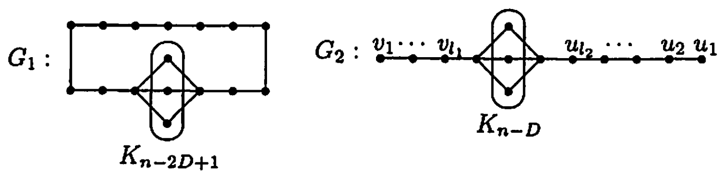

Fig 2. G_1 and G_2.

Lemma 2.3 *Let $G \in \mathcal{G}$ be a connected graph of order n with diameter $D \geq 2$. If G has the maximal spectral radius, then G is of the form like G_2 as in Fig 2.*

Proof. Let u and v be two vertices of G at distance D. We discuss in the following two cases.

If u and v lie in a cycle, then there exists a shortest cycle C containing u, v which has $2D$ or $2D + 1$ vertices. Let $U = N(C)$, the set of neighbors of C. Let $W = V - U - C$. Note that any vertex not in C has at most three consecutive neighbors on C, and note that for any two vertices in C, say c_1, c_2, $x_{c_1} \geq x_{c_2}$ or $x_{c_1} \leq x_{c_2}$ holds, where x_{c_1}, x_{c_2} correspond to vertices c_1, c_2, respectively, in the Perron vector. So by repeated use of Lemma 2.1, we can make all vertices in U be adjacent to at most three vertices in C, after adding some edges, we have, if C has $2D$ vertices, then $\rho(G) \leq \rho(G_1)$. If C has $2D + 1$ vertices, by repeating the above steps, and contracting an edge on the internal path, then adding a vertex in W, by Lemma 2.2, we also have $\rho(G) \leq \rho(G_1)$.

If u and v does not lie in a cycle C, then as above by Lemma 2.1, G must have the form like G_2.

For G_1, $\rho(G_1) < \Delta(G_1) = n - 2D + 2$. For G_2, since G_2 has an induced subgraph $K_{n-D} \bigvee \overline{K_2}$, so $\rho(G_2) \geq \delta(K_{n-D} \bigvee \overline{K_2}) = n - D$. Hence for $D \geq 2$, we have $\rho(G_2) > \rho(G_1)$. Thus we complete the proof. ■

Lemma 2.4 *[5] If H is a proper spanning subgraph of the graph G, then the characteristic polynomials satisfy*

$$P_G(x) < P_H(x),$$

for all $x \geq \rho(G)$. Moreover $\rho(H) < \rho(G)$.

Lemma 2.5 *[5] Let $e = uv$ be a cut edge of G, then*

$$P_G(x) = P_{G-e}(x) - P_{G-u-v}(x).$$

Rewrite $G(l_1, l_2) = G_2$, $l_1 + l_2 = D - 2$. We want to show that when l_1, l_2 are almost equal, the graph $G(l_1, l_2)$ has the maximal spectral radius. Without loss of generality, in the following, we assume $0 \leq l_1 < l_2 \leq D - 2$ and $l_2 - l_1 \geq 2$.

Lemma 2.6 *For $G(l_1, l_2)$ described as above, we have*

$$\rho(G(l_1, l_2)) < \rho(G(l_1 + 1, l_2 - 1)).$$

Proof. By Lemma 2.5, we have

$$P_{G(l_1,l_2)}(x) = xP_{G(l_1,l_2-1)}(x) - P_{G(l_1,l_2-2)}(x),$$

when $l_2 \geq 3$; and

$$P_{G(l_1+1,l_2-1)}(x) = xP_{G(l_1,l_2-1)}(x) - P_{G(l_1-1,l_2-1)}(x).$$

It follows that for $l_2 > l_1 \geq 1$,

$$\begin{aligned} P_{G(l_1,l_2)}(x) - P_{G(l_1+1,l_2-1)}(x) &= P_{G(l_1-1,l_2-1)}(x) - P_{G(l_1,l_2-2)}(x) \\ &\cdots \quad \cdots\cdots \\ &= P_{G(1,l_2-l_1+1)}(x) - P_{G(2,l_2-l_1)}(x) \\ &= P_{G(0,l_2-l_1)}(x) - P_{G(1,l_2-l_1-1)}(x). \end{aligned}$$

Since

$$P_{G(0,l_2-l_1)}(x) = xP_{G(0,l_2-l_1-1)}(x) - P_{G(0,l_2-l_1-2)}(x),$$
$$P_{G(1,l_2-l_1-1)}(x) = xP_{G(0,l_2-l_1-1)}(x) - P_H(x),$$

where $H = K_{n-D+1}^{l_2-l_1-1}$ is obtained by adding an edge joining a pendent vertex of a path on $l_2 - l_1 - 1$ vertices and a vertex of the complete graph K_{n-D+1}.

Hence

$$P_{G(l_1,l_2)}(x) - P_{G(l_1+1,l_2-1)}(x) = P_H(x) - P_{G(0,l_2-l_1-2)}(x).$$

Note that H is a proper subgraph of $G(0, l_2 - l_1 - 2)$ and $G(0, l_2 - l_1 - 2)$ is a subgraph of $G(l_1 + 1, l_2 - 1)$, so

$$\rho(H) < \rho(G(0, l_2 - l_1 - 2)) < \rho(G(l_1 + 1, l_2 - 1)).$$

By Lemma 2.4, we have

$$P_H(x) - P_{G(0,l_2-l_1-2)}(x) > 0 \quad for \quad x \geq \rho(G(0, l_2 - l_1 - 2)).$$

and hence

$$P_{G(l_1,l_2)}(x) - P_{G(l_1+1,l_2-1)}(x) > 0 \quad for \quad x \geq \rho(G(l_1 + 1, l_2 - 1)).$$

So we conclude that $P_{G(l_1,l_2)}(x) > 0 \quad for \quad x = \rho(G(l_1 + 1, l_2 - 1))$. That is $\rho(G(l_1, l_2)) < \rho(G(l_1 + 1, l_2 - 1))$, as stated in the lemma. ■

By the Lemma 2.6, we get the following main result of this paper.

Theorem 2.7 *Let $G \in \mathcal{G}$ be a connected graph of order n with diameter D, then $\rho(G) \leq \rho(G^*)$, where G^* is of the form like G_2 and $l_2 - l_1 \leq 1$. Equality holds if and only if $G = G^*$.*

At last, we estimate the spectral radius of G^*.

Theorem 2.8 *The spectral radius of G^* satisfies*

$$\frac{1}{2}\left(n - D - 1 + \sqrt{(n - D - 1)^2 + 8(n - D)} \right) \leq \rho(G^*) < n - D + 1.$$

The left equality holds if and only if $G^* = K_{n-D} \bigvee \overline{K_2}$.

Proof. For the right hand, first joining the two pendent vertices in G^* and then by Lemma 2.2, contracting the internal path, we have $\rho(G^*) < \rho(K_{n-D+2})$.

For the left hand, it is obvious that G^* contains $H_1 = K_{n-D} \bigvee \overline{K_2}$ as a subgraph, so $\rho(G^*) \geq \rho(H_1)$. For H_1, by symmetry, we can suppose the eigencomponents corresponding to the vertices in K_{n-D} and $\overline{K_2}$ are x_1 and x_2, respectively. So the spectral radius of H_1, $\rho(H_1)$, satisfies

$$\begin{aligned} \rho(H_1)x_1 &= (n - D - 1)x_1 + 2x_2; \\ \rho(H_1)x_2 &= (n - D)x_1. \end{aligned}$$

Simplifying the above equations, we have $\rho(H_1)^2 - (n-D-1)\rho(H_1) - 2(n-D) = 0$. So we get the right hand and complete the proof. ■

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