On the (Laplacian) spectral radius of bipartite graphs with given number of blocks

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Abstract

The (Laplacian) spectral radius of a graph is the maximum eigenvalue of its adjacency matrix (Laplacian matrix, respectively). Let $\mathscr{G}(n,k)$ be the set of bipartite graphs with n vertices and k blocks. This paper gives a complete characterization for the extremal graph with the maximum spectral radius (Laplacian spectral radius, respectively) in $\mathscr{G}(n,k)$.

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1 Introduction

All graphs considered here are connected and simple. Denote by $V(G)$ the vertex set of a graph G and $E(G)$ the edge set. For $S \subseteq V(G)$, let $G[S]$ be the subgraph induced by S. The set of the neighbors of a vertex v is denoted by $N_G(v)$ or $N(v)$. A block of a graph is a maximal 2-connected induced subgraph.

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Let $A(G)$ be the adjacency matrix of a graph G and $D(G)$ be the diagonal matrix of vertex degrees. The matrix $D(G) - A(G)$ is called the Laplacian matrix of G. The spectral radii of $A(G)$, $D(G) - A(G)$ and $D(G)+A(G)$ are denoted by $\rho(G)$, $\mu(G)$ and $\sigma(G)$, respectively. The characteristic polynomial of $A(G)$ is denoted by $P_G(x)$. It is known that $A(G)$ and $D(G) + A(G)$ are both irreducible nonnegative for a connected graph G, so from the Perron-Frobenius Theorem, there is a unique positive unit eigenvector corresponding to $\rho(G)$ and $\sigma(G)$, respectively, which is called Perron vector.

In [2], R.A. Brualdi and E.S. Solheid posed the problem of maximizing the spectral radius and determining the extremal graph for a given class of graphs. Much attention has been paid to this question in the past decades (see [1,4-6,9-13]). Specially, when the order and diameter is given, E.Dam [9] determined general graphs with the maximum spectral radius; Zhai, Shu, Liu and Lu [11, 12] characterized bipartite graphs with the maximum spectral radius and general graphs with the maximum Laplacian spectral radius. Zhang and Zhang [13] gave the graph with the maximum Laplacian spectral radius among all bipartite graphs with n vertices and k cut edges.

This paper focuses on (Laplacian) spectral radii of bipartite graphs with given number of blocks. Let $\mathscr{G}(n,k)$ be the set of connected bipartite graphs with n vertices and k blocks, and $G^{k-1}_{a_1,a_2}$ be the graph obtained from the complete bipartite graph K_{a_1,a_2} by adding $k-1$ pendant edges to a vertex of the first partition set. Section 2 determines all the extremal graphs with the maximum spectral radii in $\mathscr{G}(n,k)$. Section 3 shows that $G^{k-1}_{2,n-k-1}$ has the maximum Laplacian spectral radius in $\mathscr{G}(n,k)$ for any $2 \le k \le n-3$.

2 Maximizing the spectral radius in $\mathscr{G}(n,k)$

Denote by $G^\star_{n,k}$ the extremal graph with the maximum spectral radius in $\mathscr{G}(n,k)$. Now let us first consider two extremal cases. Clearly, $\mathscr{G}(n,n-1)$ is the set of all trees with n vertices. Since $K_{1,n-1}$ has the maximum spectral radius among all trees with n vertices, $G^\star_{n,n-1} \cong K_{1,n-1}$. Also, note that $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil} \in \mathscr{G}(n,1)$ for $n > 3$ and $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ attains the maximum spectral radius among all bipartite graphs with n vertices, $G^\star_{n,1} \cong K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$.

Proposition 2.1 *There is no connected bipartite graph with n vertices and $n-2$ blocks.*

Proof. Let $G \in \mathscr{G}(n, n-2)$. Note that any tree of order n has $n-1$ blocks. Thus G has a cycle of length at least 4. And hence there are at most $n-4$ vertices out of the cycle, which can lie in at most $n-4$ distinct blocks. Together with the cycle, we get at most $n-3$ blocks, a contradiction. $\square$

Proposition 2.1 implies that $\mathscr{G}(n, n-2)$ is an empty set. Thus we next only need consider the case $2 \le k \le n-3$.

Lemma 2.2 ([10]) *Let u, v be two vertices of a graph G and $\{v_i | i = 1, 2, \ldots, s\} \subseteq N(v) \setminus (N(u) \cup \{u\})$. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of $A(G)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \le i \le s)$. If $x_u \ge x_v$, then $\rho(G^*) > \rho(G)$.*

Lemma 2.3 *Each block of $G^*_{n,k}$ induces a complete bipartite subgraph and $G^*_{n,k}$ has only one cut vertex.*

Proof. It is known that [3] the increase of any element of an irreducible non-negative matrix increases the spectral radius. Hence the first conclusion holds. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector to $\rho(G^*_{n,k})$, where x_i corresponds to vertex i. Now assume that $G^*_{n,k}$ has two cut vertices u and v with $x_u \ge x_v$. Denote by H the component of $G^*_{n,k} - v$ that contains the vertex u. By Lemma 2.2,

$$\rho(G^*) = \rho(G^*_{n,k} - \sum_{w \in N(v) \setminus V(H)} vw + \sum_{w \in N(v) \setminus V(H)} uw) > \rho(G^*_{n,k}).$$

Moreover, the resulted graph G^* has the same number of blocks as $G^*_{n,k}$, a contradiction. $\square$

A block is said to be trivial, if it is isomorphic to $K_{1,1}$.

Lemma 2.4 *$G^*_{n,k}$ has exactly one non-trivial block.*

Proof. Since $k \le n-3$, there are at least one non-trivial block in $G^*_{n,k}$. Now assume that G_1 and G_2 are two non-trivial blocks in $G^*_{n,k}$, then G_1 and G_2 are both complete bipartite subgraphs. Let u be the unique cut vertex connecting all blocks and u_i be a vertex of G_i that lies in the different partition set from u $(i = 1, 2)$, then u_i is adjacent to u and $N_{G_i}(u_i) \setminus \{u\} \ne \emptyset$. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector corresponding to $\rho(G^*_{n,k})$. Without loss of generality, we may assume that $x_{u_1} \ge x_{u_2}$. By Lemma 2.2,

$$\rho(G^*) = \rho(G^*_{n,k} - \sum_{w \in N_{G_2}(u_2) \setminus \{u\}} u_2 w + \sum_{w \in N_{G_2}(u_2) \setminus \{u\}} u_1 w) > \rho(G^*_{n,k}).$$

To get a contradiction, it suffices to show that the resulted graph $G^\star \in \mathscr{G}(n,k)$. Clearly, $G^\star$ is a connected bipartite graph. And $G^\star$ has $G^\star[V(G_1) \cup V(G_2 - u_2)]$ and uu_2 as two blocks instead of G_1 and G_2. Now we only need to show $G^\star[V(G_1) \cup V(G_2 - u_2)]$ contains no cut vertex. Since $G^\star[V(G_2 - u_2) \cup \{u_1\}]$ is an isomorphism of G_2 and G_2 contains no cut vertex, $G^\star[V(G_2 - u_2) \cup \{u_1\}]$ has no cut vertex. Note that G_1 also has no cut vertex and G_1 has two common vertices u, u_1 with $G^\star[V(G_2 - u_2) \cup \{u_1\}]$, $G^\star[V(G_1) \cup V(G_2 - u_2)]$ contains no cut vertex. □

By Lemmas 2.3 and 2.4, the extremal graph $G^*_{n,k}$ is an isomorphism of $G^{k-1}_{a,n-k+1-a}$ for some $2 \leq a \leq n-k-1$.

Lemma 2.5

$$P_{G^{k-1}_{a_1,a_2}}(x) = x^{a_1+a_2+k-5}[x^4 - (a_1a_2 + k - 1)x^2 + (a_1 - 1)a_2(k-1)];$$

$$\rho^2(G^{k-1}_{a_1,a_2}) = \frac{1}{2}[(a_1a_2 + k - 1) + \sqrt{(a_1a_2 + k - 1)^2 - 4(a_1 - 1)a_2(k-1)}].$$

Proof. When $k = 1$, $G^{k-1}_{a_1,a_2}$ is isomorphic to K_{a_1,a_2}. And it is known that (see [3]) $P_{K_{a_1,a_2}}(x) = x^{a_1+a_2+k-3}(x^2 - a_1a_2)$.

Now let $k \geq 2$, then $G^{k-1}_{a_1,a_2}$ has $k-1$ pendant edges. It is known that (see [3]) $P_G(x) = xP_{G-u}(x) - P_{G-u-v}(x)$ for any graph G with pendant vertex u and its neighbor v. Thus

$$\begin{aligned}
P_{G^{k-1}_{a_1,a_2}}(x) &= xP_{G^{k-2}_{a_1,a_2}}(x) - P_{K_{a_1-1,a_2}}(x)x^{k-2} \\
&= x^2P_{G^{k-3}_{a_1,a_2}}(x) - 2P_{K_{a_1-1,a_2}}(x)x^{k-2} \\
&= \cdots \\
&= x^{k-1}P_{K_{a_1,a_2}}(x) - (k-1)P_{K_{a_1-1,a_2}}(x)x^{k-2} \\
&= x^{a_1+a_2+k-5}[x^4 - (a_1a_2 + k - 1)x^2 + (a_1 - 1)a_2(k-1)].
\end{aligned}$$

□

The following lemma is clear.

Lemma 2.6 *Let $P(x)$ and $Q(x)$ be two monic polynomials with real roots. And let p_i (resp. q_i) be the i-th largest root of $P(x)$ (resp. $Q(x)$). If $Q(p_1) < 0$, then $p_1 < q_1$. In particular, if $p_1 > q_2$, then $Q(p_1)$ has the same sign as $p_1 - q_1$.*

Lemma 2.7 *Let $G^*_{n,k} \cong G^{k-1}_{a_1,a_2}$, where $k \geq 2$, then $a_1 \leq a_2$.*

Proof. Assume that $a_1 > a_2$. By Lemma 2.5,

$$P_{G^{k-1}_{a_2,a_1}}(x) - P_{G^{k-1}_{a_1,a_2}}(x) = x^{a_1+a_2+k-5}(k-1)(a_2-a_1) < 0.$$

That is, $P_{G^{k-1}_{a_2,a_1}}(\rho(G^{k-1}_{a_1,a_2})) < 0$. According to Lemma 2.6, $\rho(G^{k-1}_{a_1,a_2}) < \rho(G^{k-1}_{a_2,a_1})$, a contradiction. □

By virtue of Lemma 2.7, we can denote the extremal graph $G^\star_{n,k}$ by $G^{k-1}_{a,a+t}$, where $0 \le t \le n-k-3$ and $2a+t = n-k+1$. For convenience, we next use $\langle p,q\rangle$ to denote the set of integers in the interval $[p,q]$.

Theorem 2.8 *Let n,k be two fixed positive integers with $2 \le k \le n-3$ and $G^\star_{n,k}$ be the graph with the maximum spectral radius in $\mathscr{G}(n,k)$. Define*

$$f(t) = \begin{cases} -\infty & t < 2, \\ \frac{(t-1)(n-k+t-1)(n-k+t+1)}{4t} & 2 \le t \le n-k-3, \\ +\infty & t > n-k-3. \end{cases}$$

(i) If $f(t) < k-1 < f(t+2)$ for some $t \in \langle 0, n-k-3\rangle$ and $t \equiv n-k+1 (mod 2)$, then $G^\star_{n,k} \cong G^{k-1}_{a,a+t}$, where $a = \frac{n-k+1-t}{2}$;
(ii) If $k-1 = f(t)$ for some $t \in \langle 2, n-k-3\rangle$ and $t \equiv n-k+1 (mod 2)$, then $G^\star_{n,k}$ is either $G^{k-1}_{a,a+t}$ or $G^{k-1}_{a+1,a+t-1}$, where $a = \frac{n-k+1-t}{2}$.

Proof. (i) By Lemma 2.7, $G^{k-1}_{a,a+s}$ can not be the extremal graph for any $s < 0$. Assume for a contradiction that $G^\star_{n,k} \not\cong G^{k-1}_{a,a+t}$, then there exists $i \in \langle 2-a, \lfloor \frac{t}{2} \rfloor\rangle \backslash \{0\}$ such that $\rho(G^{k-1}_{a+i,a+t-i}) \ge \rho(G^{k-1}_{a,a+t})$. Now we distinguish two cases.

Case1 $i \ge 1$.

Now $t \ge 2$. By Lemma 2.5, we can see $\rho(G^{k-1}_{a+i,a+t-i})$ is more than the second largest eigenvalue of $G^{k-1}_{a,a+t}$. Thus by Lemma 2.6, $P_{G^{k-1}_{a,a+t}}(\rho(G^{k-1}_{a+i,a+t-i})) \ge 0$. That is,

$$P_{G^{k-1}_{a,a+t}}(x) - P_{G^{k-1}_{a+i,a+t-i}}(x) = x^{2a+t+k-5} i[(t-i)x^2 - (t-i+1)(k-1)] \ge 0$$

for $x = \rho(G^{k-1}_{a+i,a+t-i})$. By direct computing, we get

$$k-1 \le \frac{(t-i)(a+t-i)(a+t)}{t-i+1} \le \frac{(t-1)(a+t-1)(a+t)}{t} = f(t),$$

a contradiction.

Case2 $i \leq -1$.

Now $a \geq 3$ and hence $t \leq n-k-5$. By Lemma 2.5, we can see $\rho(G^{k-1}_{a,a+t})$ is more than the second largest eigenvalue of $G^{k-1}_{a+i,a+t-i}$. Thus by Lemma 2.6, $P_{G^{k-1}_{a+i,a+t-i}}(\rho(G^{k-1}_{a,a+t})) \leq 0$. That is,

$$P_{G^{k-1}_{a,a+t}}(x) - P_{G^{k-1}_{a+i,a+t-i}}(x) = x^{2a+t+k-5}i[(t-i)x^2 - (t-i+1)(k-1)] \geq 0$$

for $x = \rho(G^{k-1}_{a,a+t})$. By direct computing, we get

$$k-1 \geq \frac{(t-i)(a+t-i)(a+t)}{t-i+1} \geq \frac{(t+1)(a+t+1)(a+t)}{t+2} = f(t+2),$$

also a contradiction.

(ii) If $k-1 = f(t)$ for some $t \in \langle 2, n-k-3\rangle$ and $t \equiv n-k+1 (mod 2)$, then by Lemma 2.5, we can see $\rho(G^{k-1}_{a,a+t}) = \rho(G^{k-1}_{a+1,a+t-1})$. Assume for a contradiction that there exists $i \in \langle 2-a, \lfloor \frac{t}{2} \rfloor\rangle \backslash \{0,1\}$ such that $\rho(G^{k-1}_{a+i,a+t-i}) \geq \rho(G^{k-1}_{a,a+t})$. Note that $f(t) = \frac{(t-1)(a+t-1)(a+t)}{t}$.

If $i \geq 2$, we can likewise get

$$k-1 \leq \frac{(t-i)(a+t-i)(a+t)}{t-i+1} \leq \frac{(t-2)(a+t-2)(a+t)}{t-1} < f(t),$$

a contradiction.

If $i \leq -1$, we can get

$$k-1 \geq \frac{(t-i)(a+t-i)(a+t)}{t-i+1} \geq \frac{(t+1)(a+t+1)(a+t)}{t+2} > f(t),$$

also a contradiction. □

Remark 2.9 *We observe that Theorem 2.8 characterizes the extremal graph* $G^*_{n,k}$ *for any given* n *and* $k \in \langle 2, n-3\rangle$. *In fact, we can see*

$$\bigcup_{t \in \langle 0, n-k-3\rangle, t \equiv n-k+1 (mod 2)} [f(t), f(t+2)] = (-\infty, +\infty),$$

which implies that n *and* k *always belong to one of the cases in Theorem 2.8. In addition, all cases in Theorem 2.8 indeed occur. For example,* $G^*_{24,15} \cong G^{14}_{5,5}$, $G^*_{41,32} \cong G^{31}_{4,6}$, $G^*_{56,47} \cong G^{46}_{3,7}$ *and* $G^*_{n,n-9} \cong G^{n-10}_{2,8}$ *for all* $n \geq 57$. *Besides, Theorem 2.8 (ii) implies that the extremal graphs are not unique if* $k-1 = f(t)$. *For example, both* $G^{10}_{3,5}$ *and* $G^{10}_{4,4}$ *attain the maximum spectral radius in* $\mathscr{G}(18, 11)$.

The number of cut edges is an interesting parameter of graphs. In [6], Liu, Lu and Tian determined the maximum spectral radius of general graphs with given number of cut edges. Recently, Zhang and Zhang [13] investigated the Laplacian spectral radius for bipartite graphs with given number of cut edges.

Let G be a connected graph with cut edge uv, G_1 and G_2 be two components of $G - uv$ with $u \in G_1$ and $v \in G_2$. H is the graph obtained from G by the following way: (i) contract edge uv, (ii) add a pendant edge to the vertex $u(v)$. Then by Lemma 2.2, $\rho(G) < \rho(H)$ (see Fig.1). Let $\mathscr{H}(n,k)$ be the set of bipartite graphs with n vertices and k cut edges. Similar to above, we can show the extremal graph with the maximal spectral radius in $\mathscr{H}(n,k)$ $(1 \leq k \leq n-4)$ is isomorphic to $G^k_{a,n-k-a}$ for some $a \in \langle 2, n-k-2 \rangle$. Consequently, by Theorem 2.8, the extremal graphs for $\mathscr{H}(n,k)$ are also determined.

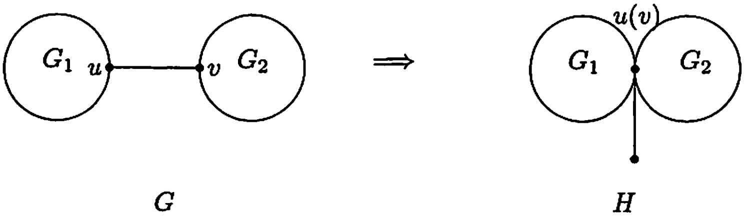

Fig.1

3 Maximizing the Laplacian spectral radius in $\mathscr{G}(n,k)$

Lemma 3.1 ([7],[8]) *For a connected graph G, $\mu(G) \leq \sigma(G)$, with equality if and only if G is bipartite.*

Lemma 3.1 implies that $\mu(G) = \sigma(G)$ for any bipartite graph G. Hence we can investigate $D(G) + A(G)$ and its spectral radius $\sigma(G)$ instead of $D(G) - A(G)$ and $\mu(G)$.

Let $G^*_{n,k}$ be the extremal graph with the maximum Laplacian spectral radius in $\mathscr{G}(n,k)$. Clearly, $G^*_{n,n-1} \cong K_{1,n-1}$. Note that the adding of any edge increases the spectral radius of $D + A$, $G^*_{n,1}$ is a complete bipartite graph. Note that $\mu(K_{a,b}) = a + b$ for any complete bipartite graph $K_{a,b}$, all $K_{a,n-a}$ $(a \in \langle 2, n-2 \rangle)$ are the only extremal graphs in $\mathscr{G}(n,1)$. Next we consider the case $2 \leq k \leq n-3$.

Lemma 3.2 ([5]) *Let u, v be two vertices of a graph G and $\{v_i | i = 1, 2, \ldots, s\} \subseteq N(v) \setminus (N(u) \cup \{u\})$. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of $D(G) + A(G)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\sigma(G^*) > \sigma(G)$.*

Similar to Lemma 2.2, Hong and Zhang gave the above result for the matrix $D(G)+A(G)$. By Lemma 3.2 and similar analysis to Section 2, we conclude that the extremal graph $G^*_{n,k}$ is also an isomorphism of $G^{k-1}_{a,n-k+1-a}$ for some $a \in \langle 2, n-k-1 \rangle$.

Lemma 3.3 ([13]) *Let a and k be integers with $a \geq 2$ and $k \geq 1$. Then*

$$\mu(G^k_{2,a}) > \mu(G^k_{3,a-1}) > \cdots > \mu(G^k_{a,2}).$$

Following from Lemma 3.3, we have

Theorem 3.4 *Among all graphs in $\mathscr{G}(n,k)$ $(k \in \langle 2, n-3 \rangle)$, $G^{k-1}_{2,n-k-1}$ is the unique graph with the maximum Laplacian spectral radius.*

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