The Signed Total Domination Number of Graphs*

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Abstract

Let G be a graph on n vertices with minimum degree r. We show that there exists a two-coloring of the vertices of G with colors, -1 and $+1$, such that all open neighborhoods contain more $+1's$ than $-1's$, and altogether the number of $+1's$ does not exceed the number of $-1's$ by more than $O(\frac{n}{\sqrt{r}})$.

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1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology is not given here we refer to [4]. Let $G = (V, E)$

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be a graph with *vertex set* V and *edge set* E, v is a vertex in V. The *order* of G is given by $n = |V|$, and r is the *minimum degree* among the vertices of G. The *open neighborhood* of v is the set consisting of all of its neighbors, denoted by $N(v)$. $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v.

Let $\chi : V \to Y$ be a function which assigns to each $v \in V$ a value in Y, where Y is a subset of real numbers. For notation convenience, we let $\chi(S) = \sum_{u \in S} \chi(u)$ for any set $S \subseteq V$. We call $\chi(V)$ the *weight* of χ. The function χ is called a *Y-dominating function* if $\chi(N[v]) \geq 1$ for each vertex $v \in V$ and Y is called the *weight set* of χ. Many dominating functions have been defined by changing the allowance weights in Y. These functional variations of domination in graphs have been studied in, e.g., [5].

A *signed domination function* of G is a function $\chi : V \to \{-1, +1\}$ such that for every vertex $v \in V(G)$, $\chi(N[v]) > 0$. The signed domination number of G, γ_s, is defined as

$$\gamma_s = \min\{\chi(V) : \chi \text{ is a signed domination function of } G\}.$$

When we simply change $N[v]$ in this definition of signed domination function to $N(v)$, we define a *signed total domination function* of G. The same as signed domination number [2, 6, 8], the signed total domination number of G, which is firstly studied by Zelinka [11], is defined as

$$\gamma_t^s = \min\{\chi(V) : \chi \text{ is a signed total domination function of } G\}.$$

For any graph G of order n with minimum degree r, several researchers have estimated γ_s, the signed domination numbers of G. For example, Füredi and Mubayi [3] showed $\gamma_s \leq (4 \cdot \sqrt{\lg r / r} + 1/r) \cdot n$; recently Matoušek [9] proved that $\gamma_s = O(n/\sqrt{r})$ by a so-called partial coloring method from combinatorial discrepancy theory [12]. For the signed total domination number, we know $\gamma_t^s(P_n) = n, n \geq 2$; $\gamma_t^s(K_{1,n-1}) = 2$ if n is even and 3 if n is odd [7]. Henning [7] also received other results on the signed total domination number, and these results are the lower bound. In this paper, we will prove that all graphs G of order n with minimum degree r have signed total domination number $O(n/\sqrt{r})$, i.e., $\gamma_t^s = O(n/\sqrt{r})$.

2 Preliminary results

In this section we firstly give some concepts and then give some lemmas to prove the main result.

Let α be a real number and suppose that $\mathcal{S}$ is a hypergraph with vertex set V and edge set $\{S_1, S_2, \ldots, S_m\}$. The function g defines an α-dominating partition of the hypergraph $\mathcal{S}$, if

$$g(S) := \sum_{a \in S} g(a) \geq \alpha,$$

for every edge S in $\mathcal{S}$, $dom_\alpha(\mathcal{S}) := \min_{g: S \to \{-1,+1\},\ g \text{ is } \alpha\text{-dominating}} \text{g(S)}.$

We denote dom_1 as dom. Clearly, we note that

Lemma 1 *For any graph $G = (V, E)$, $\gamma_t^s(G) = dom(\mathcal{N}(G))$, where $\mathcal{N}$ is the hypergraph on the vertex set V and its edges are the open neighborhoods $\{N(v) : v \in V\}$.*

We need some lemmas for the proof of theorem.

Definition 1 ([9]) *A partial coloring is a mapping $\chi : X \longrightarrow \{1, 0, +1\}$. Let substantial partial coloring be a partial coloring χ with $\chi(x) \neq 0$ for at least $\frac{1}{2}|X|$ points $x \in X$.*

Lemma 2 *Let $\mathcal{S}$ be a system of m set on an n vertex set X, $m \geq n$. Then there exists a substantial partial coloring $\chi : X \longrightarrow \{-1, 0, +1\}$ with $\chi(X) = 0$ and with*

$$|\chi(S)| \leq C \cdot \sqrt{|S| \ln \frac{2m}{n}}$$

for all $S \in \mathcal{S}$, where C is a sufficiently large constant.

Now we give two lemmas to prove Lemma 2.

Lemma 3 ([1]) *Let $X_i, 1 \leq i \leq n$, be mutually independent random variables with*

$$Pr\{X_i = 1\} = Pr\{X_i = -1\} = \frac{1}{2},$$

and set, following the usual conversion, $S_n = X_1 + X_2 + \cdots + X_n$. Let $a > 0$. Then

$$Pr[S_n > a] < e^{-\frac{a^2}{2n}}.$$

Let us think of all colorings $\chi : X \to \{-1, +1\}$ as the hamming cube $\{-1, +1\}^n$ and we define $\rho(\chi, \chi') = |\{a : \chi(a) \neq \chi'(a)\}|$. For $D \subset \{-1, +1\}^n$, we define $diam(D) = \max\{\rho(\chi, \chi'), for \chi, \chi' \in D\}$.

Lemma 4 ([*10*]) *Let* $D \subset \{-1,+1\}^n$, $r < n/2$, $|D| \geq \Sigma_{i=0}^{r}\binom{n}{i}$. *Then* $diam(D) \geq 2r$.

Proof of the Lemma 2. Label the sets of $\mathcal{S}$ by $S_1, S_2, \cdots, S_m$ for convenience. Let

$$\chi : X \to \{-1,+1\}$$

be random. For $1 \leq i \leq m$, we define

$$b_i = \text{nearest integer to } \frac{\chi(S_i)}{2C\sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}}.$$

Now we consider the probability of $b_i = 1$ and $b_i = -1$,

$$\begin{aligned}
Pr[b_i = 1] &= \Pr\left[1/2 < \frac{\chi(S_i)}{2C\sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}} < 3/2\right] \\
&= \Pr\left[\chi(S_i) < 3C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right] \\
&\quad - \Pr\left[\chi(S_i) \leq C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right] \\
&= \Pr\left[\chi(S_i) > C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right] \\
&\quad - \Pr\left[\chi(S_i) \geq 3C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right],
\end{aligned}$$

and

$$\begin{aligned}
Pr[b_i = -1] &= \Pr\left[-3/2 < \frac{\chi(S_i)}{2C\sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}} < -1/2\right] \\
&= \Pr\left[\chi(S_i) < -C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right] \\
&\quad - \Pr\left[\chi(S_i) \leq -3C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right] \\
&= \Pr\left[-\chi(S_i) \geq C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right]
\end{aligned}$$

$$- \Pr\left[-\chi(S_i) > 3C \cdot \sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right].$$

So

$$\begin{aligned}
Pr[b_i = 1] &= Pr[b_i = -1] \\
&< Pr\left[\chi(S_i) > C\sqrt{|S_i|}\sqrt{\ln\frac{2m}{n}}\right] \\
&< e^{-\frac{C^2|S_i|\ln\frac{2m}{n}}{2n}} \\
&= \left(\frac{2m}{n}\right)^{-\frac{C^2|S_i|}{2n}} \\
&= \left(\frac{n}{2m}\right)^{\frac{C^2|S_i|}{2n}}.
\end{aligned}$$

Since C is sufficiently large, we may assume

$$Pr[b_i = 1] = Pr[b_i = -1] < \left(\frac{n}{2m}\right)^{50}.$$

Now we bound the entropy $H(b_i) = \Sigma_{j=-\infty}^{+\infty} - p_j \cdot \log_2(p_j)$, and $p_j = Pr[b_i = j]$. It is clearly that the infinite sum converges and it is dominated by $Pr[b_i = 1]$. Then

$$\begin{aligned}
H(b_i) &\leq 2 \cdot -(\tfrac{n}{2m})^{50} \cdot \log_2(\tfrac{n}{2m})^{50} \\
&= 2 \cdot (\tfrac{2m}{n})^{-50} \cdot \log_2(\tfrac{2m}{n})^{50} \\
&< 2 \cdot 2^{-50} \cdot \tfrac{n}{m} \cdot (\tfrac{m}{n})^{-49} \cdot (50\log_2 \tfrac{m}{n} + 50) \\
&< 100 \cdot 2^{-50} \cdot \tfrac{n}{m}.
\end{aligned}$$

Note that $y = \frac{m}{n} \geq 1$, then $(\frac{m}{n})^{-49} \cdot (50\log_2 \frac{m}{n} + 50) \leq 50$ is naturally. Moreover, by the subadditivity of entropy, we have

$$H((b_1, b_2, \cdots, b_m)) \leq \Sigma_{i=1}^{m} H(b_i) < \varepsilon n, \quad \varepsilon = 100 \cdot 2^{-50}.$$

If we assume a random variable Z has no value with probability greater than 2^{-t}, then $H(Z) \geq t$. In contrapositive form, there exists a particular

m-tuple $(s_1, s_2, \cdots, s_m)$. So that

$$Pr[(b_1, b_2, \cdots, b_m) = (s_1, s_2, \cdots, s_m)] \geq 2^{-\varepsilon n}.$$

Probability space is composed of the 2^n possible coloring χ. Thus there is a set C' consisting of at least $2^{(1-\varepsilon)n}$ colorings $\chi : X \to \{-1, +1\}$, and all have the same value $(b_1, b_2, \cdots, b_m)$. If we choose any $\chi_1, \chi_2 \in C'$, and $\chi_1(X) \neq \chi_2(X)$, we can only get C_n^2 possibility. Because of $2^{(1-\varepsilon)n} \gg n$, it is easily to find the colorings χ_1, χ_2 satisfy $\chi_1(X) = \chi_2(X)$ in C'.

By Lemma 4, we put $C' = D$ and get $r = \alpha n$, as long as $\alpha < \frac{1}{2}$ and $2^{H(\alpha)} \leq 2^{1-\varepsilon}$. We can bound

$$\Sigma_{i=0}^{\alpha n}\binom{n}{i} \leq 2^{n \cdot H(\alpha)} \leq 2^{n(1-\varepsilon)} = |C'|.$$

For x small, $H(1/2 - x) \sim 1 - (2/\ln 2)x^2$, so $x \leq (\frac{\ln 2}{2} \cdot \varepsilon)^{1/2} = 1.75 \times 10^{-7}$, then we can take $\alpha = \frac{1}{2}(1 - 3.5 \times 10^{-7})$. Thus C' has diameter at least $(1 - 3.5 \times 10^{-7})n$.

Let $\chi_1, \chi_2 \in C'$ be $\rho(\chi_1, \chi_2) \geq |X|/2$ and satisfy $\chi_1(X) = \chi_2(X)$, we set $\chi = (\chi_1 - \chi_2)/2$ is a partial coloring of X. $\chi(a) = 0$ if and only if $\chi_1(a) = \chi_2(a)$ coordinate a, which occurs for $n - \rho(\chi_1, \chi_2) < |X|/2$. For each $1 \leq i \leq n$, the colorings χ_1, χ_2 yield the same value b_i, which means that $\chi_1(S_i), \chi_2(S_i)$ lie on a common interval of length $2C \cdot \sqrt{|S_i| \ln \frac{2m}{n}}$. Thus,

$$|\chi(S_i)| = \left|\frac{\chi_1(S_i) - \chi_2(S_i)}{2}\right| \leq C \cdot \sqrt{|S_i| \ln \frac{2m}{n}},$$

as desired. And we also have

$$\chi(X) = \frac{\chi_1(X) - \chi_2(X)}{2} = 0.$$

■

We also need another definition and lemma.

Definition 2 ([9]) *An l-transversal of hypergraphs $(X, \mathcal{S})$ is a set $T \subseteq X$ such that $|T \cap S| \geq l$ for all $S \in \mathcal{S}$.*

Lemma 5 ([3]) *Let $(X, \mathcal{S})$ be a hypergraph with n vertices and m edges, such that all edges have size at least s, and let $l \leq \frac{s}{2}$. Then there exists an l-transversal for $(X, \mathcal{S})$ of size at most*

$$\frac{2l}{s} \cdot n + \frac{l}{e^{l/4}} \cdot m.$$

3 Main results

The following theorem 6 was posed by Füredi and Mubayi in [3]. Matoušek [9] proved a special case ($m = n$) of this result, which easily leads to his conclusion $\gamma_s = O(\frac{n}{\sqrt{r}})$.

Theorem 6 *For a hypergraph (X, S) with n vertices, m edges set and every edge has at least r vertices, and r is sufficiently large, then*

$$dom(H) \leq \frac{C}{\sqrt{r}} \cdot (n + m).$$

Indeed, when both n and r are sufficiently large (otherwise, we can put $\chi(x) = 1$ for all $x \in X$, then $\gamma_t^s = O(\frac{n}{\sqrt{r}})$), we can apply Theorem 6 and Lemma 1 to the open neighborhood hypergraph and get the following obvious result.

Theorem 7 *For any graph G on n vertices with minimum degree r, the signed total domination number of G, $\gamma_t^s = O(\frac{n}{\sqrt{r}})$.*

Proof of the Theorem 6. We only need prove that there exists a mapping $\chi : V \to \{-1, +1\}$ satisfying $\chi(V) = \frac{C}{\sqrt{r}} \cdot (n + m)$ and $\chi(S) := \Sigma_{x \in S} \chi(a) \geq 1$.

We can define the coloring χ by an iterative procedure. Let $X_1 = X$ and then execute the following step for $i = 1, 2, \cdots$ until the coloring χ is fully defined. For i^{th} step, we know $X_i \subseteq X$ and suppose that the values of χ have been defined on $X \backslash X_i$ and $\chi(X \backslash X_i)$ satisfies the desired conditions, i.e.,

$$\chi(X \backslash X_i) \leq \frac{C_1}{\sqrt{r}} \cdot (n + m),$$

where C_1 is constant.

We consider the following two cases

Case 1. $|X_i| = n_i \leq \frac{1}{\sqrt{r}} \cdot (n + m)$. We can define $\chi(x) = 1$ for all $x \in X_i$, then

$$\chi(X) = \chi(X \backslash X_i) + \chi(X_i) \leq \frac{C_1 + 1}{\sqrt{r}} \cdot (n + m),$$

$C = C_1 + 1$ is constant.

Case 2. $|X_i| = n_i > \frac{1}{\sqrt{r}} \cdot (n + m)$.

Firstly, we will outline the main ideas of this case and then we will go to the details. Let $\mathcal{S}_i$ be the set system $\mathcal{S}$ restricted to X_i. For the hypergraph $(X_i, \mathcal{S}_i)$, we firstly find a suitable small enough subset $T_i \subseteq X_i$, which intersects all large enough sets in $\mathcal{S}_i$ in sufficient points, and we define $\chi(x) = 1$ for all $x \in T_i$. Then we obtain the hypergraph $(X_i', \mathcal{S}_i')$, and $X_i' = X_i \backslash T_i$, $\mathcal{S}_i'$ is the set system $\mathcal{S}_i$ restricted to the set X_i. We apply Lemma 2 to $(X_i', \mathcal{S}_i')$, finding a substantial partial coloring χ_i, $\chi_i : X_i' \to \{-1, 0, +1\}$, with $\chi_i(X_i') = 0$ and with $|\chi_i(S_i')| \le C \cdot \sqrt{|S_i'| \cdot \ln(2m'/n_i')}$ for all $S_i' \in \mathcal{S}_i'$, where C is a sufficiently large constant and $\mathcal{S}_i = m' \le m$, $n_i' = |X_i'| \le n_i$. Suppose Y_i is the set of all points of X_i' where $\chi_i(x) \neq 0$, and we define $\chi(x) = \chi_i(x)$ for all $x \in Y_i$. If $x \notin Y_i$, put $X_{i+1} = X_i' \backslash Y_i$, then we go to the next step.

In fact, because of the definition of substantial partial coloring, we can find a integer q making the $q+1$ step remain at most $\frac{1}{\sqrt{r}} \cdot (n+m)$ points. And we can define the coloring of every point by $+1's$. After $q+1$ step, we can obtain a fully coloring $\chi : X \to \{-1, +1\}$.

Now, let us describe the choice of the transversal T_i to finish the all procedure. We put $r_i = r \cdot \frac{n_i}{n}$, $s_{ij} = 2^j \cdot r_i$, $j = 1, 2, \cdots$. Let

$$\mathcal{S}_{ij} = \{S \in \mathcal{S}_i : s_{ij} \le |S| < 2s_{ij}\}.$$

Then $r_i = r \cdot \frac{n_i}{n}$, so

$$n_i > \frac{1}{\sqrt{r}} \cdot (n+m) > \frac{n}{\sqrt{r}},$$

and

$$\ln \frac{2m}{n_i} < \ln 2\sqrt{r} < \ln 2 + \frac{1}{2} \ln r,$$

thus

$$C \cdot \sqrt{s_{ij} \ln \frac{2m}{n_i}} < C \cdot \sqrt{s_{ij} \cdot (\ln 2 + \ln r/2)} \le C_2 \cdot \sqrt{s_{ij}},$$

C_2 is sufficiently large. Let $s = s_{ij}, l = l_{ij} = C_2\sqrt{s_{ij}}$,

$$s_{ij} = 2^j \cdot r_i = 2^j \cdot r \cdot \frac{n_i}{n} \ge 2^j \cdot r \cdot \frac{1}{\sqrt{r}} = 2^j \cdot \sqrt{r} \ge \sqrt{r},$$

and r is sufficiently large, we know that $l_{ij} \le \frac{1}{2} s_{ij}$. From Lemma 5, it follows that

$$|T_{ij}| \le \frac{2l_{ij}}{s_{ij}} \cdot n_i + \frac{l_{ij}}{e^{l_{ij}/4}} \cdot m.$$

Now we estimate this formula. Note that $l_{ij} = C_2\sqrt{s_{ij}} \ge C_2 \cdot 2^{j/2} \cdot r^{1/4}$. One can know

$$\frac{l_{ij}}{e^{l_{ij}/4}} \cdot m < \frac{m}{l_{ij}^4} \le \frac{m}{2^{2j} \cdot r} \quad \text{(by Talyor series)},$$

and

$$\frac{2l_{ij}}{s_{ij}} \cdot n_i = \frac{2C_2\sqrt{s_{ij}}}{s_{ij}} \cdot n_i = 2C_2 \cdot \frac{n_i}{\sqrt{s_{ij}}} = 2C_2 \cdot \frac{n_i}{2^{j/2} \cdot r_i^{1/2}}$$
$$\leq 2C_2 \cdot \frac{\sqrt{r}}{2^{j/2}} \cdot \frac{n_i}{r_i} = 2C_2 \cdot \frac{n}{2^{j/2} \cdot \sqrt{r}}.$$

Put $T_i = \bigcup_{j=1}^{\infty} T_{ij}$. By the above formula, we have

$$|T_i| \leq \Sigma_{j=0}^{\infty}|T_{ij}|$$
$$\leq \Sigma_{j=0}^{\infty} 2C_2 \tfrac{n}{\sqrt{r}} \cdot (\tfrac{\sqrt{2}}{2})^j + (1/4)^j \cdot \tfrac{m}{r}$$
$$= 2C_2 \tfrac{n}{\sqrt{r}}(1+\sqrt{2}) + \tfrac{1}{3\sqrt{r}} \cdot \tfrac{m}{\sqrt{r}}$$
$$\leq \tfrac{K}{\sqrt{r}} \cdot (n+m),$$

where $K = \max\{2C_2(1+\sqrt{2}), \frac{1}{3\sqrt{r}}\}$. Since $\chi(Y_i) = 0$ for all i, it will not have an effect on the values of $\chi(X)$, so we can estimate

$$\chi(X) = \chi(X_i) + \chi(X \backslash X_i) \leq \frac{C_1}{\sqrt{r}} \cdot (n+m) + \Sigma_{i=1}^{q}|T_i|$$
$$\leq \frac{Kq + C_1}{\sqrt{r}} \cdot (n+m),$$

$C = Kq + C_1$ is a constant.

Next we will demonstrate that χ has another property, i.e.,

$$\chi(S) > 0,$$

for all $S \in \mathcal{S}$. Let $I = \{1, 2, 3, \cdots q\}$. Then

$$\chi(S) \geq \Sigma_{i=1}^{q}|S \cap T_i| - \Sigma_{i=1}^{q}\chi_i(S \cap Y_i) = \Sigma_{i=1}^{q}(|S \cap T_i| - \chi_i(S \cap Y_i)),$$

hence, we have $|S \cap T_i| = |S \cap \bigcup_{j=1}^{\infty} T_{ij}| \geq |S \cap T_{ij}| \geq l_{ij}$, and $\chi_i(S \cap Y_i) \leq C\sqrt{|S_i'| \ln \frac{2m'}{n_i'}} < C_2\sqrt{s_{ij}} = l_{ij}$, so $\chi(S) > 0$. ■

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