Some results on neighbourhood highly irregular graphs

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Abstract

Let $G = (V, E)$ be a finite simple connected graph. For any vertex v in V, let $N_G(v)=\{u \in V : uv \in E\}$ be the open neighbourhood of v, and let $N_G[v]=N_G(v)\cup\{v\}$ be the closed neighbourhood of v. A connected graph G is said to be neighbourhood highly irregular (or simply NHI) if for any vertex $v \in V$, any two distinct vertices in the open neighbourhood of v have distinct closed neighbourhood sets. In this paper, we give a necessary and sufficient condition for a graph to be NHI. For any $n \geq 1$, we obtain a lower bound for the order of regular NHI graphs and a sharp lower bound for the order of NHI graphs with clique number n, which is better than the bound attained earlier.

Key words: Regular graphs, irregular graphs, neighbourly irregular graphs, highly irregular graphs, k-neighbourhood regular graphs, neighbourhood highly irregular graphs.

AMS subject classification code: 05C (primary)

1 Introduction

Throughout this paper, we consider only finite simple connected graphs. Notations and terminology are as in [5]. In a graph $G = (V, E)$, for any vertex $v \in V$, the open neighbourhood of v is the set of all vertices adjacent to v, that is, $N_G(v) = \{u \in V : uv \in E\}$. The closed neighbourhood of v is $N_G[v] = N_G(v) \cup \{v\}$. Degree of a vertex v is denoted by d(v). Clearly if u and v are two distinct vertices which are non – adjacent or with distinct degrees, then $N_G[u] \neq N_G[v]$. For any two subsets V_1 and V_2 of V, let $\langle V_1 \rangle$ denote the induced subgraph of G induced by V_1 and let $\langle V_1, V_2 \rangle$ denote the bipartite subgraph of G with bipartition (V_1, V_2) which contains all the edges of G having one end vertex in V_1 and the other in V_2. For example, $\langle V_1 \rangle$ and $\langle V_1, V_2 \rangle$ in a graph G, where $V_1 = \{v_2, v_4, v_5\}$ and $V_2 = \{v_1, v_3\}$ are shown in Figure 1.

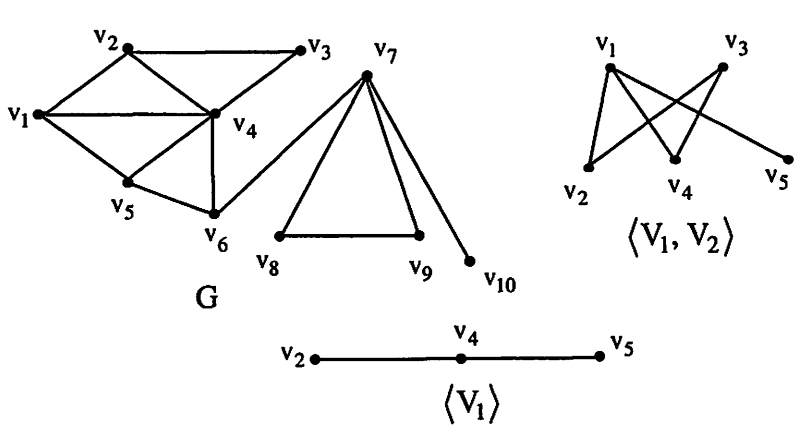

Figure 1

A connected graph G is said to be *highly irregular* (or simply HI) if each of its vertices is adjacent only to vertices with distinct degrees, that is, for any vertex u, if v and w are neighbours of u, then d(v) ≠ d(w). For example, the graphs shown in Figure 2 are highly irregular. Yousef Alavi et al. [2] introduced the concept of highly irregular graphs and established some properties of HI graphs.

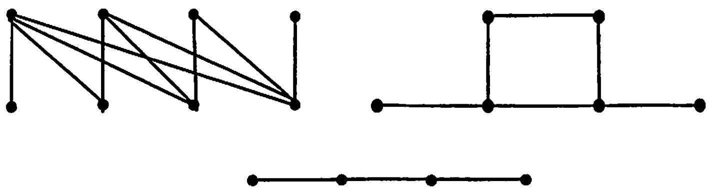

Figure 2

The concept of HI graphs has been extended to k-neighbourhood regular graphs in [6]. A connected graph G is said to be *k-neighbourhood regular* if each of its vertices is adjacent to exactly k vertices of same degree, that is, if $u \in N_G(v)$ and d(u) = m, then there are exactly k-1 other vertices of degree m in $N_G(v)$. For example, the graphs G_1 and G_2 shown in Figure 3 are 2-neighbourhood regular and 3-neighbourhood regular respectively. Note that 1-neighbourhood regular graphs are nothing but HI graphs.

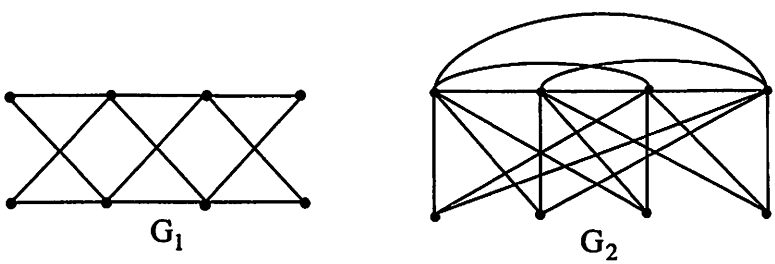

Figure 3

Inspired by these two definitions, S. Gnaana Bhragsam and S. K. Ayyaswamy [7] introduced the concept of neighbourly irregular graphs. If in a connected graph G, no two adjacent vertices have the same degree, then G is called a *neighbourly irregular graph* (or simply NI graph). The graphs shown in Figure 4 are NI graphs.

In [2], it has been noted that if v is a vertex of maximum degree d in a HI graph, then for every k, $1 \le k \le d$, v is adjacent to exactly one vertex of

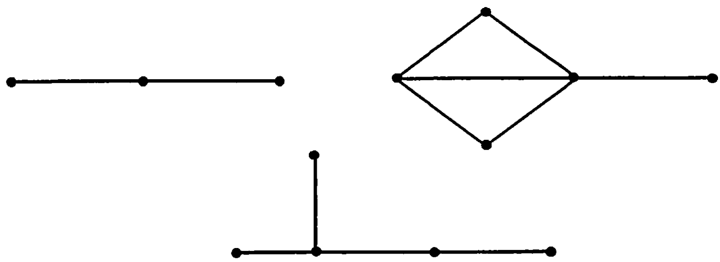

Figure 4

degree k. This forces that, any HI graph is not NI. However, the converse need not be true. For example, the graph shown in Figure 5 is neither HI nor NI.

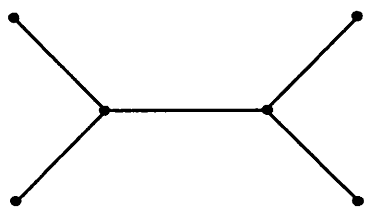

Figure 5

More results on HI graphs, k-neighbourhood regular graphs and NI graphs have been obtained in [1], [2], [3], [4], [6], [7] and [8].

Recently, V.Swaminathan and A.Subramanian [9] introduced a new class of graphs called neighbourhood highly irregular graphs. A connected graph G is said to be Neighbourhood *highly irregular* (or simply *NHI*) if for any vertex $v \in V$, $u, w \in N_G(v)$, and $u \neq w$ implies that $N_G[u] \neq N_G[w]$, that is, any two distinct neighbours of v have distinct closed neighbourhood sets.

Note that any HI graph is NHI but there are NHI graphs, which are not HI. For example, any path with at least 5 vertices is NHI but not HI.

The class of NHI graphs is wider than that of NI graphs also, that is, any NI graph is NHI. For, if a graph G is NI, then no two adjacent vertices have the same degree. Let $v \in V$, and $u, w \in N_G(v)$. If u and w are adjacent, then $d(u) \neq d(w)$ and hence $N_G[u] \neq N_G[w]$. On the other hand, if u and w are not adjacent, then $u \notin N_G[w]$, $w \notin N_G[u]$ and again $N_G[u] \neq N_G[w]$. Therefore, G is NHI. However, an NHI graph need not be NI. For example, any cycle of length $n > 3$, is NHI but not NI.

Any regular graph with at least 3 vertices is neither HI nor NI and every regular graph is k-neighbourhood regular for some k. However, for any $n > 3$, C_n is NHI where as for any $n \geq 3$, K_n is not NHI. This means that some regular graphs are NHI and some are not.

The following facts and theorems have been obtained in [9].

Fact 1 For every n, there exists an NHI graph of order n.

Fact 2 Any connected triangle free graph is NHI.

In this paper, we will prove a more general case.

Fact 3 The number of edges of an NHI graph of order $n \geq 3$ is

$$\leq \begin{cases} 2m(m-1) & \text{if } n = 2m \\ 2m^2 & \text{if } n = 2m+1 \end{cases}$$

Theorem A Every graph G of order $n \geq 2$ is an induced subgraph of an NHI graph of order 2n-k where k is the number of pendant vertices of G.

Theorem B For $n \geq 3$, the smallest order of NHI with clique number n is 2n-1.

The bound mentioned in this result is not sharp. For example, the graph shown in Figure 6 is NHI with order 6 and clique number 4. In this paper, we obtain a sharp lower bound for the order of any NHI graph with clique number n, for any $n \geq 1$.

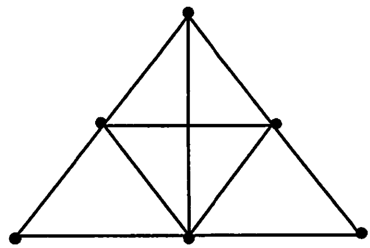

Figure 6

2 Results on NHI graphs

The following result gives a necessary and sufficient condition for a graph to be NHI.

Theorem 1 A connected graph G with $n \geq 3$ is NHI if and only if $N_G[u] \neq N_G[w]$ for any pair of adjacent vertices u and v in G with d(u) = d(v).

Proof Let G be a connected graph in which, $N_G[u] \neq N_G[w]$ for any two adjacent vertices u and v of same degree in G. However, obviously, for any two non-adjacent vertices u and v of G and for any two adjacent vertices u and v of distinct degrees, $N_G[u] \neq N_G[v]$ and therefore, G is NHI.

Conversely, assume that G is NHI. Let u and v be two adjacent vertices of same degree in V. We claim that $N_G[u] \neq N_G[v]$.

If u and v have a common neighbour w, then u and v are in $N(w)$, this implies that, $N_G[u] \neq N_G[v]$, since G is NHI. Otherwise, u and v have no common neighbour. In this case, since $n \geq 3$ and since G is connected, there is a vertex w in N(u) (in N(v)) which is not in N(v) (in N(u)). This forces that $N_G[u] \neq N_G[v]$. Hence the theorem. ■

Note that K_2 is NHI, in which $N_G[u] = N_G[v]$. In fact, K_n is the only graph in which $N_G[u] = N_G[v]$ for any two vertices u and v. For, clearly in K_n, $N_G[u] = N_G[v]$ for any two vertices u and v. In addition, if G is a graph with $N_G[u] = N_G[v]$ for any two vertices u and v, then u and v are adjacent in G. This means that, G is complete.

For any connected graph G which is not NI, let ℓ_G (or simply ℓ) denote the least positive integer such that G has two adjacent vertices of degree ℓ. Note that, $\ell \geq 2$ whenever $n \geq 3$.

Recall that for any two vertex disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the graph G = (V, E), where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \cup G_2$. The join, $G_1 \vee G_2$, of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to every vertex of G_2.

Let G be a simple graph. Then the complement G^c of G is the graph with the same vertex set V(G) in which two vertices u and v are adjacent if and only if they are non adjacent in G.

Corollary 1.1 Let G be a connected graph with $n \geq 3$. If G is NI or G contains no $K_2 \vee K^c_{\ell-1}$ as a subgraph, then G is NHI.

Proof If G is NI, then obviously G is NHI. Assume that G contains no $K_2 \vee K^c_{\ell-1}$ as a subgraph. If G is not NHI, then by the above theorem, there are two adjacent vertices u and v of same degree m in G such that $N_G[u] = N_G[v]$. Therefore, $|N_G[u]| = |N_G[v]| = m+1$ and hence $K_2 \vee K^c_{m-1}$ is a subgraph of G. Since $\ell \leq m$, this forces that G contains $K_2 \vee K^c_{\ell-1}$ as a subgraph, a contradiction. Hence G must be NHI. ∎

The converse of the above corollary need not be true. For example, the graph shown in Figure 7 is NHI but not NI with $\ell = 2$. In addition, it contains $K_2 \vee K_1^c$ as a subgraph.

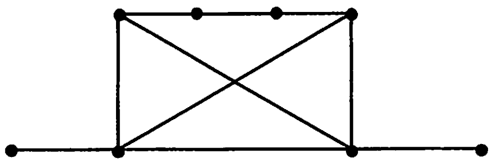

Figure 7

Corollary 1.2 Any connected triangle free graph is NHI.

Proof If n = 1 or n = 2, the result is obvious. Assume that $n \geq 3$. If G is NI, then clearly it is NHI. If G is not NI, then $\ell \geq 2$. Since G is triangle free, G contains no $K_2 \vee K_{\ell-1}^c$ and this follows that G is NHI by Corollary 1.1. ■

Corollary 1.3 Any connected bipartite graph is NHI. ■

Theorem 2 A connected graph G with $n \geq 3$ is NHI if and only if $N_{G^c}(u) \neq N_{G^c}(v)$ for any two vertices u and v.

Proof Let G be an NHI graph. Suppose there are vertices u and v such that $N_{G^c}(u) = N_{G^c}(v)$. Then u and v are not adjacent in G^c and hence adjacent in G and $N_G(u) = N_G(v)$ also. Therefore u and v have same degree in G such that $N_G[u] = N_G[v]$, which is a contradiction to Theorem 1. Hence, $N_{G^c}(u) \neq N_{G^c}(v)$ for any two vertices u and v.

Conversely, suppose G is not NHI. Again, by Theorem 1, G has two adjacent vertices u and v with same degree such that $N_G[u] = N_G[v]$. This implies that u and v are non-adjacent in G^c with $N_{G^c}(u) = N_{G^c}(v)$. That is, in G^c there are two vertices u and v such that $N_{G^c}(u) = N_{G^c}(v)$. Hence the theorem. ■

Theorem 3 For any $n \geq 5$, $K_n \setminus H$ is NHI, where H is a Hamiltonian cycle in K_n.

Proof Let the vertices of K_n be $v_0, v_1, \ldots, v_{n-1}$. Through out this proof, the operation + is addition modulo n. Let $E(H) = \{ e_i = v_i v_{i+1} : 0 \leq i \leq n\text{-}1 \}$ is an $n-3$ regular graph of order n, in which, for $0 \leq i \leq n\text{-}1$, $N(v_i) = \{v_{i+2}, v_{i+3}, \ldots, v_{n-2+i}\}$. Therefore, if v_i and v_j, $0 \leq i < j \leq n\text{-}1$, are adjacent vertices in G, then clearly for $(i, j) \notin \{(0, n\text{-}2), (1, n\text{-}1)\}$, $v_{i-1} \in N[v_j] \setminus N[v_i]$ and $v_{j+1} \in N[v_i] \setminus N[v_j]$ otherwise, $v_{i+1} \in N[v_j] \setminus N[v_i]$ and $v_{j-1} \in N[v_i] \setminus N[v_j]$, that is, $N[v_i] \neq N[v_j]$. Hence, by Theorem 1, G is NHI. ■

The above theorem can be restated as follows:

Corollary 3.1 C_n^c is NHI. ■

In a similar way, one can prove that

Corollary 3.2 $P_n^c = K_n \setminus P_n$ is NHI, for any $n > 3$. ■

For even $n \geq 4$, let the vertices of K_n be $v_1, v_2, \ldots, v_n$ and let $F = \{ v_{2i-1} v_{2i}, 1 \leq i \leq n/2 \}$ be a 1 – factor in K_n. Then in [9], it has been proved that the regular graph $K_n \setminus F$ is NHI.

Corollary 3.3 For $r \geq 2$, the smallest order of an r – regular NHI graph is

$$\begin{cases} r+2, & \text{if r is even} \\ r+3, & \text{if r is odd.} \end{cases}$$

Also the bound is strict.

Proof Let G be an r – regular NHI graph with p vertices. Then $p \geq r + 1$. If $p = r + 1$, then G is complete which is not NHI and hence $p \geq r + 2$. However, when r is even, $K_{r+2} \setminus F$ is an r – regular NHI graph on r+2 vertices, where F is a 1 – factor in K_{r+2}.

In addition, when r is odd, r + 2 is also odd and hence $p \geq r + 3$. Moreover, $K_{r+3} \setminus H$ where H is a Hamiltonian cycle in K_{r+3}, is an r – regular NHI graph on r + 3 vertices. Hence, the smallest order of the r – regular NHI graph with $r \geq 2$, is r + 2 if r is even and is r + 3 if r is odd. ■

Theorem 4 For any $n \geq 1$, the smallest order of an NHI graph with clique number n is n + m where m is the least positive integer such that $n \leq 2^m$.

To prove this theorem, we need the following two lemmas.

For any two positive integers i and k, $1 \leq k \leq i$, a $B(k, i)$ - graph is a bipartite graph with bipartition (V_1, V_2) where $|V_1| = \binom{i}{k}$ and $|V_2| = i$ in which every vertex in V_1 is of degree k and every vertex in V_2 is of degree $\binom{i-1}{k-1}$. For example, the graph shown in Figure 8 is $B(2, 4)$. The existence of such a graph is proved in Lemma 4.1. Note that when k = 1, $B(1, i)$ is a 1 - regular graph with 2i vertices.

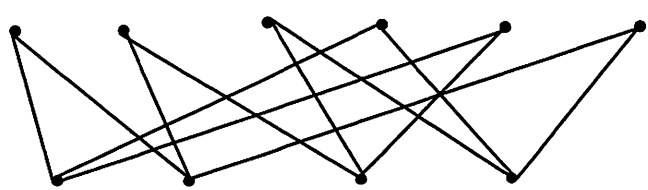

Figure 8

For $1 \leq k \leq i$, a graph is called a $B'(k, i)$ - graph if it is a bipartite graph with bipartition (V_1, V_2) where $|V_1| < \binom{i}{k}$ and $|V_2| = i$ in which every vertex in V_1 is of degree k and every vertex in V_2 is of degree less than or equal to $\binom{i-1}{k-1}$. For example, a $B'(3, 5)$ - graph is shown in Figure 9. The existence of a $B'(k, i)$ - graph is proved in Lemma 4.2.

Clearly, all the $B(k, i)$ - graphs and the $B'(k, i)$ - graphs are NHI, since they are bipartite.

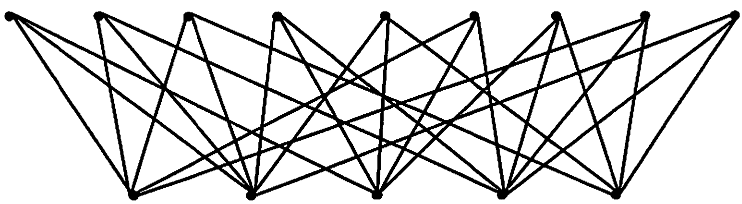

Figure 9

Lemma 4.1 For any $1 \leq k \leq i$, $B(k, i)$ - graph exists.

Proof Let $V = V_1 \cup V_2$ where V_1 contains the vertices $v_1, v_2, \ldots, v_{\binom{i}{k}}$ and $V_2 = \{u_1, u_2, \ldots, u_i\}$ and let $U_1, U_2, \ldots, U_{\binom{i}{k}}$ be the distinct k- subsets (subsets with k elements) of V_2. Join v_j with every element of U_j, for $1 \leq j \leq \binom{i}{k}$. Then the resultant graph G is bipartite with bipartition (V_1, V_2) in which $|V_1| = \binom{i}{k}$ and $|V_2| = i$. Moreover, every vertex in V_1 is adjacent to exactly k vertices of V_2 and every vertex in V_2 is adjacent to exactly $\binom{i-1}{k-1}$ vertices of V_1. Thus each vertex in V_1 is of degree k and each vertex in V_2 is of degree $\binom{i-1}{k-1}$ and hence G is a $B(k, i)$ - graph. ∎

In a similar way, one can prove the following

Lemma 4.2 For any $1 \leq k \leq i$, there is a $B'(k, i)$ - graph. ∎

Proof of Theorem 4

For any $n \geq 1$, we first construct an NHI graph G_n of order $n + m$ with clique number n.

If n = 1 or 2, then K_1 and P_3 are respectively the required graphs. So, assume that $n \geq 3$.

Let $\{v_1, v_2, ..., v_n; u_1, u_2, ..., u_m\}$ be the vertices of G_n. Take $V_1 = \{v_1, v_2, ..., v_n\}$ and $W = \{u_1, u_2, ..., u_m\}$. Suppose U_0 contains the first $\binom{m}{0}$ vertex, that is, v_1 of V_1, U_1 contains the next $\binom{m}{1}$ vertices of V_1 and so on. In general, U_k contains the $\binom{m}{k}$ vertices next to the vertices of U_{k-1} in V_1.

When $n < 2^m$, there exists j, $0 < j < m$, such that $|U_j| = \binom{m}{j}$ and $\left|V_1 \setminus \bigcup_{k=0}^{j} U_k\right| < \binom{m}{j+1}$. In this case, take $U_{j+1} = V_1 \setminus \bigcup_{k=0}^{j} U_k$ and U_{j+2}, $U_{j+3}, ..., U_m$ are all empty sets. Note that the set U_{j+1} may also be empty.

Now we define the edge set of G_n as follows:

1. Add the edges among the vertices of V_1 such that $\langle V_1 \rangle \cong K_n$.
2. When $n = 2^m$, for $1 \leq k \leq m$ add the edges between the vertices of U_k and W such that $\langle U_k, W \rangle$ is a $B(k, m)$ - graph.
3. When $n < 2^m$,
   a. For $1 \leq k \leq j < m$, add the edges between the vertices of U_k and W such that $\langle U_k, W \rangle$ is a $B(k, m)$ - graph

   and b. If U_{j+1} is nonempty then add the edges between the vertices of U_{j+1} and W such that $\langle U_{j+1}, W \rangle$ is a $B'(j+1, m)$ - graph.

The resultant graph G_n is an NHI graph of order n + m with clique number n.

For example, the graphs G_5, G_6, and G_8 are illustrated in Figure 10.

Now, it is enough to show that n + m is minimum.

Suppose that there is a graph G with clique number n and order n + s where s < m. Let $W = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G which induces K_n in G. Let $U = \{u_1, u_2, \ldots, u_s\}$ be the set of remaining vertices of G. Let W_0 be the set of all vertices of W having no neighbours in U. For $1 \leq t \leq s$, let $W_t \subseteq W$ be the set of all vertices of G with degree t in $\langle W, U \rangle$.

Claim W_t contains at most $\binom{s}{t}$ vertices, $0 \leq t \leq s$.

If W_0 contains two vertices u and v, then N[u] = N[v] = W in G. This implies that G is not NHI, which is a contradiction. Therefore W_0 contains at most one vertex, that is, $|W_0| \leq \binom{s}{0}$. Thus the result is true when t = 0.

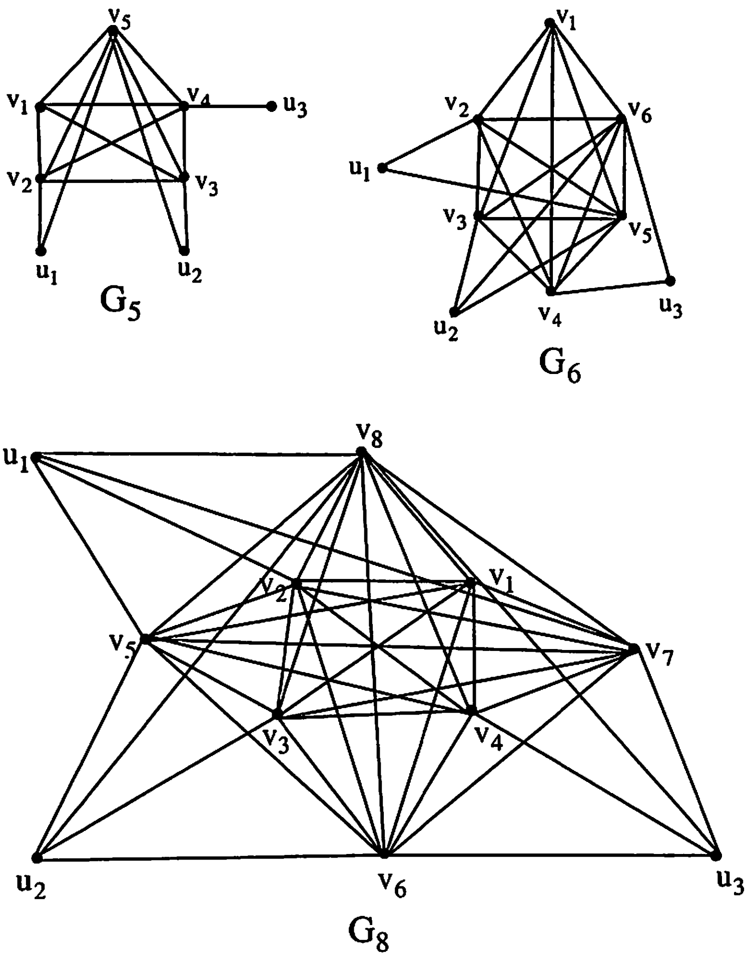

Figure 10

When $t \geq 1$, each vertex in W_t has degree t in $\langle W_t\ , U \rangle$. But $|U| = s$. Therefore, for each vertex v in W_t, N(v) in $\langle W_t\ , U \rangle$ is a t-subset (subset with t elements) of U. But the number of distinct t-subsets of U is exactly $\binom{s}{t}$. If W_t contains more than $\binom{s}{t}$ vertices, then, there are at least two vertices u and v in W_t such that N(u) = N(v) in $\langle W_t , U \rangle$ and hence in G, N[u] = N[v]. This is a contradiction to the fact that G is NHI. Hence the claim. This forces that,

$$n = |W| = |W_0| + |W_1| + \ldots + |W_s| \le \binom{s}{0} + \binom{s}{1} + \ldots + \binom{s}{s} = 2^s .$$

Thus $n \le 2^s$, where s < m. This is a contradiction, to the choice of m. Hence the theorem. ■

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REFERENCES

1. Yousef Alavi, F. Buckley, M. Shamula and S. Ruiz, *Highly irregular m-chromatic graphs*, Discrete Mathematics 72 (1988), 3-13.
2. Yousef Alavi, Gary Chartrand, F. R. K. Chung, Paul ErdÖs, R. L. Graham and O. R. Oellermann, *Highly irregular graphs*, Journal of Graph Theory 11 (1987), 235-249.
3. Yousef Alavi, J. Liu and J. Wang, *Highly irregular digraphs* Discrete Mathematics 111 (1993), 3-10.
4. Selvam Avadayappan, P. Santhi and R. Sridevi, *Some results on neighbourly irregular graphs*, International Journal of Acta Ciencia Indica, vol. XXXII M, No. 3, 1007 – 1012, (2006).

5. R. Balakrishnan and K. Ranganathan, A Text Book of Grapł Theory, Springer Verlag, (2000).
6. R. Balakrishnan and A. Selvam, k-*neighbourhood regular graphs*. Proceedings of the National Workshop on Graph Theory and it: Applications, 1996, 35-45.
7. S. Gnaana Bhragsam and S. K. Ayyaswamy, *Neighbourly irregula1 graphs*, Indian Journal of Pure and Applied Mathematics, 35(3): 389-399, March 2004.
8. A. Selvam, *Highly irregular bipartite graphs*, Indian Journal of Pure and Applied Mathematics, 27(6), 527-536, June 1996.
9. V. Swaminathan and A. Subramanian, *Neighbourhood highl) irregular graphs*, International Journal of Management and Systems, 8(2): 227-231, May-August 2002.