On k-connected restrained domination in graphs

Hongyu Chen [a*] Xuegang Chen[b] Xiang Tan[c]

[a] School of Mathematics and System Sciences, Shandong University, Jinan, Shandong Province, 250100 , China
[b] Department of Mathematics, North China Electric Power University, Beijing, 102206, China
[c] School of Statistics and Mathematics ,Shandong University of Finance, Jinan, Shandong Province, 250014, China

Abstract

In this paper we initiate the study of k-connected restrained domination in graphs. Let $G = (V, E)$ be a graph. A k-connected restrained dominating set is a set $S \subseteq V$ where S is a restrained dominating set and $G[S]$ has at most k components. The k-connected restrained domination number of G, denoted by $\gamma_r^k(G)$, is the smallest cardinality of a k-connected restrained dominating set of G. First, some exact values and sharp bounds for $\gamma_r^k(G)$ are given in Section 2. Then the necessary and sufficient conditions for $\gamma_r(G) = \gamma_r^1(G) = \gamma_r^2(G)$ are given if G is a tree or a unicyclic graph in Section 3 and Section 4.

Key words: restrained domination, 2-connected restrained domination number, tree, unicyclic graph.

1 Introduction

Graph theory terminology not presented here can be found in [1]. Let $G = (V, E)$ be a graph with $|V| = n$. The degree, neighborhood and closed neighborhood of a vertex v in a graph G are denoted by $d(v), N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The graph induced by $S \subseteq V$ is denoted

*E-mail address: hongyuchen86@163.com

by $G[S]$. The number of components of G is denoted by $\omega(G)$. A vertex of degree one is called a leaf. A vertex v of G is called a stem if it is adjacent to a leaf. Let $L(G)$ denote the set of leaves. Let $C(G) = \{(u,v)|uv \in E(G), d(u) = d(v) = 2\}$. For any connected graph G a set $S \subset V$ is called a cutset of G if $G - S$ is no longer connected. A graph G is called unicyclic graph if G contains exactly one cycle. A set S is a dominating set if for every vertex $u \in V - S$ there exists $v \in S$ such that $uv \in E$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. We call a set S a γ-set if S is a dominating set with cardinality $\gamma(G)$.

A set $S \subseteq V$ is a restrained dominating set if every vertex in $V - S$ is adjacent to a vertex in S and to another vertex in $V - S$. Let $\gamma_r(G)$ denote the size of a smallest restrained dominating set. A set S is called a γ_r-set if S is a restrained dominating set with cardinality $\gamma_r(G)$. It has been studied by G.S. Domke [2] and M.A. Henning [3].

In this paper we initiate the study of k-connected restrained domination in graphs. It is a particular case of the restrained domination. Let $G = (V, E)$ be a graph. A k-connected restrained dominating set is a set $S \subseteq V$ where S is a restrained dominating set and $G[S]$ has at most k components. The k-connected restrained domination number of G, denoted by $\gamma_r^k(G)$, is the smallest cardinality of a k-connected restrained dominating set of G. We will call a set S a γ_r^k-set if S is a k-connected restrained dominating set of cardinality $\gamma_r^k(G)$. For $k = 1$, $\gamma_r^1(G)$ is the connected restrained domination number.

For G connected and $k \geq 1$, obviously, $\gamma_r(G) \leq \gamma_r^k(G) \leq \gamma_r^1(G)$.

One possible application of the concept of k-connected restrained domination is that of prisoners and guards. Here, each vertex not belonging to the k-connected restrained dominating set corresponds to a position of a prisoner, and every vertex in the k-connected restrained dominating set corresponds to a position of a guard. Note that each prisoner's position is observed by a guard's position (to effect security) while each prisoner's position is seen by at least one other prisoner's position (to protect the rights of prisoners) and the guards can be divided into at most k parts (since there are at most k posts). To be cost-effective, it is desirable to place as few guards as possible (in the sense above).

In this paper we give the following results. First, some exact values and sharp bounds for $\gamma_r^k(G)$ are given in Section 2. Then the necessary and sufficient conditions for $\gamma_r(G) = \gamma_r^1(G) = \gamma_r^2(G)$ holding are given if G is a tree or a unicyclic graph in Section 3 and Section 4.

2 General graphs

Let K_n, C_n and P_n denote, respectively, the complete graph, the cycle and the path of order n. Also, let $K_{n_1,\ n_2,\ \ldots,\ n_t}$ denote the complete multipartite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_t$ where $|S_i| = n_i$ for $1 \le i \le t$. We call $K_{1,\ n-1}$ a star.

Lemma 2.1 can be read out of results from Domke et al.[2]

Lemma 2.1 (1) If $n \ge 1$ is an integer, then $\gamma_r(P_n) = n - 2\lfloor\frac{(n-1)}{3}\rfloor$.

(2) If $n \ge 3$, then $\gamma_r(C_n) = n - 2\lfloor\frac{n}{3}\rfloor$.

(3) Let G be a connected graph of order n. Then $\gamma_r(G) = n$ if and only if G is a star.

(4) If T is a tree of order $n \ge 3$, then $\gamma_r(T) \ge \Delta(T)$. Furthermore, $\gamma_r(T) = \Delta(T)$ if and only if T is a wounded spider which is not a star.

Lemma 2.2 $\gamma_r(G) = \gamma_r^k(G)$ if and only if there exists a γ_r-set S of G such that $\omega(G[S]) \le k$.

The following result is immediate.

Proposition 2.1 (1) If $n \ne 2$ is a positive integer, then $\gamma_r^k(K_n) = 1$.

(2) If $n \ge 2$ is an integer, then $\gamma_r^k(K_{1,\ n-1}) = n$.

(3) If n_1 and n_2 are integers such that $min\{n_1, n_2\} \ge 2$, then $\gamma_r^k(K_{n_1,\ n_2}) = 2$.

(4) If $t \ge 3$ is an integer, then $\gamma_r^k(K_{n_1,\ldots,\ n_t}) = \begin{cases} 1 & if\ min\{n_1,\ldots,n_t\} = 1, \\ 2 & otherwise \end{cases}$

Theorem 2.1 If $n \ge 1$ is an integer, then $\gamma_r^k(P_n) = \max\{n-2\lfloor\frac{n-1}{3}\rfloor, n-2(k-1)\}$.

Proof. Let S be a γ_r-set of P_n. Then $|S| = n - 2\lfloor\frac{(n-1)}{3}\rfloor$ by Lemma 2.1(1). There are now two cases to consider.

Case 1. $\omega(G[S]) \le k$. By Lemma 2.2, $\gamma_r^k(P_n) = \gamma_r(P_n) = n - 2\lfloor\frac{(n-1)}{3}\rfloor$. It is easy to prove that $n - 2\lfloor\frac{(n-1)}{3}\rfloor \ge n - 2(k-1)$.

Case 2. $\omega(G[S]) > k$. Let S' be a restrained dominating set of P_n and $\omega(G[S']) \le k$. Then $|S'| \ge |S|$. Subject to these conditions, S' has been chosen to be of minimum size. Since $\omega(G[S']) \le k$, $V - S'$ has at most $k-1$ components and any component of $V - S'$ is of size exactly two. Since S' is of minimum size, we have $\gamma_r^k(P_n) = |S'| \ge n-2(k-1) \ge |S| = n-2\lfloor\frac{(n-1)}{3}\rfloor$. The result follows.

We omit the proof of the following result as it is similar to that of Theorem 2.1.

Theorem 2.2 If $n \ge 3$, then $\gamma_r^k(C_n) = \max\{n - 2\lfloor\frac{n}{3}\rfloor, n - 2k\}$.

We close this section by providing a lower bound for the k- connected restrained domination number of a tree.

Theorem 2.3 If T is a tree of order $n \geq 3$, then $\gamma_r^k(T) \geq \Delta(T)$. Furthermore, $\gamma_r^k(T) = \Delta(T)$ if and only if $k \geq \Delta(T)$ and T is a wounded spider which is not a star.

Proof. Since $\gamma_r^k(T) \geq \gamma_r(T)$, by Lemma 2.1(4), $\gamma_r^k(T) \geq \Delta(T)$. Clearly for any wounded spider T which is not a star and $k \geq \Delta(T)$, we have $\gamma_r^k(T) = \Delta(T)$. So suppose T is a tree for which $\gamma_r^k(T) = \Delta(T)$. Then there are $\Delta(T)$ components in any γ_r^k-set of T and $\gamma_r(T) = \Delta(T)$. So $\Delta(T) \leq k$ and by Lemma 2.1(4), we have T is a wounded spider which is not a star.

3 Trees

This section is devoted to proving necessary and sufficient conditions for $\gamma_r(T) = \gamma_r^1(T)$, $\gamma_r^1(T) = \gamma_r^2(T)$ and $\gamma_r(T) = \gamma_r^2(T)$.

3.1 $\gamma_r(T) = \gamma_r^1(T)$

Lemma 3.1 If T is a tree of order n, then $\gamma_r^1(T) = n$.

The following theorem is immediate from Lemma 2.1(3) and Lemma 3.1.

Theorem 3.1 Let T be a tree of order n. Then $\gamma_r(T) = \gamma_r^1(T)$ if and only if T is a star.

3.2 $\gamma_r^1(T) = \gamma_r^2(T)$

Theorem 3.2 Let T be a tree of order $n \geq 2$. Then $\gamma_r^1(T) = \gamma_r^2(T)$ if and only if $C(T) = \emptyset$.

Proof. Let $\gamma_r^1(T) = \gamma_r^2(T)$. If $L(T) = V(T)$, then $T = K_2$ and $C(T) = \emptyset$. If $|V(T) - L(T)| = 1$, then T is a star, $C(T) = \emptyset$. Without loss of generality, we can assume that $|V(T) - L(T)| \geq 2$. If there exists $(u, v) \in C(T)$, then $V(T) - \{u, v\}$ is a 2-connected restrained dominating set of G with cardinality $n - 2$, which contradicts the fact $\gamma_r^1(T) = \gamma_r^2(T) = n$. Hence,$C(T) = \emptyset$.

Conversely, let S be a γ_r^2-set of G. Then $L(T) \subseteq S$. For any $v \in V(T) - L(T)$, if $v \notin S$, then there exists a vertex $u \notin S$ such that

$uv \in E(T)$ and $d(u) = d(v) = 2$. So $(u, v) \in C(T)$, a contradiction. Hence, $V(T) - L(T) \subseteq S$, that is, $L(T) \cup (V(T) - L(T)) \subseteq S$. It follows that $V(T) \subseteq S$. So $|S| = n$, that is, $\gamma_r^1(T) = \gamma_r^2(T)$.

3.3 $\gamma_r(T) = \gamma_r^2(T)$.

Lemma 3.3 (Domke, Hattingh et al.) If T is a tree of order $n \geq 3$, then $\gamma_r(T) = n - 2$ if and only if T is obtained from P_4, P_5, or P_6 by adding zero or more leaves to the stems of the path.

Lemma 3.4 Let T be a tree of order n, then $\gamma_r(T) = \gamma_r^2(T)$ if and only if $\gamma_r(T) = n$ or $\gamma_r(T) = n - 2$ and if $\gamma_r(T) = n - 2$, there exists a γ_r-set S such that $v_1, v_2 \in V - S$ and $d(v_1) = d(v_2) = 2$.

Proof. Suppose $\gamma_r(T) = n$. Since $\gamma_r(T) \leq \gamma_r^2(T)$, it follows that $\gamma_r(T) = \gamma_r^2(T)$.

Suppose $\gamma_r(T) = n - 2$, S is a γ_r-set, $v_1, v_2 \in V - S$ and $d(v_1) = d(v_2) = 2$. Then $V - \{v_1, v_2\}$ is a 2-connected restrained dominating set. so $\gamma_r^2(T) \leq |V - \{v_1, v_2\}| = n - 2$. Since $\gamma_r^2(T) \geq \gamma_r(T) = n - 2$, we have $\gamma_r(T) = \gamma_r^2(T)$.

Conversely, if $\gamma_r(T) = \gamma_r^2(T)$, by Lemma 2.2 , there exists a γ_r-set S of T such that $\omega(T[S]) \leq 2$. There are now two cases to consider.

Case 1. $\omega(T[S]) = 1$. Then $\gamma_r(T) = n$, for otherwise, there exist at least two vertices $v_1, v_2 \in V - S$, $v_1v_2 \in E(T)$ and v_1, v_2 are adjacent to some vertex in S. It follows that T has a cycle, which is a contradiction.

Case 2. $\omega(T[S]) = 2$. Let $T[S_1], T[S_2]$ be two components of $T[S]$. Then $|V - S| \geq 2$.

If $|V - S| \geq 3$, we may assume $v_1, v_2, v_3, \ldots, v_m \in V - S$, $m \geq 3$. We consider two cases.

Case 2.1 Each vertex in $T[V - S]$ is degree one. We suppose $v_1v_2 \in E(T), v_3v_4 \in E(T)$. Let $v_1v_1' \in E(T), v_1' \in S_1$. Then v_2 can't be adjacent to some vertex in S_1. So let $v_2v_2' \in E(T), v_2' \in S_2$. Let $v_3v_3' \in E(T), v_3' \in S_1$. Then v_4 is adjacent to some vertex in S_2, say v_4'. There forms a cycle, which is a contradiction.

Case 2.2 There exists at least one vertex in $T[V - S]$ whose degree is not less than 2. say v_2. We may assume $v_1v_2 \in E(T), v_2v_3 \in E(T)$. Let $v_1v_1' \in E(T), v_1' \in S_1$. Then v_3 is adjacent to some vertex in S_2, say v_3'. It follows that v_2 can't be adjacent to any vertex in S, which is a contradiction. Hence, $|V - S| = 2$, that is, $\gamma_r(T) = n - 2$.

If the degree of one of the two vertices in $V - S$ is more than 2, say v_1, then v_1 is adjacent to at least one vertex in S_2 or in S_1 other than v_1'. No matter which case, we can get cycles, which is a contradiction. So

$d(v_1) = d(v_2) = 2$. The result follows.

The following theorem, which is the main result of this section, now follows as a corollary from Lemma 2.1(3), Lemma 3.3 and Lemma 3.4.

Theorem 3.3 Let T be a tree of order n. Then $\gamma_r(T) = \gamma_r^2(T)$ if and only if one of cases (1)-(4) below occur.

(1) T is a star.

(2) T is P_4.

(3) T is obtained from P_5 by adding zero or more leaves to one of the stems of the path.

(4) T is obtained from P_6 by adding zero or more leaves to the stems of the path.

4 Unicyclic graphs

Let G be a unicyclic graph with cycle C_m, and let X be the set of all vertices of degree 2 in C_m. Without loss of generality, we can assume that $v_1, v_2, \ldots, v_t$ is the longest path in $G[X]$. Let $C_m = v_1, v_2, \ldots, v_t, v_{t+1}, \ldots, v_m, v$

4.1 $\gamma_r(G) = \gamma_r^1(G)$

Lemma 4.1 (Domke, Hattingh et al.) Let G be a connected graph of order n containing a cycle. Then $\gamma_r(G) = n-2$ if and only if G is C_4 or C_5 or G can be obtained from C_3 by attaching zero or more leaves to at most two of the vertices of the cycle.

Lemma 4.2 Let G be a connected unicyclic graph of order n. Then $\gamma_r(G) = \gamma_r^1(G)$ if and only if $\gamma_r(G) = n-2$ and there exists a γ_r-set S such that $V - S$ is not a cutset.

Proof. Suppose $\gamma_r(G) = n-2$, S is a γ_r-set of G, $v_1, v_2 \in V - S$ and $\{v_1, v_2\}$ is not a cutset. Then $V - \{v_1, v_2\}$ is a connected restrained dominating set. So $\gamma_r^1(G) \leq |V - \{v_1, v_2\}| = n-2$. Since $\gamma_r^1(G) \geq \gamma_r(G) = n-2$, we have $\gamma_r(G) = \gamma_r^1(G)$.

Conversely, let $\gamma_r(G) = \gamma_r^1(G)$. By Lemma 2.2, there exists a γ_r-set S of G such that $\omega(G[S]) = 1$. If $|S| = n$, by Lemma 2.1(3), G is a star, which contradicts the fact that G is a unicyclic graph. So $|V - S| \geq 2$.

If $|V - S| \geq 3$, we may assume $v_1, v_2, v_3, \ldots, v_t \in V - S$, $t \geq 3$. We consider two cases.

Case 1 $v_1v_2 \in E(G)$, $v_2v_3 \in E(G)$. Let $v_1v_1', v_2v_2', v_3v_3' \in E(G)$, $v_1', v_2', v_3' \in S$.

Case 2 $v_1v_2 \in E(G)$, $v_3v_4 \in E(G)$. Let v_1v_1', v_2v_2', v_3v_3', $v_4v_4' \in E(G)$, $v_1', v_2', v_3', v_4' \in S$.

No matter which case, G is not a unicyclic graph, a contradiction. Hence, $|V - S| = 2$, that is $\gamma_r(G) = n - 2$.

It is obvious that $V - S$ is not a cutset. The result follows.

Theorem 4.1 Let G be a connected unicyclic graph of order n. Then $\gamma_r(G) = \gamma_r^1(G)$ if and only if one of cases (1)-(3) below occur.

(1) G is C_4.

(2) G is C_5.

(3)G is obtained from C_3 by attaching zero or more leaves to at most one of the vertices of the cycle.

4.2 $\gamma_r^1(G) = \gamma_r^2(G)$

Lemma 4.3 If G is a connected unicyclic graph of order n, then $\gamma_r^1(G) = n$ or $n - 2$.

Lemma 4.4 Let C_m be a cycle with m vertices. Then $\gamma_r^1(C_m) = \gamma_r^2(C_m)$ if and only if $m = 3, 4, 5$.

Lemma 4.5 Let G be a unicyclic graph with cycle C_m and $5 \le t \le |X| \le m - 1$. Then $\gamma_r^1(G) \ne \gamma_r^2(G)$.

Lemma 4.6 Let G be a unicyclic graph with cycle C_m. If $|X| = m - 1, m \ne 5$, then $\gamma_r^1(G) = \gamma_r^2(G)$ if and only if the following conditions hold:

(a) $3 \le m \le 4$.

(b) Suppose $d(v_m) \ge 3$. Let $G' = G - \{v_1\}$. Then $C(G') = \emptyset$.

Proof. Let $\gamma_r^1(G) = \gamma_r^2(G)$. By Lemma 4.5, it follows that $3 \le m \le 4$. It is obvious that $\gamma_r^1(G') = n - 1$, $\gamma_r^1(G) = \gamma_r^2(G) = n - 2$.

Let S be a γ_r^2-set of G'. Then $v_m \in S$. For otherwise, if $v_m \in V(G') - S$, then there exists a vertex $v \in V(G') - S$ such that $v_m v \in E(G)$. So $|S| \le n - 3$ and S is a 2-connected restrained dominating set of G. Hence $\gamma_r^2(G) \le |S| \le n - 3$, which is a contradiction.

If there exists a vertex u in $V(G')$ which is not in S, then there exists another vertex $v \in V(G') - S$ such that $uv \in E(G)$. So $|S| \le n - 3$. Then $S - \{v_2\}$ is a 2-connected restrained dominating set of G. Hence, $\gamma_r^2(G) \le |S| - 1 \le n - 4$, a contradiction. So $V(G') \subseteq S$, that is, $\gamma_r^2(G') \ge$

$n-1$. Since $\gamma_r^2(G') \le \gamma_r^1(G') = n-1$, $\gamma_r^1(G') = \gamma_r^2(G')$. By Theorem 3.2, $C(G') = \emptyset$.

Conversely, let S be a γ_r^2-set of G. Then $V(G') - \{v_2, v_3\} \subseteq S$, for otherwise, if $u \in (V(G') - \{v_2, v_3\}) - S$, then there exists $v \in V(G') - S$ such that $uv \in E(G)$, $d_{G'}(u) = d_{G'}(v) = 2$, which contradicts the fact $C(G') = \emptyset$. It is obvious that G cannot be dominated by $V(G') - \{v_2, v_3\}$, so $|S| \ge n-2$. Since $\gamma_r^2(G) \le \gamma_r^1(G) = n-2$, $\gamma_r^1(G) = \gamma_r^2(G)$. The result follows.

Lemma 4.7 Let G be a unicyclic graph with cycle C_m. If $|X| = m-1, m = 5$, let $G' = G - \{v_1, v_2\}$, then $\gamma_r^1(G) = \gamma_r^2(G)$ if and only if $C(G') = \emptyset$.

Proof. Let $\gamma_r^1(G) = \gamma_r^2(G)$. It is obvious that $\gamma_r^1(G') = \gamma_r^1(G) = \gamma_r^2(G) = n-2$. Let S be a γ_r^2-set of G'. If there exists $(u,v) \in C(G')$, then $\gamma_r^2(G') < \gamma_r^1(G')$ by Theorem 3.2. If $v_m \in S$, then S is a 2-connected restrained dominating set of G. Hence, $\gamma_r^2(G) \le \gamma_r^2(G') < \gamma_r^1(G') = \gamma_r^1(G)$, which is a contradiction. If $v_m \notin S$, then $|N(v_m) - \{v_1, v_4\}| = 1$. Let $N(v_m) - \{v_1, v_4\} = \{u\}$. It is obvious either v_4 or u is not in S. So $|S| \le n-4$ and $S \cup \{v_2\}$ is a 2-connected restrained dominating set of G. Then $\gamma_r^2(G) \le |S \cup \{v_2\}| \le n-3$, which is a contradiction. Hence, $C(G') = \emptyset$.

Conversely, by Theorem 3.2, it follows that $\gamma_r^1(G') = \gamma_r^2(G') = n-2$. Since $\gamma_r^1(G') = \gamma_r^1(G)$, it follows that $\gamma_r^1(G) = \gamma_r^2(G')$. Let S be a γ_r^2-set of G. If $v_1, v_2 \notin S$, then S is a 2-connected restrained dominating set of G'. So $\gamma_r^1(G) = \gamma_r^2(G') \le \gamma_r^2(G)$. It follows that $\gamma_r^1(G) = \gamma_r^2(G)$.

If $v_1, v_2 \in S$, then there exist at least two adjacent vertices which are not in S. Since $C(G') = \emptyset$, the two vertices are v_3, v_4. So $(S - \{v_1, v_2\}) \cup \{v_3, v_4\}$ is a 2-connected restrained dominating set of G'. It follows that $\gamma_r^1(G) = \gamma_r^2(G') \le |S| = \gamma_r^2(G)$. Hence, $\gamma_r^1(G) = \gamma_r^2(G)$.

If one of the vertices v_1 and v_2 is in S, then $v_1 \in S, v_2 \notin S$. For otherwise, if $v_2 \in S, v_1 \notin S$, then $v_m \notin S$. Since $C(G') = \emptyset$, $d(v_m) \ge 4$. Then S has at least three components, which is a contradiction. So $v_3 \notin S$. Since $C(G') = \emptyset$, $v_m \in S$. It follows that $(S - \{v_1\}) \cup \{v_3\}$ is a 2-connected restrained dominating set of G'. So $\gamma_r^1(G) = \gamma_r^2(G') \le |S| = \gamma_r^2(G)$. Hence, $\gamma_r^1(G) = \gamma_r^2(G)$. The result follows.

Lemma 4.8 Let G be a unicyclic graph with cycle C_m and $|X| \le m-2$. Suppose $t = 0, 1$. Then $\gamma_r^1(G) = \gamma_r^2(G)$ if and only if $C(G) = \emptyset$ and if $v_i \in V(C_m)$, $d(v_i) = 3$, then for any vertex $u \in N(v_i)$, $d(u) \ne 2$.

Proof. Let $\gamma_r^1(G) = \gamma_r^2(G)$ and let S be a γ_r^1-set of G. Then $S = V(G)$. If there exists $(u,v) \in C(G)$, then $S - \{u,v\}$ is a 2-connected restrained dominating set of G. So $\gamma_r^2(G) \leq |S| - 2 < |S| = \gamma_r^1(G)$, a contradiction. Hence, $C(G) = \emptyset$.

Suppose $v_i \in V(C_m)$ and $d(v_i) = 3$. If $uv_i \in E(G)$ and $d(u) = 2$, then $S - \{u, v_i\}$ is a 2-connected restrained dominating set of G. So $\gamma_r^2(G) < \gamma_r^1(G)$, a contradiction. Hence, for any vertex $u \in N(v_i)$, $d(u) \neq 2$.

Conversely, let S be a γ_r^2-set of G. Then $V(G) \subseteq S$. For otherwise, if $u \in V(G) - S$, then there exists another vertex $v \in V(G) - S$ such that $uv \in E(G)$. It is obvious $2 \leq d(u) \leq 3$, $2 \leq d(v) \leq 3$. There are now three cases to consider.

Case 1. $d(u) = d(v) = 2$, which contradicts the fact $C(G) = \emptyset$.

Case 2. $d(u) = d(v) = 3$. Then S has at least three components, a contradiction.

Case 3. One of the vertices u and v is degree two, the other is degree three. We may assume $d(u) = 2$, $d(v) = 3$. It is obvious $u, v \notin V(C_m)$. Then S has at least three components, a contradiction.

So $S = V(G)$, that is $\gamma_r^1(G) = \gamma_r^2(G)$.

Lemma 4.9 Let G be a unicyclic graph with cycle C_m and $|X| \leq m-2$. Suppose $t = 2$. Let $G' = G - \{v_1\}$. Then $\gamma_r^1(G) = \gamma_r^2(G)$ if and only if $C(G') = \emptyset$ and any vertex of degree three in C_m has at most one neighbor of degree two.

Proof. Let $\gamma_r^1(G) = \gamma_r^2(G)$. The proof of $C(G') = \emptyset$ is similar to that of Lemma 4.6. If $v_i \in V(C_m)$, $d(v_i) = 3$ and $u_1, u_2 \in N(v_i)$, $d(u_1) = d(u_2) = 2$, then $V(G) - \{u_1, u_2, v_i\}$ is a 2-connected restrained dominating set of G. So $\gamma_r^2(G) \leq n - 3$, but $\gamma_r^1(G) = \gamma_r^2(G) = n - 2$, a contradiction. Hence, any vertex of degree three in C_m has at most one neighbor of degree two.

Conversely, let S be a γ_r^2-set of G. Then at least one of the vertices v_3 and v_m is in S, for otherwise, S has at least three components, a contradiction. We consider three cases:

Case 1. $v_3 \in S, v_m \notin S$. Then $d(v_m) = 3$, $v_1 \notin S$ and $N(v_m) - \{v_1\} \subseteq S$. All vertices in $V(G) - \{v_1, v_m\}$ are in S, for otherwise, S has at least three components, a contradiction. So $S = V(G) - \{v_1, v_m\}$, that is $\gamma_r^2(G) = n - 2 = \gamma_r^1(G)$.

Case 2. $v_m \in S, v_3 \notin S$. The proof is similar to that of Case 1.

Case 3. $v_3 \in S, v_m \in S$. If $v_1, v_2 \notin S$, then any vertex in $V(G) - \{v_1, v_2\}$ is in S. For otherwise, S has at least three components, a contradiction. So $S = V(G) - \{v_1, v_2\}$, that is $\gamma_r^2(G) = n - 2 = \gamma_r^1(G)$. If $v_1, v_2 \in S$, since $\gamma_r^2(G) \leq \gamma_r^1(G) = n - 2$, there exist at least two vertices which are not in S. Let $u, v \notin S$, $uv \in E(G)$. Since $C(G') = \emptyset$, one of the vertices u and v is a vertex of degree three in C_m and the other is a vertex of degree two. Assume $d(u) = 3$, $d(v) = 2$. Then $N(u) - \{v\} \subseteq S$. Any vertex in $V(G) - \{u, v\}$ is in S. For otherwise, S has at least three components, a contradiction. So $S = V(G) - \{u, v\}$, that is $\gamma_r^2(G) = \gamma_r^1(G)$. The result follows.

with a similar proof to that of Lemma 4.7 and Lemma 4.9, the following lemma holds.

Lemma 4.10 Let G be a unicyclic graph with cycle C_m and $|X| \leq m-2$. Suppose $t = 3, 4$. Let $G' = G - \{v_1, v_2\}$, then $\gamma_r^1(G) = \gamma_r^2(G)$ if and only if $C(G') = \emptyset$ and any vertex of degree three in C_m has at most one neighbor of degree two.

The following theorem, which is the main result of this section, now as a corollary from Lemmas 4.4, 4.6, 4.7, 4.8, 4.9 and 4.10.

Theorem 4.2 Let G be a unicyclic graph with cycle C_m. Then $\gamma_r^1(G) = \gamma_r^2(G)$ if and only if one of the following conditions holds:

(a) Suppose $|X| = m$. Then $m = 3, 4, 5$.

(b) Suppose $|X| = m - 1$, $m \neq 5$. Let $G' = G - \{v_1\}$. Then $3 \leq m \leq 4$ and $C(G') = \emptyset$.

(c) Suppose $|X| = m-1$, $m = 5$. Let $G' = G-\{v_1, v_2\}$. Then $C(G') = \emptyset$.

(d) Suppose $|X| \leq m - 2$, $t = 0, 1$. Then $C(G) = \emptyset$ and if $v_i \in V(C_m)$, $d(v_i) = 3$, then for any vertex $u \in N(v_i)$, $d(u) \neq 2$.

(e) Suppose $|X| \leq m - 2$, $t = 2$. Let $G' = G - \{v_1\}$. Then $C(G') = \emptyset$ and any vertex of degree three in C_m has at most one neighbor of degree two.

(f) Suppose $|X| \leq m-2, t = 3, 4$. Let $G' = G-\{v_1, v_2\}$. Then $C(G') = \emptyset$ and any vertex of degree three in C_m has at most one neighbor of degree two.

References

[1] J.A. Bondy and U.S.R. Murty, Graph theory with applications, *The Macmillan Press Ltd*, New York, 1976.

[2] G.S. Domke, J.H. Hattingh et al. Restrained domination in graphs, *Discrete Math.*, **203**(1999), 61-69.

[3] M.A. Henning, Graphs with large restrained domination number, *Discrete Math.*, **197/198**(1999), 415-429.

[4] Dexiang Ma, Xuegang Chen et al. On the total restrained domination in graphs. *Czechoslovak Math. J.*, **55(130)**(2005), 165-173.

[5] T.W. Haynes, S.T.Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[6] M.Kouider and P.D. Vestergaard, Generalized connected domination in graphs, *Theoret. Comput. Sci.*, **8**(2006), 57-64.

[7] Hartnell and P.Vestergaard, Dominating sets with at most k components, *Ars Combin.*, **74**(2005), 223-229.

[8] G.S. Domke, J.H. Hattingh et al. Restrained domination in trees, *Discrete Math.*, **211**(2000), 1-9.

[9] Xuegang Chen, On graphs with equal total domination and connected domination numbers, *Appl. Math. Lett.*, **19**(2006), 472-477.

[10] S.Arumugam and J.Paulraj Joseph, On graphs with equal domination and connected domination numbers, *Discrete Math.*, **206**(1999), 45-49.