Directed Hamilton cycle decompositions of the tensor product of symmetric digraphs

R.S. Manikandan[†], P. Paulraja and S. Sivasankar
Department of Mathematics
Annamalai University
Annamalainagar 608 002
India

Abstract

The first two authors have shown, in [13], that if $K_{r,r} \times K_m$, $m \geq 3$, is an even regular graph, then it is Hamilton cycle decomposable, where $\times$ denotes the tensor product of graphs. In this paper, it is shown that if $K_{r,r} \times K_m$ is odd regular, then $(K_{r,r} \times K_m)^*$ is directed Hamilton cycle decomposable, where $(K_{r,r} \times K_m)^*$ denotes the symmetric digraph of $K_{r,r} \times K_m$.

Keywords: Tensor product, Wreath product, Hamilton cycle decomposition.

1 Introduction

All graphs considered here are simple and finite. A k-regular graph G is called *Hamilton cycle decomposable* if G is decomposable into $k/2$ Hamilton cycles when k is even and into $(k-1)/2$ Hamilton cycles together with a perfect matching when k is odd. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k)$, then we write $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$.

For a graph G, G^* is obtained from G by replacing every edge of G by a symmetric pair of arcs. $K^*_{m,m}$ and K^*_n are the complete symmetric

[†]Department of Mathematics, Velalar college of Engineering and Technology, Erode - 638 009, India.

balanced bipartite digraph on $2m$ vertices and the complete symmetric digraph of order n, respectively.

Let C_k (resp. $\vec{C}_k$) denote a cycle (resp. directed cycle) of length k and let P_k (resp. $\vec{P}_k$) denote a path (resp. directed path) on k vertices. A C_k-*factor* of G is a spanning subgraph H of G such that each component of H is a C_k. Partitioning the edge set of G into C_k-factors is called a C_k*-factorization* of G. If G admits a C_k-factorization, then we denote it by $C_k \| G$.

For two simple graphs (resp. digraphs) G and H their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge (resp. arc) whenever (g_1, g_2) is an edge (resp. arc) in G and (h_1, h_2) is an edge (resp. arc) in H. Similarly, the *wreath product* of the graphs (resp. digraphs) G and H, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge (resp. arc) whenever (g_1, g_2) is an edge (resp. arc) in G, or $g_1 = g_2$ and (h_1, h_2) is an edge (resp. arc) in H. It is well known [9] that the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \ldots \oplus (H_k \times H)$.

We shall use the following notation throughout this note. Let G and H be simple graphs with vertex sets $V(G) = \{x_0, x_1, \ldots, x_{m-1}\}$ and $V(H) = \{y_0, y_1, \ldots, y_{n-1}\}$. Then $V(G \times H) = V(G) \times V(H)$ and for our convenience, we write $V(G) \times V(H) = \bigcup_{i=0}^{m-1} X_i$, where X_i stands for $\{x_i\} \times V(H)$. Further, in the sequel, we shall denote the vertices of X_i by $\{x_j^i \mid 0 \le j \le n-1\}$, where x_j^i stands for the vertex (x_i, y_j). We shall call X_i, $0 \le i \le m-1$, the *ith layer of* $G \times H$. It is clear that $G \times H$ is an m-partite graph with parts $X_0, X_1, \ldots, X_{m-1}$ (It can also be considered as an n-partite graph with parts $Y_0, Y_1, \ldots, Y_{n-1}$, where $Y_i = V(G) \times \{y_i\}$).

If G is a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, \ldots, x_{n-1}\}, Y = \{y_0, y_1, \ldots, y_{n-1}\}$ and if G contains the set of edges $F_i(X, Y) = \{x_j y_{i+j} \mid 0 \le j \le n-1$, where addition in the subscript is taken modulo $n\}$, $0 \le i \le n-1$, then we say that G has the *1-factor of distance* i *from* X *to* Y. Note that $F_i(X, Y) = F_{n-i}(Y, X)$, $0 \le i \le n-1$. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$. In a bipartite graph with bipartition (X, Y) with $|X| = |Y|$, if $x_i y_j$ is an edge, then $x_i y_j$ is called an *edge of distance* $j - i$ if $i \le j$, or $n - (i - j)$, if $i > j$, *from* X *to* Y (The same edge is said to be of *distance* $i - j$ if $i \ge j$ or $n - (j - i)$, if $i < j$, *from* Y *to* X). For $S \subseteq V(G)$, $\langle S \rangle$ denotes the *subgraph of* G *induced by* S. Definitions which are not seen here can be found in [5] or [8].

It is well known [11] that K_n is Hamilton cycle decomposable. Also, the k-cycle system problem of decomposing K_n into cycles of length k if $k \mid \binom{n}{2}$ when n is odd, or if $k \mid (\binom{n}{2} - (\frac{n}{2}))$ when n is even has been settled recently [2, 18]. More recently, Alspach et al. [3] have solved the problem of decomposing K_n^* into $\vec{C}_k$ if $k \mid n(n-1)$.

For the odd regular graph K_{2m}, Tillson [20] obtained a directed Hamilton cycle decomposition of K_{2m}^*, for $2m \geq 8$. In this note, we discuss the directed Hamilton cycle decomposition of $(K_{r,r} \times K_m)^*$. The problem of finding Hamilton cycle decompositions of product graphs is not new. Hamilton cycle decompositions of various products have been studied in [1, 4, 6, 7, 10, 13]. In [16], Ng has obtained a partial solution to the following conjecture of Alspach et al. [1]: If D_1 and D_2 are Hamilton cycle decomposable digraphs, then $D_1 \circ D_2$ is Hamilton cycle decomposable. Also, Ng [17] has proved that the complete symmetric r-partite regular digraph $K_r^*(s)$ is decomposable into directed hamiltonian cycles if and only if $(r, s) \neq (4, 1)$ or $(6, 1)$. It has been shown that $K_r \times K_s$ is Hamilton cycle decomposable [6]. Manikandan and Paulraja [13] have proved that $K_{r,r} \times K_s$ is Hamilton cycle decomposable. If $K_{r,r} \times K_s$ is an even regular graph, then it is easy to see that $(K_{r,r} \times K_s)^*$ is directed Hamilton cycle decomposable. However, it is not trivial to see if $(K_{r,r} \times K_s)^*$ is directed Hamilton cycle decomposable when $K_{r,r} \times K_s$ is odd regular. In this note, we prove that $(K_{r,r} \times K_s)^*$ is directed Hamilton cycle decomposable. We prove the following

Theorem 1.1. *For $r \geq 2$, $(K_{r,r} \times K_m)^*$ admits a directed Hamilton cycle decomposition except possibly when r is odd and $m = 4$, and $(r, m) = (3, 6)$.*

2 Proof of the Theorem

First we prove a few lemmas, then using them we prove Theorem 1.1. The following theorems will be used to prove the main result of this note.

Theorem 2.1. *[3] For positive integers m and n, with $2 \leq m \leq n$, the digraph K_n^* can be decomposed into directed cycles of length m if and only if m divides the number of arcs in K_n^* and $(n, m) \neq (4, 4), (6, 3), (6, 6)$.* ∎

Theorem 2.2. *[13] For $m \geq 3$, $K_{r,r} \times K_m$ has a Hamilton cycle decomposition.* ∎

Theorem 2.3. *[14] For $m \geq 3$ and $k \geq 2$, $C_{2k+1} \parallel C_{2k+1} \times K_m$.* ∎

Lemma 2.4. *For $r \geq 2$, the complete symmetric bipartite digraph, $K_{r,r}^*$, is directed Hamilton cycle decomposable.*

Proof. Let the bipartition of $K^*_{r,r}$ be (X, Y), with $X = \{x_0, x_1, \ldots, x_{r-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{r-1}\}$. Let $\vec{H}_i = (x_0, y_{1+i}, x_1, y_{2+i}, x_2, y_{3+i}, x_3, \ldots, x_{r-3}, y_{r+i-2}, x_{r-2}, y_{r+i-1}, x_{r-1}, y_{0+i}, x_0)$, $0 \le i \le r-1$, where the additions in the subscripts are taken modulo r. Clearly, these r directed Hamilton cycles are arc-disjoint. ■

The above lemma can also be obtained from [19].

Lemma 2.5. *For $m \ge 3$ and $k \ge 2$, $P_{2k+2} \parallel P_{2k+2} \times K_m$.*

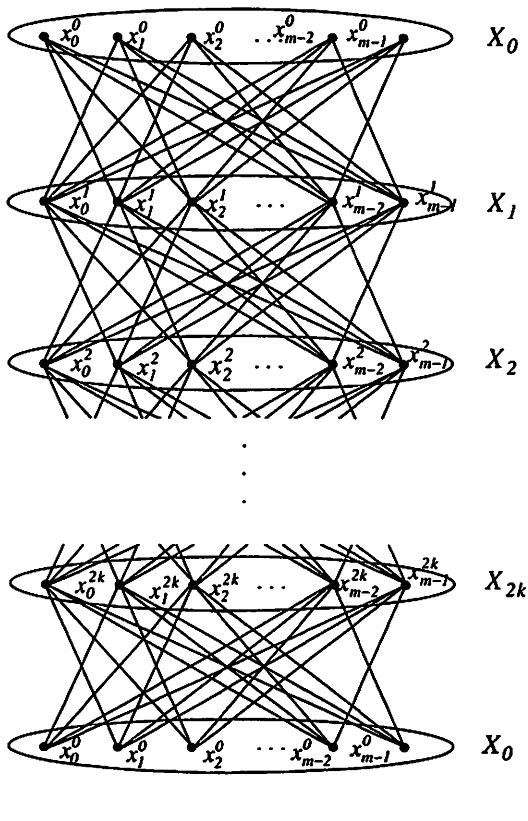
Proof. Let $V(C_{2k+1}) = \{x_0, x_1, x_2, \ldots, x_{2k}\}$. View the graph $C_{2k+1} \times K_m$ as follows: assume that the layers $X_0, X_1, X_2, \ldots, X_{2k}$ of $C_{2k+1} \times K_m$ are arranged one after the other, beginning with X_0, see Figure 2.1; instead of joining the edges from X_{2k} to X_0 upwards, keep a copy of X_0 after X_{2k} and the edges are joined from X_{2k} to that X_0 (that is, an identification of the last layer X_0 with the first layer X_0 we get $C_{2k+1} \times K_m$). The resulting graph is precisely $P_{2k+2} \times K_m$ with layers $X_0, X_1, X_2, \ldots, X_{2k}$ and X_0. Clearly, $C_{2k+1} \times K_m$ admits a C_{2k+1}-factorization, by Theorem 2.3, and any $2k+1$ cycle of a C_{2k+1}-factor of $C_{2k+1} \times K_m$ becomes a path P_{2k+2} in this drawing (observe that any $2k+1$ cycle of $C_{2k+1} \times K_m$ must meet all of its layers). Hence corresponds to a C_{2k+1}-factor of $C_{2k+1} \times K_m$ there is a P_{2k+2} factor of $P_{2k+2} \times K_m$ and hence the result follows. ■

Remark 2.6. *By the construction of the P_{2k+2}-factorization obtained in the above Lemma 2.5, it is clear that the origin and terminus of any path of a P_{2k+2}-factor are in the single column of $P_{2k+2} \times K_m$.* ■

Lemma 2.7. *For $m \ge 3$ and $m \ne 4, 6$, $\vec{C}_{2r} \times K^*_m$ is directed Hamilton cycle decomposable.*

Proof. Let $V(C_{2r}) = \{x_0, x_1, \ldots, x_{2r-1}\}$. Let $V(C_{2r} \times K_m) = \bigcup_{i=0}^{2r-1} X_i$, where $X_i = \{x^i_0, x^i_1, \ldots, x^i_{m-1}\}$. By Theorem 2.1, $K^*_m = \vec{H}_1 \oplus \vec{H}_2 \oplus \ldots \oplus \vec{H}_{m-1}$, where $\vec{H}_i$ is a directed Hamilton cycle of K^*_m. Let $V(K^*_m) = \{y_0, y_1, \ldots, y_{m-1}\}$; to each directed Hamilton cycle $\vec{H}_i$ of K^*_m, we obtain the 1-factor F'_i of the induced subgraph $\langle X_{2r-1} \cup X_0 \rangle$ of $C_{2r} \times K_m$, as follows: $F'_i = \{(x^{2r-1}_i, x^0_j) \mid (y_i, y_j) \in \vec{H}_i\}$, $1 \le i \le m-1$.

Clearly, $(C_{2r} \times K_m) \backslash E(\langle X_{2r-1} \cup X_0 \rangle) \cong P_{2r} \times K_m$; obtain a P_{2r}-factorization $\mathcal{F} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{m-1}\}$ of $P_{2r} \times K_m$ as in Lemma 2.5. Each factor $\mathcal{P}_i$ has m path-components, and each path-component of $\mathcal{P}_i$ has its origin x^0_l and terminus x^{2r-1}_l for some $l \in \{0, 1, \ldots, m-1\}$, see the Remark 2.6. Clearly, $H'_i = \mathcal{P}_i \cup F'_i$, $1 \le i \le m-1$, is a Hamilton cycle of $C_{2r} \times K_m$. Thus $\{H'_1, H'_2, \ldots, H'_{m-1}\}$ is a Hamilton cycle decomposition of $C_{2r} \times K_m$. Orient all edges of H_i', $1 \leq i \leq m-1$, from the layer X_i to X_{i+1} for $i = 0, 1, \ldots, 2r-1$, where addition in the subscript is taken modulo $2r$. Let $\vec{H}_i'$ be the corresponding directed Hamilton cycle in the resulting oriented graph. The resulting directed graph is $\vec{C}_{2r} \times K_m^*$ and $\vec{H}_1', \vec{H}_2', \ldots, \vec{H}_{m-1}'$ is a directed Hamilton cycle decomposition of it. ∎



$C_{2k+1} \times K_m$

Figure 2.1

Lemma 2.8. *For odd $r \geq 5$, $C_{2r} \times K_6$ has a Hamilton cycle decomposition.*

Proof. Let $V(C_{2r} \times K_6) = \bigcup_{i=0}^{2r-1} X_i$, where $X_i = \{x_0^i, x_1^i, \ldots, x_5^i\}$. By Lemma 2.5, the subgraph of $C_{2r} \times K_6$ induced by $\left\langle \bigcup_{i=0}^{2r-3} X_i \right\rangle$ has a path factorization $\mathscr{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5\}$ such that for each i, $1 \leq i \leq 5$, each path-component of $\mathcal{P}_i$ has its end vertices x_l^0 and x_l^{2r-3}, for some l, $0 \le l \le 5$, see the Remark 2.6. Let $F_k(X_i, X_j)$ denote the 1-factor of distance k from X_i to X_j. Let

$H_1 = \mathcal{P}_1 \cup F_1(X_{2r-3}, X_{2r-2}) \cup F_1(X_{2r-2}, X_{2r-1}) \cup F_3(X_{2r-1}, X_0)$,
$H_2 = \mathcal{P}_2 \cup F_2(X_{2r-3}, X_{2r-2}) \cup F_2(X_{2r-2}, X_{2r-1}) \cup F_1(X_{2r-1}, X_0)$,
$H_3 = \mathcal{P}_3 \cup F_3(X_{2r-3}, X_{2r-2}) \cup F_3(X_{2r-2}, X_{2r-1}) \cup F_5(X_{2r-1}, X_0)$,
$H_4 = \mathcal{P}_4 \cup F_4(X_{2r-3}, X_{2r-2}) \cup F_5(X_{2r-2}, X_{2r-1}) \cup F_4(X_{2r-1}, X_0)$ and
$H_5 = \mathcal{P}_5 \cup F_5(X_{2r-3}, X_{2r-2}) \cup F_4(X_{2r-2}, X_{2r-1}) \cup F_2(X_{2r-1}, X_0)$.

Clearly, $\{H_1, H_2, \ldots, H_5\}$ is a Hamilton cycle decomposition of $C_{2r} \times K_6$. ∎

The proof of the following lemma follows from the above lemma, since $\vec{C}_{2r} \times K_6^*$ is nothing but orienting the edges of $C_{2r} \times K_6$ from X_i to X_{i+1}, where addition in the subscript of X_{i+1} is taken modulo $2r$. The Hamilton cycles of $C_{2r} \times K_6$ obtained in Lemma 2.8 with this orientation become directed Hamilton cycles of $\vec{C}_{2r} \times K_6^*$.

Lemma 2.9. *For odd $r \ge 5$, $\vec{C}_{2r} \times K_6^*$ has a directed Hamilton cycle decomposition.* ∎

Proof of Theorem 1.1.

Case 1: $K_{r,r} \times K_m$ is an even regular graph.

The result follows by a Hamilton cycle decomposition of $K_{r,r} \times K_m$, see [13].

Case 2: $K_{r,r} \times K_m$ is an odd regular graph.

Observe that $(K_{r,r} \times K_m)^* = K_{r,r}^* \times K_m^*$
$= (\vec{H}_1 \oplus \vec{H}_2 \oplus \ldots \oplus \vec{H}_r) \times K_m^*$, by Lemma 2.4,
$= (\vec{H}_1 \times K_m^*) \oplus \ldots \oplus (\vec{H}_r \times K_m^*)$.

By Lemmas 2.7 and 2.9, each $\vec{H}_i \times K_m^*$ is directed Hamilton cycle decomposable. Thus $(K_{r,r} \times K_m)^*$ is directed Hamilton cycle decomposable. ∎

Conclusion: It is shown, in [4], that if both G and H are Hamilton cycle decomposable graphs, then $G \times H$ need not be Hamilton cycle decomposable. Hence it is interesting to find pairs of Hamilton cycle decomposable graphs G_1 and G_2 such that $G_1 \times G_2$ is Hamilton cycle decomposable if $G_1 \times G_2$ is even regular or, if $G_1 \times G_2$ is odd regular, then $(G_1 \times G_2)^*$ is directed Hamilton cycle decomposable. Recently, it has been shown, see [15], that the tensor product of a pair of regular complete multipartite graphs is Hamilton cycle decomposable. In fact, R.S. Manikandan and P. Paulraja [12] have conjectured that if both G and H

are Hamilton cycle decomposable circulants and $G \times H$ is connected, then $G \times H$ is Hamilton cycle decomposable. Further, R. Balakrishnan and P. Paulraja have conjectured that if G and H are r and s regular Hamilton cycle decomposable graphs, respectively, and $G \times H$ is rs-edge connected, then it is Hamilton cycle decomposable.

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