Embeddings of Maximum Packings of Triples

Renwang Su*
College of Statistics and Mathematics
Zhejiang Gongshang University
Hangzhou 310018, P. R. China
srwang123@sina.com

Hung-Lin Fu
Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan
hlfu@math.nctu.edu.tw

Abstract

Let $MPT(v,\lambda)$ denote a maximum packing of triples of order v with index λ. For $\lambda > 1$ and $v \geq 3$, it is proved in this paper that the necessary and sufficient condition for the embedding of an $MPT(v,\lambda)$ in an $MPT(u,\lambda)$ is $u \geq 2v+1$.

Keywords: maximum packing; difference triple; embedding

1 Introduction

A *maximum packing of triples* (or simply a *maximum packing*) of order v and index λ, denoted by $MPT(v,\lambda)$, is a pair (V,B) where V is a v-set and B is a collection of 3-subsets (called *blocks* or *triples*) of V such that (1) each 2-subset of V is contained in at most λ triples, and (2) if C is any collection of 3-subsets satisfying (1) then $|B| \geq |C|$.

Let (V,B) be an $MPT(v,\lambda)$, the *leave* of (V,B), denoted by $\mathrm{L}(v,\lambda)$, is a *multigraph* (V,E) where an edge $\{x,y\} \in E$ with multiplicity m if and only if the corresponding 2-subset $\{x,y\}$ is contained in exactly $\lambda - m$ *triples* of B. It is well-known (Hanani [3]) that the *leave* of an $MPT(v,\lambda)$ is empty if and only if $\lambda(v-1) \equiv 0 (\mathrm{mod}\ 2)$ and $\lambda v(v-1) \equiv 0 (\mathrm{mod}\ 6)$. In

*corresponding author

this case, the $MPT(v, \lambda)$ is called a *triple system* and denoted by $TS(v, \lambda)$. For fixed λ, if there exists a $TS(v, \lambda)$, then v is called *λ-admissible*. *Triple systems* with $\lambda = 1$ are also known as *Steiner triple systems* and a $TS(v, \lambda)$ is also denoted by $STS(v)$.

For $v \geq 3$, by Mendelsohn, Shalaby and Shen [7], the only *graphs* which can be *leaves* are shown in Table 1 (where v and λ are reduced modulo 6) with the following abbreviations, where E_4 and 06 refer to the families of graphs listed next to them and the notations are from [7].

Graphs of odd degrees

1F — a matching on v vertices

1FY — a matching on $v - 4$ vertices and a tree on 4 vertices with one vertex of degree 3

06 (a) $1FH$ — a matching on $v - 6$ vertices and a graph induced by AB, BC, BD, DF, DG

(b) $1F_5$ — a matching on $v - 6$ vertices and a tree on 6 vertices with one vertex of degree 5

(c) $1FYY$ — a matching on $v - 8$ vertices and two vertex-disjoint trees each on 4 vertices with one vertex of degree 3

(d) $1F_3$ — a matching on $v - 2$ vertices and a triple edge AB, AB, AB

(e) $1F_{-0-}$ — a matching on $v - 4$ vertices and a graph induced by AB, BC, BC, CD

Graphs of even degrees

2 — a double edge AB, AB

E_4 (a) C_4 — a 4-cycle

(b) 4 — a quadruple edge AB, AB, AB, AB

(c) 2^2 — 2 double edges AB, AB, CD, CD

(d) ∞ — AB, AB, BC, BC

$\lambda \setminus v$	0	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$1F$	$\emptyset$	$1F$	$\emptyset$	$1FY$	E_4
2	$\emptyset$	$\emptyset$	2	$\emptyset$	$\emptyset$	2
3	$1F$	$\emptyset$	06	$\emptyset$	$1FY$	$\emptyset$
4	$\emptyset$	$\emptyset$	E_4	$\emptyset$	$\emptyset$	E_4
5	$1F$	$\emptyset$	$1FY$	$\emptyset$	$1FY$	2

Table 1. leaves of maximum packings

Now let (X, A) be an $MPT(v, \lambda)$. (X, A) is said to be *embedded* in (Y, B) ($MPT(u, \lambda)$) provided that $X \subset Y$ and $A \subseteq B$. We also say that (X, A) is a *subdesign* of (Y, B). The *embedding* problem is one of the fundamental problems in design theory. In 1979, Stern proved the following result:

Theorem 1.1 *(Stern [10]) Suppose both u and v are λ-admissible, and let $v \geq 3$. Then a $TS(v, \lambda)$ can be embedded in a $TS(u, \lambda)$ if and only if $u \geq 2v + 1$.*

For the *embedding* of *maximum packings of triples* with $\lambda = 1$, we have the following results:

Theorem 1.2 *(Fu, Lindner and Rodger [2]; Hartman [4]; Hartman, Mendelsohn and Rosa [5]; Mendelsohn and Rosa [6]) Let $u > v \geq 3$. Any $MPT(v, 1)$ can be embedded in an $MPT(u, 1)$ if and only if*
(1) if $v \in \{3, 4, 5\}$ then $u > v$,
(2) if $v = 6$ then $u = 7$ or $u \geq 10$,
(3) if $v > 6$ and v is even then $u = v + 1$ or $u \geq 2v$, and
(4) if $v > 6$ and v is odd then $u \geq 2v$.

An $MPT(v, \lambda)$ is called *simple* if it contains no repeated *triples*. In 1995, Milici, Quattrocchi and Shen [8] proved that for $v \geq 3$ and any even λ, a *simple* $MPT(v, \lambda)$ can be *embedded* in a *simple* $MPT(u, \lambda)$ if and only if $u \geq 2v + 1$. As a consequence of this result, we have the following theorem:

Theorem 1.3 *(Milici, Quattrocchi and Shen [8]) Let $v \geq 4$ and $\lambda \equiv 0 \pmod 2$. An $MPT(v, \lambda)$ can be embedded in an $MPT(u, \lambda)$ if and only if $u \geq 2v + 1$.*

For the *embedding* of *maximum packings of triples* with $\lambda > 1$, Su, Fu and Shen [11] have recently proved the following theorem:

Theorem 1.4 *(Su, Fu and Shen [11]) Let $u > v \geq 6$ and $\lambda > 1$. Then an $MPT(v, \lambda)$ can be embedded in a $TS(u, \lambda)$ if and only if $\lambda(u - 1) \equiv 0 \pmod 2$, $\lambda u(u - 1) \equiv 0 \pmod 6$ and $u \geq 2v + 1$.*

In general, for which v and u can any $MPT(v, \lambda)$ be *embedded* in an $MPT(u, \lambda)$ with $\lambda > 1$? The main purpose of the present paper is to complete the solution to the *embedding* problem of *maximum packings of triples* for all λ. First, we prove the following necessary condition:

Lemma 1.5 *Let $u > v \geq 3$ and $\lambda > 1$. If (X, A) is an $MPT(v, \lambda)$ embedded in an $MPT(u, \lambda)$ (Y, B), then $u \geq 2v + 1$.*

Proof: Let $L(v,\lambda)$ denote the leave of (X,A), and $L(u,\lambda)$ the leave of (Y,B). Suppose there are exactly s edges of $L(v,\lambda)$ such that each of them is one edge of some triple in B. Suppose there are exactly t edges $\{x,y\}$ of vertices $x \in X$, y$\in$ Y\X such that each of these edges is one edge of $L(u,\lambda)$. Let $n = 2s + t$. Then we must have the following inequality:

$$\lambda v(u-v) - max(n) \leq \lambda(u-v)(u-v-1).$$

Checking cases one by one, it can be seen that for any choice of the parameters u, v and λ, this inequality does not hold for $v \geq 3, \lambda > 1$ and $2v \geq u$. This completes the proof.

Suppose that $u > v \geq 3$ and $\lambda > 1$. In this paper, we will show that the necessary condition of Lemma 1.5 for *embedding* of an $MPT(v,\lambda)$ in an $MPT(u,\lambda)$ is also sufficient.

2 Preliminary results

Let $A_1 + A_2 + \cdots + A_n$ denote the *union* of multisets A_1, A_2, $\cdots$, A_n (so if e occurs k_i times in A_i for each i where $1 \leq i \leq n$, then it occurs $k_1+k_2+\cdots+k_n$ times in $A_1+A_2+\cdots+A_n$). If $A_1 = A_2 = \cdots = A_n = A$, then let nA denote $A_1 + A_2 + \cdots + A_n$. Let λK_n denote the *multigraph* on n vertices in which each pair of vertices is joined by exactly λ edges. Let $V(G)$ denote the vertex set of a *multigraph* G, and $E(G)$ denote the collection of *edges* in G.

Given two *multigraphs* G and H, the *union* $G \cup H$ is the *graph* with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) + E(H)$. If $V(G) \cap V(H) = \emptyset$, then the *join* $G \bigvee H$ is the *graph* with $V(G \bigvee H) = V(G) \cup V(H)$ and $E(G \bigvee H) = E(G) + E(H) + \{\{x,y\} | x \in V(G), y \in V(H)\}$. If G is a *multigraph*, then let λG denote the *graph* with $V(\lambda G) = V(G)$ and $E(\lambda G) = \lambda E(G)$. In addition, let $\overline{K}_v$ denote the *graph* on v vertices with $E(\overline{K}_v) = \emptyset$.

The following construction is essentially the same as Construction 2.1 in [11].

Construction 2.1 *Let $u > v$, X be a v-set, Y be a u-set and $X \subseteq Y$. Suppose (X, A_1) is an $MPT(v,\lambda_1)$ with leave $L(v,\lambda_1)$ and can be embedded in (Y, B_1), an $MPT(u,\lambda_1)$ with leave $L(u,\lambda_1)$; (X, A_2) is an $MPT(v,\lambda_2)$ with leave $L(v,\lambda_2)$ and can be embedded in (Y, B_2), an $MPT(u,\lambda_2)$ with leave $L(u,\lambda_2)$. If $L(v,\lambda_1) \cup L(v,\lambda_2)$ is a $L(v,\lambda_1+\lambda_2)$, and $L(u,\lambda_1) \cup L(u,\lambda_2)$ can be partitioned into a $L(u,\lambda_1+\lambda_2)$ and triples, then an $MPT(v,\lambda_1+\lambda_2)$ with leave $L(v,\lambda_1+\lambda_2)$ can be embedded in an $MPT(u,\lambda_1+\lambda_2)$ with leave $L(u,\lambda_1+\lambda_2)$.*

By Theorems 1.1-1.4 and Construction 2.1, we have the following theorem:

Theorem 2.2 *Suppose u, v and λ are positive integers and $v \geq 4$. If $u \geq 2v+1$, then an $MPT(v,\lambda)$ can be embedded in an $MPT(u,\lambda)$ with the following possible exceptions:*

(1) $v \equiv 2 (mod\ 3)$, $u \equiv 0 (mod\ 2)$ and $\lambda \equiv 3 (mod\ 6)$,

(2) $v \equiv 2 (mod\ 6)$, $u \equiv 0,2,4,5 (mod\ 6)$ and $\lambda \equiv 5 (mod\ 6)$,

(3) $v \equiv 5 (mod\ 6)$, $u \equiv 0 (mod\ 2)$, $\lambda \equiv 1,5 (mod\ 6)$ and $\lambda > 1$.

In this paper, we eliminate all of these possible exceptions. In fact, by Construction 2.1, we only need to embed an $MPT(v,\lambda)$ in an $MPT(u,\lambda)$ for $\lambda = 3, 5$ or 7, and all $u \geq 2v+1$. In addition, the following two lemmas are also useful.

Lemma 2.3 *(Fu, Lindner and Rodger [2]) Let $v = 6h+5 \geq 11$, $u > v$, $X = \{a_i | 0 \leq i \leq 6h+4\}$ and $Y = X \cup Z_{u-v}$.*

(1) If $u = 6t+6h+7$, $t \geq 2$ and $u \geq 2v+1$, then any $MPT(v,1)$ (X,A) with leave $\{\{a_0,a_1\},\{a_1,a_2\},\{a_2,a_3\},\{a_0,a_3\}\}$ can be embedded in an $STS(u)$ (Y,B) such that $\{\{0,a_0,a_1\},\{1,a_0,a_3\},\{1,a_1,a_2\},\{3,a_2,a_3\}\} \subseteq B$,

(2) If $u = 6t+6h+9$, $t \geq 2$ and $u \geq 2v+1$, then any $MPT(v,1)$ (X,A) with leave $\{\{a_0,a_1\},\{a_1,a_2\},\{a_2,a_3\},\{a_0,a_3\}\}$ can be embedded in an $STS(u)$ (Y,B) such that $\{\{0,a_0,a_1\},\{1,a_0,a_3\},\{1,a_1,a_2\},\{2,a_2,a_3\}\} \subseteq B$.

Lemma 2.4 *(Chetwynd and Hilton [1]) A regular graph with an even number n of vertices and high degree (at least $(\sqrt{7}-1)n/2$) has a 1-factorization.*

3 The case $\lambda = 3$

Let $D(n,\lambda)$ be the following *multiset* with elements from Z_n.

$$D(n,\lambda) = \begin{cases} \{\lambda \cdot d | 1 \leq d \leq (n-1)/2\}, & if\ n \equiv 1 (mod 2), \\ \{\lambda \cdot d | 1 \leq d \leq (n-2)/2\} + \{\lambda/2 \cdot n/2\}, & if\ n \equiv 0 (mod 2). \end{cases}$$

The elements of $D(n,\lambda)$ are called *differences*. The symbol $\lambda \cdot d$ means that the *difference* d appears λ times. We remark that we also use $n-d$ to represent the *difference* d.

Let $a,b,c \in D(n,\lambda)$, if $a+b+c \equiv 0 (\text{mod } n)$ or one is the sum of the others, say, $a+b \equiv c (\text{mod } n)$, then $D = (a,b,c)$ is called a *difference triple*, and let (D) denote the *block set* $\{\{0,a,a+b\}+i | 0 \leq i \leq n-1\}$

or $\{\{0, b, a+b\}+i | 0 \le i \le n-1\}$, and we say that (D) is induced by the *difference triple* D. If $n \equiv 0 \pmod 3$, and $a = b = c = n/3$, then $\{\{0, n/3, 2n/3\}+i | 0 \le i \le n/3-1\}$ can form a *2-regular spanning subgraph* of K_n on the vertex set Z_n. In this case, the *difference* $n/3$ is used once. Otherwise the *block set* induced by the *difference triple* (a, b, c) can form a *6-regular spanning subgraph* of K_n.

Suppose m and n are positive integers and $\lambda \ge 2$. let Z be an n-set. A collection F of 2-subsets (called *pairs*) of Z is called an *m-factor* if each vertex of Z is contained in exactly m *pairs* of F. If F is an *m-factor* and $a \in Z$, then let $a * F$ denote the *multiset* $\{\{a, x, y\} | \{x, y\} \in F\}$.

In the following two sections, we always suppose $u > v$, $X = \{a_i | 0 \le i \le v-1\}$, $Y = X \cup Z_{u-v}$, $V(K_v) = V(\overline{K}_v) = X$ and $V(K_{u-v}) = Z_{u-v}$.

Lemma 3.1 *Let $v \equiv 5 (mod\ 6)$ and $u \equiv 0 (mod\ 6)$. If $u \ge 2v+1$, then a $TS(v, 3)$ can be embedded in an $MPT(u, 3)$.*

Proof: Write $v = 6h+5$ and $u - v = 6t+1$. Since $u \ge 2v+1$, we must have $2 \le h+1 \le t$. Let (X, A) be a $TS(v, 3)$.

Case 1. $1 \le h+1 = t$.

For each i where $0 \le i \le 6h+3$, let $F_i = \{\{i, 6h+5\}, \{3h+i+3, 6h+6\}\} \cup \{\{6h+i-j+4, i+j+2\} | 0 \le j \le 3h\}$, where $3h+i+3$, $6h+i-j+4$ and $i+j+2$ are reduced modulo $6h+5$. Let $F_{6h+4} = \{\{6h-i+4, i\} | 0 \le i \le 3h+1\} \cup \{\{3h+2, 6h+6\}\}$, $B_1 = \bigcup_{i=0}^{6h+4} (a_i * F_i)$, $B_2 = \{\{1, 3, 5\} + 6i | 0 \le i \le h\}$ and $B_3 = \{\{0, 1, 2\} + 2i | 0 \le i \le 3h+2\}$. It can be checked that $G = 3K_{6h+7} - (B_2 + B_3 + \{\{0, 6h+6\}\} + F_0 + F_1 + \cdots + F_{6h+4})$ is a $(12h+10)$-regular spanning subgraph of $3K_{6h+7}$ on the vertex set Z_{6h+7}. So G can be partitioned into $6h+5$ 2-factors $F^{(0)}$, $F^{(1)}$, $\cdots$, $F^{(6h+4)}$. Let $B_4 = a_0 * F^{(0)} + a_1 * F^{(1)} + \cdots + a_{6h+4} * F^{(6h+4)}$ and $B_4 = A + B_1 + B_2 + B_3 + B_4$. Then (Y, B) is an $MPT(12h+12, 3)$ with leave $\{\{a_i, i+1\} | 0 \le i \le 6h+4\} \cup \{\{0, 6h+6\}\}$ and contains (X, A) as a subdesign. The conclusion follows.

Case 2. $1 \le h+1 < t$.

Let $Z = Y \cup \{\infty\}$. By Lemma 2.3, an $MPT(v, 1)$ (X, A_1) with leave $\{\{a_0, a_1\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_0, a_3\}\}$ can be embedded in an $STS(6t+6h+7)$ (Z, C_1) such that $C_0 = \{\{0, a_0, a_1\}, \{1, a_0, a_3\}, \{1, a_1, a_2\}, \{3, a_2, a_3\}\} \subseteq C_1$. Let $C_2 = \{\{\infty, x, y\} | \{\infty, x, y\} \in C_1\}$ and $C_3 = C_1 - (C_0 + C_2 + A_1)$. Choose differences of $D(6t+1, 2)$ to form the following collection T of difference triples:

$$T=\begin{cases} \{(1,2,3,);(3,4,6)\}, & if\ t=2,\\ \{(1,2,3,);(4,5,9);(4,5,9);(3,8,8)\}, & if\ t=3,\\ \{(1,2,3);(t+2,2t-3,3t-1); & \\ (2t,3t,t+1);(t,2t-1,3t-1);(t-1,2t-1,3t-2); & \\ (2i,3t-i,3t-i+1):3\le i\le t-1; & \\ (2i-1,t+i,t-i+1):3\le i\le t-2,t\ge 5\}, & if\ t\ge 4. \end{cases}$$

Choose $2t-2h-3$ difference triples $D_1, D_2, \cdots, D_{2t-2h-3}$ from $T\backslash\{(1,2,3)\}$. Let $B_1 = (D_1)+(D_2)+\cdots+(D_{2t-2h-3})$, $B_2 = \{\{4,5,6\}+3i|0 \le i \le 2t-2\}$, $B_3 = \{\{0,1,a_0\},\{0,1,a_1\},\{1,3,a_2\},\{1,3,a_3\}\}$ and $B_4 = \{\{0,1,3\}+i|1 \le i \le 6t\} \cup \{\{0,2,3\}\}$. Then it can be checked that $G = 2K_{u-v} - (B_1 + B_2 + B_4 + \{\{0,1\},\{0,1\},\{1,3\},\{1,3\}\})$ is a $(12h+10)$-regular spanning subgraph of $2K_{u-v}$ on the vertex set Z_{u-v}. So G can be partitioned into $6h+5$ 2-factors $F_0, F_1, \cdots, F_{6h+4}$. Let $B_5 = a_0 * F_0 + a_1 * F_1 + \cdots + a_{6h+4} * F_{6h+4}$ and $B = A + B_1 + B_2 + B_3 + B_4 + B_5 + C_3$. Then (Y, B) is an $MPT(u,3)$ with leave $\{\{x,y\}|\{\infty,x,y\} \in C_1\}$ and contains (X, A) as a subdesign. The conclusion follows.

Lemma 3.2 *Let* $v \equiv 5 (mod\ 6)$ *and* $u \equiv 2 (mod\ 6)$*. Then a* $TS(v,3)$ *can be embedded in an* $MPT(u,3)$ *if* $u \ge 2v+1$*.*

Proof: Suppose $v = 6h+5$, $u - v = 6t+3$ and $1 \le h+1 \le t$.
Case 1. $t = 1$.

In this case, $h = 0$ and $u = 14$. Let (X, A) be a $TS(5,3)$, $B_1 = \{\{a_0, 2i+1, 2i+2\}|0 \le i \le 3\} + \{\{a_1, 2i, 2i+1\}|1 \le i \le 4\} + \{\{a_2, 2i+1, 2i+2\}|1 \le i \le 4\} + \{\{a_3,6,7\},\{a_3,0,8\},\{a_3,1,2\}\} + \{\{a_4,0,1\},\{a_4,2,3\},\{a_4,4,5\}\}$; $B_2 = 2\{\{0,3,6\}+i|0 \le i \le 3\} + \{\{0,1,3\}+i|i \in Z_9\}$. Then $G = 3K_9 - (B_2 + 2\{\{0,1\}+i|i \in Z_9\})$ is a 10-regular spanning subgraph of $3K_9$ on the vertex set Z_9. So G can be partitioned into five 2-factors $F_i : 0 \le i \le 4$. Let $B_3 = a_0 * F_0 + a_1 * F_1 + \cdots + a_4 * F_4$ and $B = A + B_1 + B_2 + B_3$. Then (Y, B) is an $MPT(14,3)$ with leave $1FYY$ of type 06 and contains (X, A) as a subdesign. The conclusion follows.
Case 2. $t \ge 2$.

Let (X, A) be a $TS(v,3)$, and $Z = Y \cup \{\infty\}$. By Lemma 2.3, any $MPT(v,1)$ (X, A_1) with leave $\{\{a_0,a_1\},\{a_1,a_2\},\{a_2,a_3\},\{a_0,a_3\}\}$ can be embedded in an $STS(6t+6h+9)$ (Z, C_1) such that $C_0 = \{\{0,a_0,a_1\}, \{1,a_0,a_3\}, \{1,a_1,a_2\},\{2,a_2,a_3\}\} \subseteq C_1$. Let $C_2 = C_1 - (C_0 + C_3 + A_1)$, where $C_3 = \{\{\infty,x,y\}|\ \{\infty,x,y\} \in C_1\}$. Choose differences of $D(6t+3,2)$

to form the following collection T of difference triples:

$$T = \begin{cases} \{(2,4,6);(3,4,7)\}, & if\ t=2, \\ \{(4,6,10);(4,6,10);(5,8,8);(3,9,9)\}, & if\ t=3, \\ \{(2t,t+1,3t+1);(t,2t,3t); & \\ (2t+2,2t+2,2t-1);(2t+1,t+2,t-1); & \\ (2i,3t-i+1,3t-i+2):3\leq i\leq t-2, t\geq 5; & \\ (2i-1,t+i,t-i+1):3\leq i\leq t-1\}, & if\ t\geq 4. \end{cases}$$

Choose $2t-2h-2$ difference triples $D_1, D_2, \cdots, D_{2t-2h-2}$ from T. Let $B_1 = (D_1)+(D_2)+\cdots+(D_{2t-2h-2})$, $B_2 = \{\{0,2,3\},\{3,4,6\},\{6,7,8\},\{5,7,8\}\} \cup \{\{9,10,11\}+6i|0\leq i\leq t-2\} \cup \{\{9,10,12\}+6i|0\leq i\leq t-2\} \cup \{\{12,13,14\}+6i|0\leq i\leq t-2\} \cup \{\{11,13,14\}+6i|0\leq i\leq t-2\}$, $B_3 = \{\{0,1\},\{0,1\},\{1,2\},\{1,2\},\{4,5\},\{4,5\}\}$ and $B_4 = \{\{0,2t+1,4t+2\}+i|0\leq i\leq 2t\}\}$. It can be checked that $G = 2K_{u-v}-(B_1+B_2+B_3+B_4)$ is a $(12h+10)$-regular spanning subgraph of $2K_{u-v}$ on the vertex set Z_{u-v}. So G can be partitioned into $6h+5$ 2-factors $F_0, F_1, \cdots, F_{6h+4}$. Let $B_5 = a_0 * F_0 + a_1 * F_1 + \cdots + a_{6h+4} * F_{6h+4}$, $B_6 = \{\{0,1,a_0\},\{0,1,a_1\},\{1,2,a_2\},\{1,2,a_3\}\}$ and $B = A+B_1+B_2+B_4+B_5+B_6+C_2$. Then (Y,B) is an $MPT(u,3)$ with leave $1F_{-0-}$ or $1F_3$ of type 06 and contains (X,A) as a subdesign. The conclusion follows.

Lemma 3.3 *Let $v\equiv 5 (mod\ 6)$ and $u\equiv 4 (mod\ 6)$. Then a $TS(v,3)$ can be embedded in an $MPT(u,3)$ if $u\geq 2v+1$.*

Proof: Suppose $v = 6h+5$, $u-v = 6t+5$ and $1\leq h+1\leq t$. Let (X,A) be a $TS(v,3)$, $V(\overline{K}_v) = X$, and $V(K_{u-v+1}) = Z_{6t+5}\cup\{\infty\}$. By [2], the graph $(\overline{K}_v + K_{u-v+1})$ can be partitioned into a collection B' of triples. Let $B_1 = B' - \{\{\infty,x,y\}|\{\infty,x,y\}\in B'\}$. It is easy to see that there exist 4 different vertices $i,j,k,l\in Z_{6t+5}$ such that $\{\{i,j\},\{k,l\}\}\subseteq \{\{x,y\}|\{\infty,x,y\}\in B'\}$. Choose differences of $D(6t+5,2)$ to form the following collection T of difference triples:

$$T = \begin{cases} \{(2,4,5)\}, & if\ t=1, \\ \{(4,6,7);(2,3,5);(4,5,8)\}, & if\ t=2, \\ \{(2t,2t+2,2t+3);(t+1,t+2,2t+3); & \\ (3,2t-2,2t+1);(2t,2t+1,2t+4); & \\ (2i,3t-i+2,3t-i+3):1\leq i\leq t-2; & \\ (2i-1,t+i+1,t-i+2):2\leq i\leq t-2, t\geq 4\}, & if\ t\geq 3. \end{cases}$$

Choose $2t-2h-1$ difference triples $D_1, D_2, \cdots, D_{2t-2h-1}$ from T. Let $B_2 = (D_1)+(D_2)+\cdots+(D_{2t-2h-1})$, $B_3 = \{\{2,3,4\}+3i|0\leq i\leq 2t\} \cup \{\{3,4,5\}+3i|0\leq i\leq 2t\}$ and $B_4 = \{\{0,1\},\{0,1\},\{1,2\},\{1,2\}\}$. Then $G = 2K_{u-v}-(B_2+B_3+B_4)$ is a $(12h+10)$-regular spanning subgraph of

$2K_{u-v}$ on the vertex set Z_{u-v}. So G can be partitioned into $6h+5$ 2-factors $F_0, F_1, \cdots, F_{6h+4}$. Clearly there exists a permutation σ of Z_{6t+4} such that $\sigma(1) = i$, $\sigma(3) = j$, $\sigma(0) = k$ and $\sigma(2) = l$. Let $B_5 = a_0 * \sigma(F_0) + a_1 * \sigma(F_1) + \cdots + a_{6h+4} * \sigma(F_{6h+4})$ and $B = A + B_1 + \sigma(B_2) + \sigma(B_3) + B_5 + \{\{i, k, l\}\}$. Then (Y, B) is an $MPT(u, 3)$ and contains (X, A) as a subdesign. The conclusion follows.

Lemma 3.4 *Let $v \equiv 2 (mod\ 6)$, $v \geq 8$ and $u \equiv 2 (mod\ 6)$. Then an $MPT(v, 3)$ can be embedded in an $MPT(u, 3)$ if $u \geq 2v + 1$.*

Proof: Write $v = 6h + 2$ and $u - v = 6t$. Since an $MPT(6h + 2, 3)$ can be embedded in an $MPT(12h + 5, 3)$ and an $MPT(12h + 5, 3)$ can be embedded in an $MPT(24h + 14, 3)$, we can suppose $2 \leq h + 1 \leq t \leq 3h + 1$. Let (X, A) be an $MPT(v, 3)$. Choose differences of $D(6t, 3)$ to form the following collection T of difference triples:

$(1, 2, 3)$;

$(2i, 3t - i, 3t - i) : 1 \leq i \leq t - 1$;

$(2i - 1, t + i, t - i + 1) : 1 \leq i \leq t$.

Choose $3t - 3h - 2$ difference triples $D_1, D_2, \cdots, D_{3t-3h-2}$ from T. Let $B_1 = (D_1) + (D_2) + \cdots + (D_{3t-3h-2})$ and $B_2 = \{\{0, 2t, 4t\} + i | 0 \leq i \leq 2t - 1\}$. Then $G = 3K_{u-v} - (B_1 + B_2)$ is a $(18h + 7)$-regular spanning subgraph of $3K_{u-v}$ on the vertex set Z_{6t}. Since $18h + 7 \geq 36t/7 \geq 3t(\sqrt{7} - 1)$, by Lemma 2.4, G can be partitioned into $18h + 7$ 1-factors $F, F_0, F_1, \cdots, F_{18h+5}$. Let $B_3 = B_3^0 + B_3^1 + \cdots + B_3^{6h+1}$, where $B_3^i = a_i * F_{3i} + a_i * F_{3i+1} + a_i * F_{3i+2}$ for any $0 \leq i \leq 6h + 1$. Set $B = A + B_1 + B_2 + B_3$. Then (Y, B) is an $MPT(u, 3)$ with leave of type 06 and contains (X, A) as a subdesign. This completes the proof.

Lemma 3.5 *Let $v \equiv 2 (mod\ 6)$, $v \geq 8$ and $u \equiv 4 (mod\ 6)$. Then an $MPT(v, 3)$ can be embedded in an $MPT(u, 3)$ if $u \geq 2v + 1$.*

Proof: Write $v = 6h + 2$ and $u - v = 6t + 2$. We can suppose $2 \leq h + 1 \leq t$. Let (X, A) be an $MPT(v, 3)$ with leave $L(v, 3)$ of type 06. If $L(v, 3)$ is $1F_3$ or $1F_{-0-}$, then the conclusion follows from Construction 2.1 with $(\lambda_1 = 1) + (\lambda_2 = 2)$. If $L(v, 3)$ is $1FH$, $1F_5$ or $1FYY$, we prove the lemma for the case $L(v, 3) = 1FH$, the other two cases can be dealt with in a similar way. Now suppose $L(v, 3) = 1FH = \{\{a_{2i}, a_{2i+1}\} | 3 \leq i \leq 3h\} \cup \{\{a_0, a_1\}, \{a_0, a_2\}, \{a_0, a_3\}, \{a_3, a_4\}, \{a_3, a_5\}\}$. Let $H_1 = \{\{0, 2\}, \{1, 2\}, \{2, 3\}\} \cup \{\{2i, 2i+1\} | 2 \leq i \leq 3t\}$. By [2], the graph $G = (\overline{K}_v \bigvee K_{u-v}) - H_1$ can be partitioned into a collection A_1 of triples.

Let $H_2 = \{\{0, a_0\}, \{1, a_0\}, \{1, a_0\}, \{0, 3\}, \{1, 2\}, \{0, a_2\}, \{0, a_1\}, \{1, a_1\}, \{3, a_2\}, \{2, a_0\}\}$. By [8], $2(\overline{K}_v \bigvee K_{u-v}) - H_2$ can be partitioned into a collection A_2 of triples. Let $A_3 = \{\{0, 2, 3\}, \{0, a_0, a_2\}, \{1, a_0, a_1\}, \{1, 2, a_0\}\}$. Set $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 3)$ with leave $\{\{a_0, a_3\}, \{a_3, a_4\}, \{a_3, a_5\}\} \cup \{\{0, a_1\}, \{1, 2\}, \{3, a_2\}\} \cup \{\{2i, 2i+1\} | 2 \le i \le 3t\} \cup \{\{a_{2i}, a_{2i+1}\} | 3 \le i \le 3h\}$ and contains (X, A) as a subdesign. This completes the proof.

Lemma 3.6 *Let $v \equiv 2 (mod\ 6)$, $v \ge 8$ and $u \equiv 0 (mod\ 6)$. Then an $MPT(v, 3)$ can be embedded in an $MPT(u, 3)$ if $u \ge 2v + 1$.*

Proof: Write $v = 6h + 2$ and $u - v = 6t + 4$. We must have $1 \le h \le t$. Let (X, A) be an $MPT(v, 3)$ with leave $L(v, 3)$ of type 06.

Case 1. $L(v, 3) = 1F_3$ or $1F_{-0-}$.

The conclusion follows from Construction 2.1 with $(\lambda_1 = 1) + (\lambda_2 = 2)$.

Case 2. $L(v, 3) = 1FH$. Let $1FH = \{\{a_{2i}, a_{2i+1}\} | 3 \le i \le 3h\} \cup \{\{a_0, a_2\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_3, a_5\}\}$. $H_1 = \{\{1, a_1\}, \{3, a_1\}, \{1, 3\}, \{1, a_2\}, \{2, a_2\}, \{1, a_3\}, \{2, a_3\}\}$. By [8], the graph $G = 2(\overline{K}_v \bigvee K_{u-v}) - H_1$ can be partitioned into a collection A_1 of triples. Let $H_2 = \{\{2i, 2i+1\} | 0 \le i \le 3t+1\} \cup \{\{1, a_4\}, \{2, a_4\}, \{1, 2\}\}$. By [2], $(\overline{K}_v \bigvee K_{u-v}) - H_2$ can be partitioned into a collection A_2 of triples. Let $A_3 = \{\{1, 2, 3\}, \{1, a_1, a_2\}, \{1, a_3, a_4\}, \{2, a_2, a_3\}\}$ and $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 3)$ with leave $\{\{a_0, a_2\}, \{a_3, a_5\}, \{2, a_4\}, \{3, a_1\}, \{0, 1\}\} \cup \{\{2i, 2i+1\} | 2 \le i \le 3t+1\} \cup \{\{a_{2i}, a_{2i+1}\} | 3 \le i \le 3h\}$ and contains (X, A) as a subdesign. The conclusion follows.

Case 3. $L(v, 3) = 1FYY$.

Suppose

$$1FYY = \{\{a_0, a_1\}, \{a_1, a_2\}, \{a_1, a_3\}\} \cup \{\{a_4, a_5\}, \{a_5, a_6\}, \{a_5, a_7\}\} \cup \{\{a_{2i}, a_{2i+1}\} | 4 \le i \le 3h\}.$$

Let $H_1 = \{\{1, a_3\}, \{2, a_3\}, \{1, 2\}\} \cup \{\{2i, 2i+1\} | 0 \le i \le 3t+1\}$. By [8], $(\overline{K}_v \bigvee K_{u-v}) - H_1$ can be partitioned into a collection A_1 of triples. Let

$$H_2 = \{\{1, a_1\}, \{2, a_1\}, \{1, a_2\}, \{2, a_2\}, \{1, 3\}, \{1, 3\}, \{1, a_5\}, \{3, a_5\}, \{1, a_6\}, \{3, a_6\}, \{0, a_7\}, \{3, a_7\}, \{0, 3\}\}.$$

By [8], $2(\overline{K}_v \bigvee K_{u-v}) - H_2$ can also be partitioned into a collection A_2 of triples. Set $A_3 = \{\{1, a_1, a_2\}, \{2, a_1, a_3\}, \{1, a_5, a_6\}, \{3, a_5, a_7\}, \{1, 2, 3\}, \{0, 1, 3\}\}$ and $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 3)$ with leave $\{\{2i, 2i+1\} | 2 \le i \le 3t+1\} \cup \{\{a_0, a_1\}, \{a_4, a_5\}, \{2, a_2\}, \{1, a_3\}, \{3, a_6\}, \{0, a_7\}\} \cup \{\{a_{2i}, a_{2i+1}\} | 4 \le i \le 3h\}$ and contains (X, A) as a subdesign. The conclusion follows.

Case 4. $L(v, 3) = 1F_5$.

Suppose $1F_5 = \{\{a_0, a_i\}|1 \leq i \leq 5\} \cup \{\{a_{2i}, a_{2i+1}\}|3 \leq i \leq 3h\}$. Let $H_1 = \{\{0, a_0\}, \{3, a_0\}, \{0, 3\}\} \cup \{\{2i, 2i + 1\}|0 \leq i \leq 3t + 1\}$. By [8], $(\overline{K}_v \bigvee K_{u-v}) - H_1$ can be partitioned into a collection A_1 of triples. Let

$$H_2 = \{\{1, a_0\}, \{1, a_1\}, \{2, a_0\}, \{2, a_1\}, \{0, a_3\}, \{2, a_3\}, \{0, 2\}, \{1, a_4\}, \{3, a_4\}, \{1, 3\}, \{0, a_5\}, \{3, a_5\}, \{0, 3\}\}.$$

By [8], $2(\overline{K}_v \bigvee K_{u-v}) - H_2$ can also be partitioned into a collection A_2 of triples. Set $A_3 = \{\{3, a_0, a_5\}, \{1, a_0, a_4\}, \{0, a_0, a_3\}, \{2, a_0, a_1\}, \{0, 1, 3\}, \{0, 2, 3\}\}$ and $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 3)$ with leave $\{\{2i, 2i + 1\}|2 \leq i \leq 3t + 1\} \cup \{\{a_0, a_2\}, \{1, a_1\}, \{0, a_5\}, \{2, a_3\}, \{3, a_4\}\} \cup \{\{a_{2i}, a_{2i+1}\}|3 \leq i \leq 3h\}$ and contains (X, A) as a subdesign. The conclusion follows.

4 The cases $\lambda = 5$ or $\lambda = 7$

In this section, we consider the *embedding* problem of *maximum packings of triples* for $\lambda = 5$ or $\lambda = 7$. First, we have the following lemma:

Lemma 4.1 *Let $v \equiv 5 (\text{mod } 6)$, $u \equiv 0, 4 (\text{mod } 6)$ and $\lambda = 5$ or 7. Then an $MPT(v, \lambda)$ can be embedded in an $MPT(u, \lambda)$ if $u \geq 2v + 1$.*

Proof: We take $(\lambda_1 = 2) + (\lambda_2 = 3)$ for $\lambda = 5$ and $(\lambda_1 = 3) + (\lambda_2 = 4)$ for $\lambda = 7$. The conclusion then follows from Theorem 2.2, Lemmas 3.1, 3.3 and Construction 2.1.

Lemma 4.2 *Let $v \equiv 2 (\text{mod } 6)$, $v \geq 8$ and $u \equiv 2, 4 (\text{mod } 6)$. Then an $MPT(v, 5)$ can be embedded in an $MPT(u, 5)$ if $u \geq 2v + 1$.*

Proof: We prove the lemma for the case $u \equiv 2$ (mod 6), the case $u \equiv 4$ (mod 6) can be dealt with in a similar way. Write $v = 6h + 2$ and $u - v = 6t$. Since $u \geq 2v + 1$, an $MPT(6h + 2, 5)$ can be embedded in an $MPT(12h + 7, 5)$ and an $MPT(12h + 7, 5)$ can be embedded in an $MPT(24h + 20, 5)$, we can suppose $1 < h + 1 \leq t \leq 3h + 2$. Let (X, A) be an $MPT(v, 5)$ with leave $L(v, 5)$=$1FY$. Choose differences of $D(6t, 5)$ to form the following collection T of difference triples:

$(1, 2, 3)$;

$(2i, 3t - i, 3t - i) : 1 \leq i \leq t - 1$;

$(2i, 3t - i, 3t - i) : 1 \leq i \leq t - 1$;

$(2i-1, t+i, t-i+1): 1 \le i \le t$;

$(2i-1, t+i, t-i+1): 1 \le i \le t$.

From T choose $5t-5h-3$ difference triples $D_1, D_2, \cdots, D_{5t-5h-3}$. Let $B_1 = (D_1) + (D_2) + \cdots + (D_{5t-5h-3})$, $B_2 = \{\{0, 2t, 4t\} + i | 0 \le i \le 2t-1\}$. Clearly $G = 5K_{u-v} - (B_1 + B_2)$ is a (30h+11)-regular spanning subgraph of $5K_{u-v}$ on the vertex set Z_{u-v}. Since $30h + 11 \ge 36t/7 \ge 3t(\sqrt{7}-1)$, by Lemma 2.4, G can be partitioned into $30h+11$ 1-factors F, F_0, F_1, $\cdots$, F_{30h+9}. Let $B_3 = B_3^0 + B_3^1 + \cdots + B_3^{6h+1}$, where $B_3^i = a_i * F_{5i} + a_i * F_{5i+1} + a_i * F_{5i+2} + a_i * F_{5i+3} + a_i * F_{5i+4}$ for each i where $0 \le i \le 6h+1$. Set $B = A + B_1 + B_2 + B_3$. Then (Y, B) is an $MPT(u, 5)$ with leave $L(v, 5) \cup F$ and contains (X, A) as a subdesign. This completes the proof.

Lemma 4.3 *Let $v \equiv 2 (mod\ 6)$, $v \ge 8$ and $u \equiv 0 (mod\ 6)$. Then an $MPT(v, 5)$ can be embedded in an $MPT(u, 5)$ if $u \ge 2v + 1$.*

Proof: Write $v = 6h + 2$ and $u - v = 6t + 4$. Suppose $1 \le h \le t$. Let (X, A) be an $MPT(v, 5)$ with leave $L(v, 5) = \{\{a_0, a_i\} | 1 \le i \le 3\} \cup \{\{a_{2i}, a_{2i+1}\} | 2 \le i \le 3h\}$. Let $F = \{\{2i, 2i+1\} | 0 \le i \le 3t+1\}$. By [2], $(\overline{K}_v \bigvee K_{u-v}) - F$ can be partitioned into a collection A_1 of triples. Let $G = \{\{0, a_i\} | 0 \le i \le 3\} \cup \{\{1, a_1\}, \{1, a_2\}, \{3, a_0\}, \{3, a_3\}\}$. By [8], $4(\overline{K}_v \bigvee K_{u-v}) - G$ can be partitioned into a collection A_2 of triples. Set $B = A + A_1 + A_2 + \{\{0, a_0, a_1\}, \{3, a_0, a_3\}, \{0, 1, a_2\}\}$. Then (Y, B) is an $MPT(u, 5)$ with leave $\{\{a_0, a_2\}, \{0, a_3\}, \{1, a_1\}\} \cup \{\{a_{2i}, a_{2i+1}\} | 2 \le i \le 3h\} \cup F$ and contains (X, A) as a subdesign. This completes the proof.

Lemma 4.4 *Let $v \equiv 2 (mod\ 6)$, $v \ge 8$ and $u \equiv 5 (mod\ 6)$. Then an $MPT(v, 5)$ can be embedded in an $MPT(u, 5)$ if $u \ge 2v + 1$.*

Proof: Let $v = 6h + 2$, $u - v = 6t + 3$ and $1 \le h \le t$. Let (X, A) be an $MPT(v, 5)$ with leave $L(v, 5)$=$\{\{a_0, a_i\} | 1 \le i \le 3\} \cup \{\{a_{2i}, a_{2i+1}\} | 2 \le i \le 3h\}$. Let $K_{1,v} = \{\{0, a_i\} | 0 \le i \le 6h+1\}$ and $K_{2,2} = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{0, 2\}\}$. By[6], $(\overline{K}_v \bigvee K_{u-v}) - (K_{1,v} \cup K_{2,2})$ then can be partitioned into a collection A_1 of triples. Let $G = \{\{0, a_0\}, \{1, a_0\}, \{1, a_1\}, \{2, a_1\}, \{0, 1\}, \{1, 2\}\}$. By [2], $4(\overline{K}_v \bigvee K_{u-v}) - G$ can be partitioned into a collection A_2 of triples. Let $A_3 = \{\{0, a_{2i}, a_{2i+1}\} | 2 \le i \le 3h\} \cup \{\{0, a_0, a_3\}, \{0, a_0, a_2\}, \{1, a_0, a_1\}, \{0, 2, a_1\}, \{1, 2, 3\}\}$. Set $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 5)$ with leave the double edge $\{\{0, 1\}, \{0, 1\}\}$ and contains (X, A) as a subdesign. This completes the proof.

Lemma 4.5 *Let $v \equiv 5 (mod\ 6)$ and $u \equiv 2 (mod\ 6)$. Then an $MPT(v, 5)$ can be embedded in an $MPT(u, 5)$ if $u \ge 2v + 1$.*

Proof: Suppose $v = 6h + 5$, $u - v = 6t + 3$ and $1 \leq h + 1 \leq t$. Let (X, A) be an $MPT(v, 5)$ with leave $\{\{a_0, a_1\}, \{a_0, a_1\}$. By [8], $4(\overline{K}_v \bigvee K_{u-v})$ can be partitioned into a collection A_1 of triples. Let $H = \{\{0, a_i\} | 0 \leq i \leq 3\} \cup \{\{i, a_i\} | 4 \leq i \leq 6h + 4\} \cup \{\{0, 1\}, \{2, 3\}\} \cup \{\{2i + 1, 2i + 2\} | 3h + 2 \leq i \leq 3t\}$. By [2], $(\overline{K}_v \bigvee K_{u-v}) - H$ can be partitioned into a collection A_2 of triples. Let $A_3 = \{\{0, a_0, a_1\}\}$, $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 5)$ with leave $1FY = \{\{i, a_i\} | 4 \leq i \leq 6h + 4\} \cup \{\{0, a_2\}, \{0, a_3\}, \{0, 1\}, \{2, 3\}, \{a_0, a_1\}\} \cup \{\{2i + 1, 2i + 2\} | 3h + 2 \leq i \leq 3t\}$ and contains (X, A) as a subdesign. This completes the proof.

Lemma 4.6 *Let $v \equiv 5 (mod\ 6)$ and $u \equiv 2 (mod\ 6)$. Then an $MPT(v, 7)$ can be embedded in an $MPT(u, 7)$ if $u \geq 2v + 1$.*

Proof: Let $v = 6h + 5$, $u - v = 6t + 3$ and $1 \leq h + 1 \leq t$. Let (X, A) be an $MPT(v, 7)$ with leave $L(v, 7)$ of type E_4.

Case 1. $L(v, 7) = C_4$ or 2^2.

Suppose $C_4 = \{\{a_0, a_1\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_0, a_3\}\}$ and $2^2 = \{\{a_0, a_1\}, \{a_0, a_1\}, \{a_2, a_3\}, \{a_2, a_3\}\}$. Let $H = \{\{0, a_i\} | \ 0 \leq i \leq 4\} \cup \{\{i - 4, a_i\} | 5 \leq i \leq 6h + 4\} \cup \{\{2i + 1, 2i + 2\} | 3h \leq i \leq 3t\}$. By [2, 8], $7(\overline{K}_v \bigvee K_{u-v}) - H$ can be partitioned into a collection A_1 of triples. Let $A_2 = \{\{0, a_0, a_1\}, \{0, a_2, a_3\}\}$ and $B = A + A_1 + A_2$. Then (Y, B) is an $MPT(u, 7)$ with leave a 1-factor on Y and contains (X, A) as a subdesign. The conclusion follows.

Case 2. $L(v, 7) = \infty = \{\{a_0, a_1\}, \{a_0, a_1\}, \{a_1, a_2\}, \{a_1, a_2\}\}$.

Let $H_1 = \{\{0, a_i\} | 0 \leq i \leq 3\} \cup \{\{1, a_i\} | 0 \leq i \leq 3\}$. By [2], there exists a perfect matching F on Y such that $(\overline{K}_v \bigvee K_{u-v}) - (H_1 + F)$ can be partitioned into a collection A_1 of triples. Let $H_2 = \{\{1, a_1\}, \{0, a_1\}, \{0, 1\}\}$. Clearly $6(\overline{K}_v \bigvee K_{u-v}) - H_2$ can also be partitioned into a collection A_2 of triples. Let $A_3 = \{\{0, a_0, a_1\}, \{0, a_1, a_2\}, \{2, a_0, a_1\}, \{0, 1, a_3\}, \{1, a_1, a_2\}\}$ and $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 7)$ with leave a 1-factor on Y and contains (X, A) as a subdesign. The conclusion follows.

Case 3. $L(v, 7) = 4 = \{\{a_2, a_3\}, \{a_2, a_3\}, \{a_2, a_3\}, \{a_2, a_3\}\}$.

Let $H_1 = \{\{0, a_i\} | 0 \leq i \leq 3\} \cup \{\{1, a_i\} | 0 \leq i \leq 3\}$. By [2], there exists a perfect matching F on Y such that $(\overline{K}_v \bigvee K_{u-v}) - (H_1 + F)$ can be partitioned into a collection A_1 of triples. Let $H_2 = \{\{0, 1\}, \{0, a_3\}, \{0, 1\}, \{1, a_3\}, \{0, a_2\}, \{1, a_2\}\}$. Clearly $6(\overline{K}_v \bigvee K_{u-v}) - H_2$ can also be partitioned into a collection A_2 of triples. Let $A_3 = \{\{0, 1, a_0\}, \{0, a_2, a_3\}, \{0, 1, a_1\}, \{0, a_2, a_3\}, \{1, a_2, a_3\}, \{1, a_2, a_3\}\}$, and $B = A + A_1 + A_2 + A_3$. Then (Y, B) is an $MPT(u, 7)$ with leave a 1-factor F on Y and contains (X, A) as a subdesign. The conclusion follows.

5 Main results

In this section, we will prove that the necessary condition of Lemma 1.5 is also sufficient for the *embedding* of an $MPT(v,\lambda)$ in an $MPT(u,\lambda)$ with index $\lambda > 1$.

Lemma 5.1 *If $u \geq 7$, then any $TS(3,2)$ can be embedded in an $MPT(u,2)$.*

Proof: Suppose $7 \leq u \leq 14$. By Theorem 1.1, we only need to consider the cases for $u \in \{8,11,14\}$. We prove the case $u = 8$, the other cases can be dealt with in a similar way. Let (X,A) be a $TS(3,2)$, $X = \{a_i | 0 \leq i \leq 2\}$. Let $Y = X \cup Z_5$, $B_1 = \{\{0,1,2\}\}$, $L = 2\{\{3,4\}\}$, then $G = 2K_5 - (B_1 + L)$ is a 6-regular spanning subgraph of $2K_5$ on the vertex set Z_5. So G can be partitioned into three 2-factors F_0, F_1, and F_2. Let $B = A + B_1 + a_0 * F_0 + a_1 * F_1 + a_2 * F_2$, then (Y,B) is an $MPT(8,2)$ containing (X,A) as a subdesign. This completes the proof.

Now we are ready to prove our main theorem.

Theorem 5.2 *Let $v \geq 3$ and $\lambda > 1$. Then an $MPT(v,\lambda)$ can be embedded in an $MPT(u,\lambda)$ if and only if $u \geq 2v+1$.*

Proof: The necessity comes from Lemma 1.5. The sufficiency is proved by combining Theorem 2.2, Lemmas 3.1-3.6, 4.1-4.6, 5.1, and using Table 1 and Construction 2.1 repeatedly.

Acknowledgment

The authors would like to thank Prof. Hao Shen and the referees for their helpful comments and suggestions. The research was supported by the National Natural Science Foundation of China under Grant No. 60873267 and Zhejiang Provincial Natural Science Foundation of China under Grant No. Y6100126 for the first author, and NSC 90- 2115-M-009-027 for the second author.

References

[1] A.G. Chetwynd and A.J.W. Hilton, 1-factorizing regular graphs of high degree-an improved bound, Discrete Math. 75 (1989), 103-112.

[2] H.L. Fu, C.C. Lindner and C.A. Rodger, Two Doyen-Wilson theorems for maximum packings with triples, Discrete Math. 178 (1998), 63-71.

[3] H. Hanani, Balanced imcomplete block designs and related designs, Discrete Math. 11 (1975), 255-369.

[4] A. Hartman, Partial triple systems and edge colourings, Discrete Math. 62 (1986), 183-196.

[5] A. Hartman, E. Mendelsohn and A. Rosa, On the strong Lindner conjecture, Ars Combin. 18 (1984), 139-150.

[6] E. Mendelsohn and A. Rosa, Embedding maximal packings of triples, Congr. Numer. 40 (1983), 235-247.

[7] E. Mendelsohn, N. Shalaby and H. Shen, Nuclear designs, Ars Combin. 32 (1991), 225-238.

[8] S. Milici, G. Quattrocchi and H. Shen, Embedding of simple maximum packings of triples with λ even, Discrete Math. 145 (1995), 191-200.

[9] H. Shen, Embeddings of simple triple systems, Science in China, Ser.A 35 (1990), 283-291.

[10] G. Stern, Triple systeme mit untersystemen, Arch. Math. 33 (1979), 204-208.

[11] R.W. Su, H.L. Fu and H. Shen, Embeddings of maximum packings in triple systems, Discrete Math. 284 (2004), 235-245.