H-kernels in the D-join

Hortensia Galeana-Sánchez, Rocío Sánchez-López
Instituto de Matemáticas, U.N.A.M. Área de la investigación científica.
Circuito Exterior. Ciudad Universitaria. Coyoacán 04510.
México, D. F. México

Abstract

In [8] it was introduced the concept of H-kernel, which generalizes the concepts of kernel and kernel by monochromatic paths. In this paper we prove necessary and sufficient conditions for the existence of H-kernels in the D-join of digraphs and consequently we will give a sufficient condition for D-join to be H-kernel perfect.

Keywords: H-kernel, kernel by monochromatic paths, D-join

1 Introduction

For general concepts we refer the reader to [2] and [3]. For a digraph D, the vertex set is denoted by $V(D)$ and the arc set by $A(D)$.

1.1 Operations on Graphs and Digraphs.

Since it is often more easy to work with graphs whose structure can be characterized in terms of smaller and simpler graphs, authors such as Harary, Trauth, Weichsel, Ore, and Zykov, among others, defined binary operations on two vertex disjoint graphs such as: the *union*, the *join*, the *composition*, the *cartesian product*, the *lexicographic product*, by mentioning some operations, see page 21 of [20] for a summary.

Because of that in graph theory there are several open problems, which are characterized by their complexity, several authors decided to study each problem in particular when the graph is the result of two graphs and a binary operation between them. For example, in Theory of Domination in Graphs, W.E. Clark and S. Suen[7] studied the number of domination for the cartesian product of two graphs. In [10] B. Effantin and H. Kheddouci studied the Grundy number for the cartesian product of a complete graph by another graph. Beside, the operations on graphs allow us to obtain a large family of examples and counterexamples in graph theory.

On the other hand, due to the similarity between graph theory and digraphs theory, operations defined on graphs were extended to digraphs. For example, in [5] the generalized cartesian product and the generalized lexicographical product with respect to graphs were introduced. And in [32] these concepts were extended to digraphs.

Let a digraph D be defined on the vertex set $V(D) = \{1, 2, ..., p\}$, and $\alpha = (D_i)_{i \in \{1,...,p\}}$ be a sequence of digraphs where the digraphs D_1,..., D_p are of the same order n with vertex set $V(D_i) = \{y_1,..., y_n\} = V$, $n \geq 1$ for each $i \in \{1, ..., p\}$.

The *generalized lexicographical product* of the digraph D and the sequence α is the digraph $D[D_1,..., D_p]$ such that:

$$V(D[D_1,...,D_p]) = V(D) \times V \quad \text{and}$$
$$A(D[D_1,...,D_p]) = \{((s,y_i),(r,y_j)) \mid (s = r \text{ and } (y_i,y_j) \in A(D_s)) \text{ or } ((s,r) \in A(D))\}.$$

Notice that Putting $D_i = H$, for each $i = 1, ..., p$, we obtain the lexicographical product of D and H which is denoted by $D[H]$.

Now, the following digraph generalizes the previous construction. Let D be a digraph with $V(D) = \{1, 2, ..., p\}$, $p \geq 2$ and $\alpha = (D_i)_{i \in \{1,...,p\}}$ be a sequence of vertex disjoint digraphs on $V(D_i) = \{i_1,...,i_{p_i}\}$, $p_i \geq 1$ for each $i \in \{1, ..., p\}$. The *D-join* of the digraph D and the sequence α is the digraph $\sigma(\alpha,D)$ such that:

$$V(\sigma(\alpha,D)) = \bigcup_{i=1}^{p} (\{i\} \times V(D_i)) \quad \text{and}$$
$$A(\sigma(\alpha,D)) = \{((s,s_l),(r,r_t)) \mid (s = r \text{ and } (s_l,r_t) \in A(D_s)) \text{ or } ((s,r) \in A(D))\}.$$

It may be noted that if all digraphs from the sequence have the same vertex set, then from the D-join we obtain the generalized lexicographic product of the digraph D and the sequence α, i.e., $\sigma(\alpha,D) = D[D_1,...,D_p]$. If all digraphs from the sequence α are isomorphic to the same digraph H, then from the D-join we obtain the composition of the digraphs D and H.

Since many mathematical results assert that some properties are preserved under certain operations, then several authors are interested in knowing what kind of digraph constructions preserve their properties. In particular we will study the D-join.

1.2 Kernels, (k,l)-kernels, Kernels by monochromatic paths and H-kernels.

A set $K \subseteq V(D)$ is said to be a *kernel* if it is both independent (a vertex in K has no successor in K) and absorbing (a vertex not in K has a successor in K). This concept was introduced in [31] by Von Neumann and Morgenstern. A classical problem in Digraph Theory is to find sufficient conditions for the existence of a kernel in a digraph, since not every digraph has a kernel. Moreover, in [6] Chvátal showed that deciding if a graph possesses a kernel is an NP-complete problem; and in [11] Fraenkel showed that it remains NP-complete for planar directed graphs with indegrees less or equal to 2, outdegrees less or equal to 2 and degrees less or equal to 3. Several authors have been investigating sufficient conditions for the existence of kernels in digraphs, see for example [9], [12], [26], [27]. Such conditions are usually hereditary and so they also imply the existence of a kernel for every induced subdigraph. A digraph such that every induced subdigraph has a kernel is called *kernel-perfect*. The existence of kernels in digraphs formed by some operations from another digraphs have been studied by M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel [4], J. Topp [30], Galeana-Sánchez [14], by mentioning some. In particular, in [15] Galeana-Sánchez and Neumann-Lara proved that the D-join, $\sigma(\alpha, D)$, is a kernel-perfect digraph whenever D and D_i are kernel-perfect digraphs for each $D_i \in \alpha$.

Let k, l be fixed integers, $k \geq 2$ and $l \geq 1$. We say that a subset $J \subseteq$ V(D) is a (k, l)-kernel of D if

(i) for each x, y in J and $x \neq y$, $d_D(x, y) \geq k$ and

(ii) for each $x \in$ V(D) $\setminus J$, $d_D(x, J) \leq l$.

The concept of a (k, l)-kernel was introduced by M. Kwaśnik in [23] and the existence of (k, l)-kernels in digraphs was studied, for example, in [17], [21], [32]. Note that if $k = 2$ and $l = 1$, then we obtain the definition of a kernel of a digraph. Therefore, since the concept of (k, l)-kernel generalizes that of kernel, in [22] M. Kucharska studied necessary and sufficient conditions for the existence of (k, l)-kernels in the D-join, where D is a digraph without circuits of length less than k; and in [29] W. Szumny, A. Włoch and I. Włoch studied necessary and sufficient conditions for the existence of (k, l)-kernels in the D-join if D is an arbitrary digraph on $p \geq 2$ vertices and $\alpha = (D_i)_{i \in \{1,\dots,p\}}$ is an arbitrary sequence of vertex disjoint digraphs.

Let D be an m-coloured digraph. A set $N \subseteq$ V(D) is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

1. for every pair of diferent vertices u, $v \in N$ there is no monochromatic directed path between them and,
2. for every vertex $x \in V(D) \setminus N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

Another generalization of the concept of kernel is the concept of kernel by monochromatic paths, which was introduced by H. Galeana-Sánchez in [16], since a digraph D has a kernel if and only if the m-coloured digraph D, in which every two different arcs have different colours, has a kernel by monochromatic paths. The existence of kernels by monochromatic paths in edge-coloured digraphs was studied primarily by Sauer, Sands and Woodrow in [28], where they proved that any 2-coloured digraph has a kernel by monochromatic paths. Sufficient conditions for the existence of kernels by monochromatic paths in m-coloured digraphs have also been investigated by several authors, see for example [1], [13], [18], [19], [25]. And as expected, in [33] I. Włoch showed necessary and sufficient conditions for guarantee the existence of kernels by monochromatic paths in the D-join, $\sigma(\alpha,D)$, and consequently she exhibited a sufficient condition for $\sigma(\alpha,D)$ to be a monochromatic kernel perfect.

Let H be a digraph possibly with loops and D a digraph whose arcs are colored with the vertices of H. For an arc (z_1,z_2) of D we will denote by $c(z_1,z_2)$ its color. A directed path $P = (z_0, z_1,..., z_t)$ in D will be called an H-path if $(c(z_0, z_1), c(z_1, z_2),..., c(z_{t-1}, z_t))$ is a directed walk in H. A set $S \subseteq V(D)$ is said to be an H-kernel if it satisfies the following two conditions:

1. for every two different vertices in S there is no H-path in D joining them and,
2. for every $x \in V(D) \setminus S$ there exists an H-path in D from x to S.

In [8] H. Galeana-Sánchez and P. Delgado-Escalante introduced the concept of H-kernel, which generalizes that of kernel by monochromatic paths; since an H-kernel is a kernel by monochromatic paths when H consists only of loops. The concept of H-kernel was motived by the work done by P. Arpin and V. Linek [3]. It is worth mentioning that in [3] Arpin and Linek define what is an H-independent set by walks and an H-absorbent set by walks and these concepts were later used by H. Galeana-Sánchez and P. Delgado-Escalante [8] in order to introduce formally the concept of H-kernel.

In this paper we will give a generalization on all the results exposed by I. Włoch in [33] making use of the concept of H-kernel.

2 Preliminaries

If $S \subseteq V(D)$ is a nonempty set, then the subdigraph of D induced by the vertex set S is that digraph having vertex set S, whose arc set consist of all those arcs of D joining vertices of S. The arc $(z_1,z_2) \in A(D)$ is called as an SS^*-arc whenever $z_1 \in S \subseteq V(D)$ and $z_2 \in S^* \subseteq V(D)$. If C is a directed walk, directed path or directed cycle in D, then $l(C)$ will denote its length. For $\{z_1,z_2\} \subseteq V(D)$ a z_1z_2-walk(path) is a directed walk(path) from z_1 to z_2 in D and if we restrict z_1 and z_2 to $V(C)$, then the z_1z_2-walk(path) contained in C will be denoted by (z_1, C, z_2). If $S \subseteq V(D)$ and $z \in V(D)$, then a directed walk(path) from z to S is a zx-walk(path) for some $x \in S$. A digraph H, possibly with loops, is a *transitive digraph* if for any u, v, w in $V(D)$ we have $\{(u,v), (v,w)\} \subseteq A(D)$ implies $(u,w) \in A(D)$.

A digraph D is said to be edge-coloured if its arcs are coloured. A digraph D is said to be m-colored if the arcs of D are colored with m colors. Let D be an m-colored digraph. For an arc (z_1,z_2) of D we will denote by $c(z_1,z_2)$ its color. A directed path (or directed cycle) is called monochromatic if all of its arcs are coloured alike. A set $I \subseteq V(D)$ is said to be independent by monochromatic paths (or independent by monochromatic directed paths) if for every pair of different vertices $u,v \in I$ there is no monochromatic path between them. The set $A \subseteq V(D)$ is absorbent by monochromatic paths (or absorbent by monochromatic directed paths) if for every vertex $x \in V(D) \setminus A$ there exist a vertex $y \in A$ such that there is a monochromatic path from x to y. A set $N \subseteq V(D)$ is called a kernel by monochromatic paths of the m-coloured digraph D if N is an absorbent and independent set by monochromatic paths. A digraph D such that every induced subdigraph in D has a kernel by monochromatic paths is called monochromatic kernel perfect digraph.

Let H be a digraph possibly with loops and D a digraph whose arcs are colored with the vertices of H (this is what we call an H-colored digraph). A directed path (walk) $P = (z_0, z_1,..., z_t)$ in D will be called an H-restricted path (walk) (or H-path(walk)) if $(c(z_0, z_1), c(z_1, z_2),..., c(z_{t-1}, z_t))$ is a directed walk in H. We will say that an arc is an H-restricted path (walk). We remark that in the general case the existence of an H-walk between two vertices does not guarantee the existence of an H-path between those vertices, although for some H this is true. Furthermore, there are examples where the concatenation of two H-paths is an H-walk, but no H-path exists between the endpoints. Let $z \in V(D)$, a closed directed walk $C = (z = z_0, z_1,...,z_t = z)$ will be called an H-quasirestricted closed walk on z if $(c(z_i, z_{i+1}), c(z_{i+1}, z_{i+2})) \in A(H)$ for each $i \in \{0, 1, ..., t-2\}$. By $C_D(z)$ we denote the set of all H-quasirestricted closed walks on z. We will say that $S \subseteq V(D)$ is H-independent by paths (walks) if for every two different vertices in S there is no H-restricted path (walk) in D joining them. We will say that $S \subseteq V(D)$ is H-absorbent by paths (walks) if for every $x \in V(D) \setminus S$ there exists an H-restricted path (walk) in D from x to S. A set $N \subseteq V(D)$ is called an H-kernel if N is H-independent by paths (H-independent) and H-absorbent by paths (H-absorbent). An H-colored digraph D will be called an H-kernel perfect digraph if every induced subdigraph of D has an H-kernel.

Let D be an H-colored digraph, $V(D) = \{1, 2, ..., p\}$ $p \geq 2$, and $\alpha = (D_i)_{i \in \{1,...,p\}}$ be a sequence of vertex disjoint H-colored digraphs where $V(D_i) = \{i_1, ..., i_{p_i}\}$, $p_i \geq 1$ for each $i \in \{1, ..., p\}$. For the rest of the work $\sigma(\alpha, D)$ is the H-colored digraph such that:

$$V(\sigma(\alpha,D)) = \bigcup_{i=1}^{p} (\{i\} \times V(D_i)) \qquad \text{and}$$

$$A(\sigma(\alpha,D)) = \{((s,s_l),(r,r_t)) \text{ coloured } k \mid (s = r \text{ and } (s_l,r_t) \in A(D_s) \text{ coloured } k) \text{ or } ((s,r) \in A(D) \text{ coloured } k)\}.$$

By D_i^c we mean a copy of the digraph D_i in $\sigma(\alpha,D)$.

3 Main Results

The Theorem 3.1 was considered of great importance in [33], since from it I. Włoch proved of a very natural way the existence of kernel by monochromatic paths in the D-join.

Theorem 3.1. *Let D be an edge coloured digraph, $\alpha = (D_q)_{q \in \{1,...,p\}}$ a sequence of edge coloured vertex disjoint digraphs and (i,n), $(j,m) \in V(\sigma(\alpha,D))$ two different vertices. There is a monochromatic path in $\sigma(\alpha,D)$ from (i,n) to (j,m) if and only if*

(a) for $i \neq j$, there exists a monochromatic path in D from i to j

or

(b) for $i = j$, there exists a monochromatic path in D_i from n to m or $C_D(i) \neq \emptyset$

Since the concatenation of two H-walks(paths) is not always an H-walk(path), then guarantee the existence of H-walks in D from the existence of H-walks in $\sigma(\alpha,D)$ might appear to be a problem quite complicated. For example, in the digraph $\sigma(\alpha,D)$ of Figure 1 we have that $((1,u_1), (2,v_1), (2,v_2), (3,w_2))$ and $((1,u_1), (2,v_1), (2,v_2), (3,w_2), (1,u_2))$ are H-paths, while in D there are no H-walks from 1 to 3 and there are no H-quasirestricted closed walks on 1.

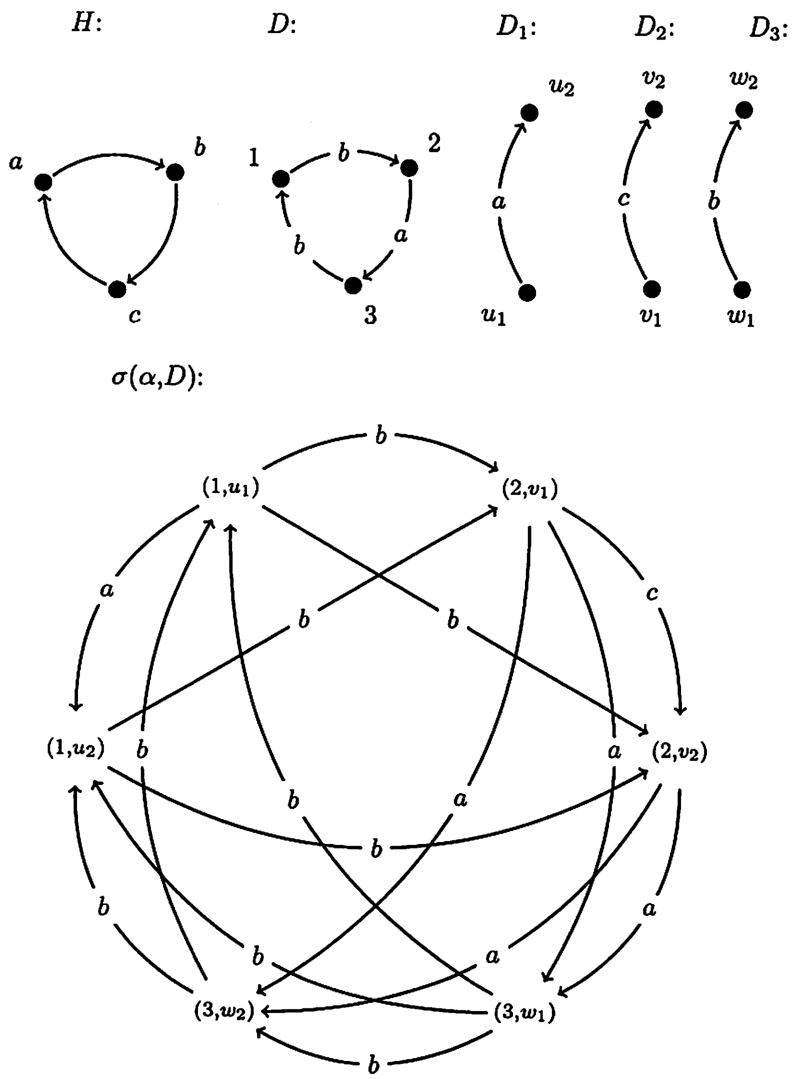

Figure 1:

Therefore, if H is a transitive digraph, then our problem is solved and so the Theorem 3.5 generalizes the Theorem 3.1.

The following pair of results will be useful for the rest of the work.

Lemma 3.2. *Let H be a transitive digraph possibly with loops. If C is a directed walk from u to v of length at least one in H, possibly $u = v$, then $(u,v) \in A(H)$.*

Lemma 3.3. *Let H be a transitive digraph and D an H-colored digraph. For u, v in $V(D)$ with $u \neq v$ every H-walk in D from u to v contains an H-path in D from u to v.*

Proof. Let C be an H-walk in D from u to v. We proceed by induction on $l(C)$, the length of C.

If $l(C) = 1$, then C is already an H-path.

Assume that the statement holds for every H-walk from u to v of length less than n.

Let $C = (u = v_0, v_1,...,v_n = v) \subseteq D$ be an H-walk in D from u to v with length n.

If $v_i \neq v_j$ for every $i \neq j$, then C is the desire H-path. Suppose that there exist i and j, $i \neq j$, such that $v_i = v_j$. Without loss of generality let us suppose that $i \leq j - 1$.

Being as C is an H-walk, we get that $(c(v_{h-1},v_h),c(v_h,v_{h+1})) \in A(H)$ for each $h \in \{1, ..., n-1\}$. Even more, since $(c(v_{i-1},v_i), c(v_i,v_{i+1}),...,c(v_{j-1},v_j), c(v_j,v_{j+1}))$ is a directed walk in H, then it follows from Lemma 3.2 that $(c(v_{i-1},v_i),c(v_j,v_{j+1})) \in A(H)$. Hence $C' = (u = v_0, C, v_i) \cup (v_i = v_j, C, v_n = v)$ is an H-walk from u to v of length less than n, so it follows from the induction hypothesis that C' contains an H-path T from u to v. Finally notice that $T \subset C' \subset C$. □

The previous result allow us to establish the following Theorem.

Theorem 3.4. *Let H be a transitive digraph and D an H-colored digraph. $N \subseteq V(D)$ is an H-kernel by walks in D if and only if N is an H-kernel in D.*

□

Theorem 3.5. *Let H be a transitive digraph, D an H-colored digraph, $\alpha = (D_q)_{q \in \{1,...,p\}}$ a sequence of H-colored vertex disjoint digraphs and $(i,n),(j,m) \in V(\sigma(\alpha,D))$ two different vertices. There is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m) if and only if*

(a) for $i \neq j$, there exists an H-walk in D from i to j

or

(b) for $i = j$, there exists an H-walk in D_i from n to m or $C_D(i) \neq \emptyset$

Proof. **(sufficiency)** If $i \neq j$ and there exists an H-walk in D from i to j, say $(i = i_0, i_1, \ldots, i_k = j)$, then by the definition of $\sigma(\alpha,D)$ we have that for each $(i_h,s) \in V(D^c_{i_h})$ and for each $(i_{h+1},l) \in V(D^c_{i_{h+1}})$ $((i_h,s),(i_{h+1},l)) \in A(\sigma(\alpha,D))$ and $c(i_h,i_{h+1}) = c((i_h,s),(i_{h+1},l))$ for every $h \in \{0,\ldots,k-1\}$. Hence, there is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m).

If $i = j$ and there exists an H-walk in D_i from n to m, say $(n = v_1,v_2,\ldots,v_k = m)$, then from the definition of $\sigma(\alpha,D)$ we get that $((i,n), (i,v_2),\ldots, (i,m))$ is an H-walk in $\sigma(\alpha,$D$)$ from (i,n) to (i,m), since $c((i,v_r), (i,v_{r+1})) = c(v_r,v_{r+1})$ for each $r \in \{1,\ldots,k-1\}$.

Suppose that $i = j$, $C_D(i) \neq \emptyset$ and there are no H-walks in D_i from n to m. If $(i = i_0,i_1,\ldots,i_k = i)$ is an H-quasirestricted closed walk on i contained in D, then from the definition of $\sigma(\alpha,$D$)$ we get that for each $(i_h,s) \in V(D^c_{i_h})$ and for each $(i_{h+1},l) \in V(D^c_{i_{h+1}})$ $((i_h,s),(i_{h+1},l)) \in A(\sigma(\alpha,D))$ and $c(i_h,i_{h+1}) = c((i_h,s),(i_{h+1},l))$ for every $h \in \{0,\ldots,k-1\}$, which means that there is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (i,m).

(necessity) Let $C = ((i,n), (r_1,s_1), (r_2,s_2),\ldots, (r_{k-1},s_{k-1}), (j,m))$ be an H-walk in $\sigma(\alpha,D)$.

We consider the two possible cases:

Case 1. $i \neq j$
In this case we will prove that there is an H-walk in D from i to j by induction on $l(C)$, the length of C.

If $l(C) = 1$, since $i \neq j$, from the definition of $\sigma(\alpha,D)$ we get that $(i,j) \in A(D)$. Hence, there is an H-walk in D from i to j.

If $l(C) = 2$, three possibilities will be analyzed:

(1) $r_1 = i$.

Since $((r_1,s_1), (j,m)) \in A(\sigma(\alpha,D))$, from the definition of $\sigma(\alpha,D)$ we get that $((i,n), (j,m)) \in A(\sigma(\alpha,D)$, which means that $(i,j) \in A(D)$ (because $i \neq j$). Hence, there is an H-walk in D from i to j.

(2) $r_1 = j$.

Since $((i,n),(r_1,s_1)) \in A(\sigma(\alpha,D))$, from the definition of $\sigma(\alpha,D)$ we get that $((i,n), (j,m)) \in A(\sigma(\alpha,D)$, which means that $(i,j) \in A(D)$ (because $i \neq j$). Hence, there is an H-walk in D from i to j.

(3) $r_1 \notin \{i, j\}$.

In this case, from the definition of $\sigma(\alpha,D)$ we get that (i, r_1, j) is an H-walk in D from i to j, since $c(i,r_1) = c((i,n),(r_1,s_1))$, $c(r_1,j) = c((r_1,s_1),(j,m))$, $i \neq j$ and $r_1 \notin \{i, j\}$.

Now assume that if C' is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m) of length less than k, with $i \neq j$, then there is an H-walk in D from i to j.

Let $C = ((i,n), (r_1,s_1), (r_2,s_2),..., (r_{k-1},s_{k-1}), (j,m))$ be an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m) of length k, with $k \geq 3$.

If C is not an H-path, then from Lemma 3.3 we get that C contains an H-path from (i,n) to (j,m) of length less than k. It follows from the inductive hypothesis that there is an H-walk in D from i to j.

Suppose that C is an H-path.

If $V(C) \cap V(D^c_{r_h}) = \{(r_h,s_h)\}$ for each $h \in \{1,...,k-1\}$, then from the definition of $\sigma(\alpha,D)$ we get that $(i = r_0, r_1, r_2,..., r_{k-1}, j = r_k)$ is an H-walk in D from i to j; since $c(i,r_1) = c((i,n),(r_1,s_1))$, $c(r_1,r_2) = c((r_1,s_1),(r_2,s_2)),..., c(r_{k-1},j) = c((r_{k-1},s_{k-1}),(j,m))$, $i \neq j$ and $r_h \neq r_{h+1}$ for each $h \in \{1,...,k-2\}$.

Suppose that there exists $h \in \{1,...,k-1\}$ such that $| V(C) \cap V(D^c_{r_h}) | \geq 2$.

Nex, consider

$$l = min\ \{h \in \{1,...,k-1\} \ / \ | V(C) \cap V(D^c_{r_h}) | \geq 2\}$$

and let

(r_{w_1},s_{w_1}) the first vertex in C that appears in $D^c_{r_l}$
(r_{w_2},s_{w_2}) the second vertex in C that appears in $D^c_{r_l}$

.

.

.

$(r_{w_\beta},s_{w_\beta})$ the last one vertex in C that appears in $D^c_{r_l}$

with $\beta \geq 2$.

We will analyze the following possibilities for $D^c_{r_l}$.

(i) $D^c_{r_l} = D^c_i$

Since $(j,m) \notin V(D^c_i)$ (because $i \neq j$); in particular $(j,m) \neq (r_{w_\beta},s_{w_\beta})$, which means that there exists $(r_{w_\beta+1}, s_{w_\beta+1}) \in V(C)$ such that $((r_{w_\beta},s_{w_\beta}), (r_{w_\beta+1},s_{w_\beta+1})) \in A(C)$, and since $(r_{w_\beta},s_{w_\beta})$ is the last vertex in C that appears in D^c_i, then $(r_{w_\beta+1},s_{w_\beta+1}) \notin V(D^c_i)$. Hence, from the definition of $\sigma(\alpha,D)$ we get that $((i,n),(r_{w_\beta+1},s_{w_\beta+1})) \in A(\sigma(\alpha,D))$ and $c((i,n),(r_{w_\beta+1},s_{w_\beta+1})) = c((r_{w_\beta},s_{w_\beta}),(r_{w_\beta+1},s_{w_\beta+1}))$; thus $C' = ((i,n), (r_{w_\beta+1},s_{w_\beta+1})) \cup ((r_{w_\beta+1},s_{w_\beta+1}), C, (j,m))$ is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m) of length less than k, so it follows from the induction hypothesis that there exists an H-walk in D from i to j.

(ii) $D^c_{r_l} \neq D^c_i$

Since $(i,n) \notin V(D^c_{r_l})$ (because $D^c_{r_l} \neq D^c_i$); in particular $(i,n) \neq (r_{w_1},s_{w_1})$, which means that there exists $(r_{w_1-1},s_{w_1-1}) \in V(C)$ such that $((r_{w_1-1}, s_{w_1-1}),(r_{w_1},s_{w_1})) \in A(C)$, and since (r_{w_1},s_{w_1}) is the first vertex in C that appears in $D^c_{r_l}$, then $(r_{w_1-1},s_{w_1-1}) \notin V(D^c_{r_l})$.

If $(j,m) \in V(D^c_{r_l})$, then from the definition of $\sigma(\alpha,D)$ we get that $((r_{w_1-1}, s_{w_1-1}),(j,m)) \in A(\sigma(\alpha,D))$ and $c((r_{w_1-1},s_{w_1-1}), (j,m)) = c((r_{w_1-1}, s_{w_1-1}), (r_{w_1},s_{w_1}))$. Hence, $C' = ((i,n),C,(r_{w_1-1},s_{w_1-1})) \cup ((r_{w_1-1},s_{w_1-1}), (j,m))$ is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m) of length less than k, so it follows from the induction hypothesis that there exists an H-walk in D from i to j.

Suppose that $(j,m) \notin V(D^c_{r_l})$. Due to $|\, V(C) \cap V(D^c_{r_l}) \,| \geq 2$ we can choose $(r_{w_\gamma},s_{w_\gamma}) \in V(D^c_{r_l})$ such that $(r_{w_\gamma},s_{w_\gamma}) \neq (r_{w_1},s_{w_1})$. Since $(j,m) \notin V(D^c_{r_l})$; in particular $(j,m) \neq (r_{w_\gamma},s_{w_\gamma})$, which means that there exists $(r_{w_\gamma+1},s_{w_\gamma+1}) \in V(C)$ such that $((r_{w_\gamma},s_{w_\gamma}),(r_{w_\gamma+1}, s_{w_\gamma+1})) \in A(C)$. Being as $P = ((r_{w_1-1},s_{w_1-1}),(r_{w_1},s_{w_1})) \cup ((r_{w_1}, s_{w_1}), C, (r_{w_\gamma},s_{w_\gamma})) \cup ((r_{w_\gamma},s_{w_\gamma}),(r_{w_\gamma+1},s_{w_\gamma+1}))$ is an H-walk in $\sigma(\alpha,D)$ (because $P \subseteq C$), it follows from the definition of H-walk that $(c((r_{w_1-1},s_{w_1-1}), (r_{w_1},s_{w_1})), c((r_{w_1},s_{w_1}), (r_{w_1+1},s_{w_1+1})),\ldots, c((r_{w_\gamma}, s_{w_\gamma}),(r_{w_\gamma+1},s_{w_\gamma+1})))$ is an directed walk in H, and so from Lemma 3.2 we get that $(c((r_{w_1-1},s_{w_1-1}), (r_{w_1},s_{w_1})), c((r_{w_\gamma}, s_{w_\gamma}), (r_{w_\gamma+1}, s_{w_\gamma+1}))) \in A(H)$. Hence, $C' = ((i,n), C, (r_{w_1-1},s_{w_1-1})) \cup ((r_{w_1-1},$

$s_{w_1-1}),(r_{w_\gamma},s_{w_\gamma})) \cup ((r_{w_\gamma},s_{w_\gamma}),\ C,\ (j,m))$ is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m), since $c((r_{w_1-1},\ s_{w_1-1}),(r_{w_\gamma},\ s_{w_\gamma})) = c((r_{w_1-1},\ s_{w_1-1}),\ (r_{w_1},s_{w_1}))$. Being as $l(C') < k$, it follows from the induction hypothesis that there exists an H-walk in D from i to j.

Case 2. $i = j$

If $C = ((i,n),\ (r_1,s_1),\ (r_2,s_2),\dots,\ (r_{k-1},s_{k-1}),\ (i,m))$ is contained in D_i^c, then from the definition of $\sigma(\alpha,D)$ we get that there is an H-walk in D_i from n to m.

Suppose that there are no H-walks in D_i from n to m.

Remark 1. Since there are no H-walks in D_i from n to m, then $l(C) \geq 2$ and there exists $\epsilon \in \{1,\dots,k-1\}$ such that $(r_\epsilon,s_\epsilon) \notin \mathrm{V}(D_i^c)$.

We will prove that $C_D(i) \neq \emptyset$ by induction on $l(C)$, the length of C.

If $l(C) = 2$, then it follows from the Remark 1 that $(r_1,s_1) \notin \mathrm{V}(D_i^c)$ and from the definition of $\sigma(\alpha,D)$ we get that $(i,\ r_1,\ i)$ is an H-quasirestricted closed walk on i, since $c(i,r_1) = c((i,n),\ (r_1,s_1))$ and $c(r_1,i) = c((r_1,s_1),\ (i,m))$. Hence, $C_D(i) \neq \emptyset$.

Assume that if C' is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (i,m) of length less than k, then $C_D(i) \neq \emptyset$.

Let $C = ((i,n),\ (r_1,s_1),\ (r_2,s_2),\dots,\ (r_{k-1},s_{k-1}),\ (i,m))$ be an H-walk in $\sigma(\alpha,D)$ from (i,n) to (i,m) of length k, with $k \geq 3$.

If C is not an H-path, then from Lemma 3.3 we get that C contains an H-path from (i,n) to (i,m) of length less than k. It follows from the inductive hypothesis that $C_D(i) \neq \emptyset$.

Suppose that C is an H-path.

By Remark 1 we can choose $t \in \{1,\dots,k-1\}$ such that (r_t,s_t) is the first vertex in C that does not appear in $\mathrm{V}(D_i^c)$.

If $(r_t,s_t) \neq (r_1,s_1)$, then it follows from the choice of t that $(r_{t-1},s_{t-1}) \in \mathrm{V}(D_i^c)$ and $((r_{t-1},s_{t-1}),(r_t,s_t)) \in \mathrm{A}(C)$, moreover from the definition of $\sigma(\alpha,D)$ we have that $c((i,n),(r_t,s_t)) = c((r_{t-1},s_{t-1}),(r_t,s_t))$. Hence, $((i,n),(r_t,s_t)) \cup ((r_t,s_t),\ C,\ (i,m))$ is an H-walk in $\sigma(\alpha,D)$ from (i,n) to

(i,m) of length less than k, so it follows from the induction hypothesis that $C_D(i) \neq \emptyset$.

If $(r_t,s_t) = (r_1,s_1)$, then proceeding as in Case 1 we have that $C_D(i) \neq \emptyset$.

□

The previous results allow us establish the following Theorems.

Theorem 3.6. *Let H be a transitive digraph, D an H-colored digraph and $\alpha = (D_q)_{q\in\{1,\dots,p\}}$ a sequence of H-colored vertex disjoint digraphs. $S^* \subseteq V(\sigma(\alpha,D))$ is an H-independent set by walks in $\sigma(\alpha,D)$ if and only if there exists an H-independent set by walks $S \subseteq V(D)$ in D such that $S^* = \bigcup_{i\in S} S_i$, where $S_i \subseteq V(D_i^c)$ and for every $i \in S$*

$$(1)\quad S_i - \begin{cases} \text{is an } H\text{-independent set by walks in } D_i^c & \text{if } C_D(i) = \emptyset \\ \qquad\qquad\qquad\qquad \text{or} & \\ \text{contains exactly one vertex from } V(D_i^c) & \text{if } C_D(i) \neq \emptyset \end{cases}$$

Proof. **(necessity)** Let $S^* \subseteq V(\sigma(\alpha,D))$ be an H-independent set by walks in $\sigma(\alpha,D)$.

Consider $S = \{\, i \in V(D) \mid (\, S^* \cap V(D_i^c)\,) \neq \emptyset \,\}$.

Claim 1. S is an H-independent set by walks of D.

We proceed by contradiction, suppose that S is not an H-independent set by walks in D. Then there exists $i,j \in S$, $i \neq j$, such that there is an H-walk in D from i to j. By Theorem 3.5 (a) we have that for each $(i,n) \in V(D_i^c)$ and for each $(j,m) \in V(D_j^c)$ there is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m). Since $i,j \in S$, we have that $(S^* \cap V(D_i^c)) \neq \emptyset$ and $(S^* \cap V(D_j^c)) \neq \emptyset$. Therefore, in particular, there are H-walks in $\sigma(\alpha,D)$ from $(S^* \cap V(D_i^c))$ to $(S^* \cap V(D_j^c))$, which is not possible because S^* is H-independent by walks in $\sigma(\alpha,D)$.

On the other hand, if we denote by $S_i = S^* \cap V(D_i^c)$ (for each i such that $(S^* \cap V(D_i^c)) \neq \emptyset$), then it follows from the definition of S that $S^* = \bigcup_{i\in S} S_i$.

Claim 2. For every $i \in S$, S_i satisfies (1).

Let $i \in S$ be. We distinguish two possible cases:

Case 1. $C_D(i) = \emptyset$.
We will prove that S_i is an H-independent set by walks in D_i^c.

Since S^* is H-independent by walks in $\sigma(\alpha,D)$, then S_i must be H-independent by walks in D_i^c (because $D_i^c \subseteq \sigma(\alpha,D)$).

Case 2. $C_D(i) \neq \emptyset$.
We will prove that $\mid S_i \mid = 1$.

Since $i \in S$, then $S_i \neq \emptyset$. Now, we proceed by contradiction. Let us suppose that $\mid S_i \mid \geq 2$.

Since $C_D(i) \neq \emptyset$, it follows from Theorem 3.5 (b) that for each pair of vertices $(i,n),(i,m) \in V(D_i^c)$ there is an H-walk in $\sigma(\alpha,D)$ between them. So, in particular, there are H-walks between every pair of elements of S_i, which can not happen because S^* is H-independent by walks in $\sigma(\alpha,D)$ and $S_i \subseteq S^*$. Hence $\mid S_i \mid = 1$.

(sufficiency) Let $S \subseteq V(D)$ be an H-independent set by walks of D and $S_i \subseteq V(D_i^c)$ as in the hypothesis of Theorem 3.6 for each $i \in S$.

We will prove that $S^* = \bigcup_{i \in S} S_i$ is an H-independent set by walks of $\sigma(\alpha,D)$.

If $\mid S^* \mid = 1$, then S^* is an H-independent set by walks in $\sigma(\alpha,D)$. Hence, suppose that $\mid S^* \mid \geq 2$.

Let $(i,n),(j,m) \in S^*$ be distinct vertices. We will prove that there are no H-walks between (i,n) and (j,m) in $\sigma(\alpha,D)$.

We proceed by contradiction. Let us suppose that there exists an H-walk from (i,n) to (j,m) in $\sigma(\alpha,D)$.

Case a. $i \neq j$.
In this case, it follows from Theorem 3.5 (a) that there is an H-walk in D from i to j, which is not possible because S is H-independent by walks in D and $\{i,j\} \subseteq S$.

Case b. $i = j$.

Since $| S_i | \geq 2$ (because $\{(i,n),(i,m)\} \subseteq S_i$), then the choice of S_i implies that $C_D(i) = \emptyset$. On the other hand, since there is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (j,m), it follows from Theorem 3.5 (b) that there is an H-walk in D_i from n to m, which means that there is an H-walk in D_i^c from (i,n) to (i,m), which is not possible because S_i is H-independent by walks in D_i^c.

Hence, S^* is an H-independent set by walks in $\sigma(\alpha,D)$. □

Theorem 3.7. *Let H be a transitive digraph, D an H-colored digraph and $\alpha = (D_q)_{q\in\{1,\dots,p\}}$ a sequence of H-colored vertex disjoint digraphs. $S^* \subseteq V(\sigma(\alpha,D))$ is an H-absorbent set by walks in $\sigma(\alpha,D)$ if and only if there exists an H-absorbent set by walks $S \subseteq V(D)$ in D such that $S^* = \bigcup_{i\in S} S_i$, where $S_i \subseteq V(D_i^c)$ and for every $i \in S$*

$$(1) \quad S_i - \begin{cases} \text{is an } H\text{-absorbent set by walks in } D_i^c, & \text{if } C_D(i) = \emptyset \text{ and for each } j \in S\setminus\{i\} \text{ there are no } H\text{-restricted walks in } D \text{ from } i \text{ to } j \\ & \text{or} \\ \text{is an nonempty subset of } V(D_i^c), & \text{otherwise} \end{cases}$$

Proof. **(necessity)** Let $S^* \subseteq V(\sigma(\alpha,D))$ be an H-absorbent set by walks in $\sigma(\alpha,D)$.

Consider $S = \{ i \in V(D) \mid (S^* \cap V(D_i^c)) \neq \emptyset \}$.

Claim 1. S in an H-absorbent set by walks of D.

Let $k \in V(D) \setminus S$. Since $k \notin S$, then $(S^* \cap V(D_k^c)) = \emptyset$, and since S^* is H-absorbent by walks in $\sigma(\alpha,D)$, we get that for each $(k,n) \in V(D_k^c)$ there exists $(j,m) \in S^*$ such that there is an H-walk in $\sigma(\alpha,D)$ from (k,n) to (j,m). Therefore, from the definition of S we get that $j \in S$ and it follows from Theorem 3.5 (a) that there is an H-walk in D from k to j (because $k \neq j$).

On the other hand, if we denote by $S_i = S^* \cap V(D_i^c)$ (for each i such that $(S^* \cap V(D_i^c)) \neq \emptyset$), then it follows from the definition of S that $S^* = \bigcup_{i \in S} S_i$.

Claim 2. for every $i \in S$, S_i satisfies (1).

Let $i \in S$ be. We distinguish two possible cases:

Case 1. $C_D(i) = \emptyset$ and for each $j \in S \setminus \{i\}$ there are no H-walks in D from i to j.

We will prove that S_i is an H-absorbent set by walks in D_i^c.

Since there are no H-walks in D from i to j for each $j \in S\setminus\{i\}$, it follows from Theorem 3.5 (a) that for each $(i,n) \in V(D_i^c)$ and for each $(j,m) \in V(D_j^c)$ there are no H-walks in $\sigma(\alpha,D)$ from (i,n) to (j,m) for each $j \in S\setminus\{i\}$. Since S^* is an H-absorbent set by walks in $\sigma(\alpha,D)$, if there exists $(i,n) \in V(D_i^c) \setminus S_i$, then there exists $(i,m) \in S_i$ such that there is an H-walk in $\sigma(\alpha,D)$ from (i,n) to (i,m). Hence, since $C_D(i) = \emptyset$, it follows from Theorem 3.5 (b) that there is an H-walk in D_i from n to m, which means that there is an H-walk in D_i^c from (i,n) to (i,m). Thus, S_i is an H-absorbent set by walks in D_i^c.

Case 2. either $C_D(i) \neq \emptyset$ or there exists $j \in S\setminus\{i\}$ such that there is an H-walk in D from i to j.

Because of the definition of S_i we get that $S_i \neq \emptyset$.

(sufficiency) Let $S \subseteq V(D)$ be an H-absorbent set by walks of D and $S_i \subseteq V(D_i^c)$ as in the hypothesis of Theorem 3.7 for each $i \in S$.

We will prove that $S^* = \bigcup_{i \in S} S_i$ is an H-absorbent set by walks in $\sigma(\alpha,D)$.

Let $(k,n) \in V(\sigma(\alpha,D)) \setminus S^*$.

Case a. $k \notin S$.

Since $k \notin S$ and S is H-absorbent by walks in D, there exists $i \in S$ such that there is an H-walk in D from k to i. Hence, it follows from Theorem 3.5 (a) that for each $(k,l) \in V(D_k^c)$ and for each $(i,m) \in V(D_i^c)$ there is an H-walk in $\sigma(\alpha,D)$ from (k,l) to (i,m), and so there is an H-walk in $\sigma(\alpha,D)$ from (k,n) to $S_i \subseteq S^*$ (because $S_i \subseteq V(D_i^c)$).

Case b. $k \in S$.

If S_k is an H-absorbent set by walks in D_k^c, then there exists an H-walk in $\sigma(\alpha,D)$ from (k,n) to $S_k \subseteq S^*$ (because $(k,n) \notin S_k$).

If S_k is an nonempty subset of $V(D_k^c)$, then the choice of S_k implies that either $C_D(k) \neq \emptyset$ or there exists $j \in S\setminus\{k\}$ such that there is an H-walk in D from k to j. If $C_D(k) \neq \emptyset$, then it follows from Theorem 3.5 (*b*) that for each pair of vertices (k,n), $(k,m) \in V(D_k^c)$ there is an H-walk between them, so in particular there is an H-walk in $\sigma(\alpha,D)$ from (k,n) to $S_k \subseteq S^*$. If there exists $j \in S\setminus\{k\}$ such that there is an H-walk in D from k to j, then it follows from Theorem 3.5 (*a*) that there is an H-walk in $\sigma(\alpha,D)$ from (k,n) to $S_j \subseteq S^*$.

Hence S^* is an H-absorbent set by walks in $\sigma(\alpha,D)$. □

Theorem 3.8. *Let H be a transitive digraph, D an H-colored digraph and $\alpha = (D_q)_{q\in\{1,\dots,p\}}$ a sequence of H-colored vertex disjoint digraphs. $N^* \subseteq V(\sigma(\alpha,D))$ is an H-kernel by walks in $\sigma(\alpha,D)$ if and only if there exists an H-kernel by walks $N \subseteq V(D)$ in D such that $N^* = \bigcup_{i\in N} N_i$, where $N_i \subseteq V(D_i^c)$ and for every $i \in N$*

$$(1)\ N_i - \begin{cases} \textit{is an } H\textit{-kernel by walks of } D_i^c, & \textit{if } C_D(i) = \emptyset \\ \qquad\qquad\qquad\qquad \textit{or} & \\ \textit{contains exactly one vertex from } V(D_i^c), & \textit{if } C_D(i) \neq \emptyset \end{cases}$$

Proof. (necessity) Let $N^* \subseteq V(\sigma(\alpha,D))$ be an H-kernel by walks in $\sigma(\alpha,D)$.

Consider $N = \{\, i \in V(D) \mid (\, N^* \cap V(D_i^c)\,) \neq \emptyset \,\}$.

Claim 1. N is an H-kernel by walks in D.

It follows from Theorems 3.6 and 3.7 that N is H-independent by walks and H-absorbent by walks in D, respectively. Hence, N is an H-kernel by walks in D.

On the other hand, if we denote by $N_i = N^* \cap V(D_i^c)$ (for each i such that $(\, N^* \cap V(D_i^c)\,) \neq \emptyset$), then it follows from the definition of N that N^*

$$= \bigcup_{i \in N} N_i.$$

Claim 2. for every $i \in$ N, N_i satisfies (1).

Let $i \in N$ be. We distinguish two possible cases:

Case 1. $C_D(i) = \emptyset$

We will prove that N_i is an H-kernel by walks in D_i^c.

It follows from Theorem 3.6 that N_i is an H-independent set by walks in D_i^c. Since there are no H-walks in D from i to j for each $j \in N\backslash\{i\}$ (because N is H-independent by walks in D), then it follows from Theorem 3.7 that N_i is an H-absorbent set by walks in D_i^c. Hence, N_i is an H-kernel by walks in D_i^c.

Case 2. $C_D(i) \neq \emptyset$

We will prove that $\mid N_i \mid = 1$.

Since N* is an H-independent set by walks in $\sigma(\alpha,D)$ and $C_D(i) \neq \emptyset$, it follows from Theorem 3.6 that $\mid N_i \mid = 1$.

(sufficiency) Let $N \subseteq$ V(D) be an H-kernel by walks of D and $N_i \subseteq$ V(D_i^c) as in the hypothesis of Theorem 3.8 for each $i \in N$.

We will prove that $N^* = \bigcup_{i \in N} N_i$ is an H-kernel by walks of $\sigma(\alpha,D)$.

Since N is an H-independent set by walks in D, we have that for each $i,j \in N$, $i \neq j$, there are no H-walks in D from i to j. On the other hand, since N_i is H-independent by walks and H-absorbent by walks in D_i^c, if $C_D(i) = \emptyset$ (because N_i is an H-kernel by walks in D_i^c); or $\mid N_i \mid = 1$, if $C_D(i) \neq \emptyset$, for each $i \in N$. Then

$$N_i - \begin{cases} \text{is an } H\text{-independent set by walks in } D_i^c, & \text{if } C_D(i) = \emptyset \\ & \text{or} \\ \text{contains exactly one vertex from V}(D_i^c), & \text{if } C_D(i) \neq \emptyset \end{cases}$$

and

$$N_i - \begin{cases} \text{is an } H\text{-absorbent set by walks in } D_i^c, & \text{if } C_D(i) = \emptyset \text{ and for each } j \in N \setminus \{i\} \text{ there are no } H\text{-restricted walks in } D \text{ from } i \text{ to } j \\ \text{or} & \\ \text{is an nonempty subset of } V(D_i^c), \text{ (due to } \mid N_i \mid = 1) & \text{otherwise} \end{cases}$$

Therefore, it follows from Theorems 3.6 and 3.7 that N^* is H-independent by walks and H-absorbent by walks in $\sigma(\alpha,D)$, respectively. □

The following Lemma will be useful:

Lemma 3.9. *Every induced subdigraph of $\sigma(\alpha,D)$ is*

1. *a digraph of the form $\sigma(\alpha',D')$, where D' is an induced subdigraph of D, with $\mid V(D') \mid \geq 2$, and α' is an sequence of induced subdigraphs of D_i, for each $i \in V(D')$,*
2. *an induced subdigraph of D_i^c for some $i \in \{1, ..., p\}$*

or

3. *the union of the digraphs as in 1 and 2*

Theorem 3.10. *Let H be a transitive digraph, D an H-colored digraph and $\alpha = (D_q)_{q \in \{1,\ldots,p\}}$ a sequence of H-colored vertex disjoint digraphs. If $\sigma(\alpha,D)$ is H-kernel perfect, then D and D_i are H-kernel perfect digraphs, for each $i \in \{1, ..., p\}$.*

Proof. (I) We will prove that D is H-kernel perfect.
Let G be an induced subdigraph of D. We will see that G contains an H-kernel.

If $\mid V(G) \mid = 1$, then G contains an H-kernel. Let us assume that G has at least two vertices.

Let $\alpha' = (D_r)_{r \in V(G)} \subseteq \alpha$. Since $\sigma(\alpha',G)$ is an induced subdigraph of $\sigma(\alpha,D)$, then $\sigma(\alpha',G)$ contains an H-kernel (because $\sigma(\alpha,D)$ is H-kernel perfect). Hence, it follows from Theorem 3.8 that G contains an H-kernel.

(**II**) We will prove that D_i is an H-kernel perfect digraph, for each $i \in \{1, ..., p\}$.

Let $i \in \{1, ..., p\}$ and G an induced subdigraph of D_i. We will see that G contains an H-kernel.

If $| V(G) | = 1$, then G contains an H-kernel. Let us assume that G has at least two vertices

Let $\alpha' = (G)$ and $D' = (\{i\}, \emptyset)$. Since $\sigma(\alpha',D')$ is an induced subdigraph of $\sigma(\alpha,D)$, then $\sigma(\alpha',D')$ contains an H-kernel (because $\sigma(\alpha,D)$ is an H-kernel perfect digraph). Hence, it follows from Theorem 3.8 that G contains an H-kernel. □

Theorem 3.11. *Let H be a transitive digraph, D an H-colored digraph and $\alpha = (D_q)_{q \in \{1,...,p\}}$ a sequence of H-colored vertex disjoint digraphs. If D is an H-kernel perfect digraph and for each $i \in \{1, ..., p\}$*

(1) D_i – { *is H-kernel perfect,* *if $C_D(i) = \emptyset$* *or* *has the property that all of its induced subdigraphs have an H-kernel and each of them contains exactly one vertex,* *if $C_D(i) \neq \emptyset$*

then $\sigma(\alpha,D)$ is an H-kernel perfect digraph.

Proof. We will prove that every induced subdigraph of $\sigma(\alpha,D)$ has an H-kernel.

Let G be an induced subdigraph of $\sigma(\alpha,D)$. It follows from Lemma 3.9 that G has three possibilities for its characterization.

I. G is a digraph of the form $\sigma(\alpha',D')$, where D' is an induced subdigraph of D (with $| V(D') | \geq 2$) and $\alpha' = (D'_i)_{i \in V(D')}$ is an suquence of induced subdigraphs of D_i, for each $i \in V(D')$.

Since D is H-kernel perfect, then D' contains an H-kernel N. On the other hand, it follows from (1) that for each $i \in N$ there exists $N_i \subseteq V(D_i'^c)$ such that

$$(2) \ldots N_i - \begin{cases} \text{is an } H\text{-kernel of } D_i'^c, & \text{if } C_D(i) = \emptyset \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{or} & \\ \text{is an } H\text{-kernel of } D_i'^c, \\ \text{which contains exactly} & \text{if } C_D(i) \neq \emptyset \\ \text{one vertex of } V(D_i'^c), \end{cases}$$

Now, we will see that for each $i \in N$

$$(3) \ldots N_i - \begin{cases} \text{is an } H\text{-kernel of } D_i'^c, & \text{if } C_{D'}(i) = \emptyset \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{or} & \\ \text{contains exactly} & \text{if } C_{D'}(i) \neq \emptyset \\ \text{one vertex of } V(D_i'^c), \end{cases}$$

If $C_{D'}(i) = \emptyset$, then either $C_D(i) = \emptyset$ or $C_D(i) \neq \emptyset$ and in both cases it follows from (2) that N_i is an H-kernel of $D_i'^c$. On the other hand, observe that if $C_{D'}(i) \neq \emptyset$, then $C_D(i) \neq \emptyset$, and it follows from (2) that in particular N_i contains exactly one vertex of $V(D_i'^c)$. Therefore (3) is satisfied.

Hence, it follows from Theorem 3.8 that $\sigma(\alpha', D')$ contains an H-kernel.

II. G is an induced subdigraph of $D_i{}^c$, for some $i \in \{1, \ldots, p\}$.

In this case, it follows from (1) that G contains an H-kernel, since G is isomorphic to an induced subdigraph of D_i (because of the definition of $\sigma(\alpha, D)$).

III. G is the union of the digraphs as in I and II.

In this case G contains an H-kernel; namely, the union of each of the H-kernels in each digraph as in I or II (due to $V(D_i^c) \cap V(D_j^c) = \emptyset$ for each $i \neq j$).

□

Corollary 3.12. *Let H be a transitive digraph, D an H-colored digraph and $\alpha = (D_q)_{q \in \{1,...,p\}}$ a sequence of H-colored vertex disjoint digraphs. If $C_D(i) = \emptyset$ for each $i \in V(D)$, then $\sigma(\alpha, D)$ is an H-kernel perfect digraph if and only if D and D_i are H-kernel perfect digraphs, for each $i \in \{1, ..., p\}$.*

As a consequence of the previous results is obtained a generalization of the work made by I. Włoch in [33] in the case that the set $A(H)$ consists only of loops.

Theorem 3.13. *Let D be an edge coloured digraph and $\alpha = (D_q)_{q \in \{1,...,p\}}$ a sequence of edge coloured vertex disjoint digraphs. If $\sigma(\alpha, D)$ is monochromatic kernel perfect, then D and D_i are monochromatic kernel perfect digraphs, for each $i \in \{1, ..., p\}$.*

Theorem 3.14. *Let D be an edge coloured digraph and $\alpha = (D_q)_{q \in \{1,...,p\}}$ be an sequence of edge coloured vertex disjoint digraphs. If D is an monochromatic kernel perfect digraph and for each $i \in \{1, ..., p\}$*

$$D_i - \begin{cases} \text{is monochromatic kernel perfect,} & \text{if } C_D(i) = \emptyset \\ & \text{or} \\ \text{has the property that all of its induced subdigraphs have a kernel by monochromatic paths and each of them contains exactly one vertex,} & \text{if } C_D(i) \neq \emptyset \end{cases}$$

then $\sigma(\alpha, D)$ is an monochromatic kernel perfect digraph.

Corollary 3.15. *Let D be an edge coloured digraph and $\alpha = (D_q)_{q \in \{1,...,p\}}$ a sequence of edge coloured vertex disjoint digraphs. If $C_D(i) = \emptyset$ for each $i \in V(D)$, then $\sigma(\alpha, D)$ is an monochromatic kernel perfect digraph if and only if D and D_i are monochromatic kernel perfect digraphs, for each $i \in \{1, ..., p\}$.*

ACKNOWLEDGEMENT. The authors are grateful with the anonymous referees for their corrections and suggestions that helped to improve the final version of this paper.

References

[1] P. Arpin, V. Linek, Reachability problems in edge-colored digraphs, Discrete Mathematics, 307 (2007) 2276-2289.

[2] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer-Verlag, London, 2000.

[3] C. Berge, Graphs, North-Holland Mathematical Library, The Netherlands, 1989.

[4] M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel, Some operations preserving the existence of kernels, Discrete Math. 205 (1999) 211-216.

[5] M. Borowiecki and A. Szelecka, One-factorizations of the generalized Cartesian product and of the x-join of regular graphs, Discuss. Math. 13 (1993) 15-19.

[6] V. Chvátal, On the computational complexity of finding a kernel, Report CRM300, Centre de Recherches Mathématiques, Université de Montréal, 1973.

[7] W.E. Clark and S. Suen, An inequality related to Vizing's conjecture, Elec. J. Combin. 7 (#N4) (2000) 1-3.

[8] P. Delgado-Escalante and H. Galeana-Sánchez, Restricted domination in arc-colored digraphs, Discrete Appl. Math., in print.

[9] P. Duchet, A sufficient condition for a digraph to be kernel perfect, J. Graph Theory 11 (1) (1987) 81-85.

[10] B. Effantin and H. Kheddouci, Grundy Number of graphs, Discuss. Math. Graph Theory 27(2007) 5-18.

[11] A.S. Fraenkel, Planar Kernel and Grundy with $d \leq 3$, $d_{out} \leq 2$, $d_{in} \leq 2$, are NP-complete, Discrete Appl. Math. 3 (1981) 257-262.

[12] H. Galeana-Sánchez, V. Neumann-Lara, On kernels and semikernels of digraphs, Discrete Math. 48 (1984) 67-76.

[13] H. Galeana-Sánchez, On monochromatic paths and monochromatic cycles in edge coloured tournaments, Discrete Math. 156 (1996) 103-112.

[14] H. Galeana-Sánchez, A new method to extend KP-digraphs to CKI-digraphs, Discrete Math. 69 (1988) 207-209.

[15] H. Galeana-Sánchez and V. Neumann-Lara, On the dichromatic number in kernel theory, Math. Slovaca 48 (1998) 213-219.

[16] H. Galeana-Sánchez, Kernels in edge coloured digraphs, Discrete Math. 184 (1998) 87-99.

[17] H. Galeana-Sánchez, On the existence of (k,l)-kernels in digraphs, Discrete Math. 85 (1990) 99-102.

[18] H. Galeana-Sánchez, L. Pastrana-Ramírez, Kernels in edge coloured line digraphs, Discuss. Math. Graph Theory 18 (1998) 91-98.

[19] G. Hahna, P. Ille, Robert E. Woodrow, Absorbing sets in arc-coloured tournaments, Discrete Math. 283 (2004) 93-99.

[20] F. Harary, Graph Theory, Addison-Wesley, Reading (1972) 274pp.

[21] M. Kucharska, On (k,l)-kernels of orientations of special graphs, Ars Combin. 60 (2001) 137-147.

[22] M. Kucharska, On (k,l)-kernel perfectness of special classes of digraphs, Discuss. Math. Graph Theory 25 (2005) 103-119.

[23] M. Kwaśnik, On (k,l)-kernels in graphs and their products (PhD Thesis, Technical University of Wrocław, Wrocław, 1980).

[24] V. Linek, B. Sands, A note on paths in edge-colored tournaments, Ars Combin. 44 (1996) 225-228.

[25] S. Minggang, On monochromatic paths in m-coloured tournaments, J. Combin. Theory (B) 45 (1988) 108-111.

[26] M. Richardson, Solutions of irreflexive relations, Ann. Math. 58 (2) (1953) 573.

[27] M. Richardson, Extensions theorems for solutions of irreflexive relations, Proc. Natl. Acad. Sci. USA 39 (1953) 649.

[28] B. Sands, N. Sauer and R.Woodrow, On monochromatic paths in edge coloured digraphs, J. Combin. Theory (B) 33 (1982) 271-275.

[29] W. Szumny, A. Włoch and Iwona Włoch, On (k,l)-kernels in D-join of digraphs, Discuss. Math. Graph Theory 27 (2007) 457-470.

[30] J. Topp, Kernels of digraphs formed by some unary operations from other digraphs, J. Rostock Math. Kolloq. 21 (1982) 73-81.

[31] J. Von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton University Press, Princeton, NJ, 1944.

[32] A. Włoch and I. Włoch, On (k,l)-kernels in generalized products, Discrete Math. 164 (1997) 295-301.

[33] I. Włoch, On kernels by monochromatic paths in D-join, Ars Combinatoria 99 (2011) - in print.