On the Maximum Number of Disjoint Chorded Cycles in Graphs*

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Abstract

Let k be a positive integer and let $G = (V(G), E(G))$ be a graph with $|V(G)| \geq 4k$. In this paper it is proved that if the minimum degree sum is at least $6k - 1$ for each pair of nonadjacent vertices in $V(G)$, then G contains k vertex disjoint chorded cycles. This result generalizes the main Theorem of Finkel. Moreover, the degree condition is sharp in general.

Key words: Cycles with chords; Ore-type; Quadrilateral.
AMS subject classification: 05C35, 05C38.

1 Terminology and Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges and we use Bondy and Murty [2] for terminology and notation not defined here. Let $G = (V, E)$ be a graph, the order of G is $|G| = |V|$ and its size is $e(G) = |E|$. A set of subgraphs is said to be vertex-disjoint or independent if no two of them have any common vertex in G, and we use disjoint or independent to stand for vertex-disjoint throughout this paper. Let G_1 and G_2 be two subgraphs of G or subsets of $V(G)$. If G_1 and G_2 have no any common vertex in G, we define $E(G_1, G_2)$ to be the set of edges of G between G_1 and G_2, and let $e(G_1, G_2) = |E(G_1, G_2)|$. Let H be a subgraph of G and $u \in V(G)$ a vertex, $N(u, H)$ is the set of neighbors of u contained in H. We let $d(u, H) = |N(u, H)|$. Clearly, $d(u, G)$ is the degree of u in G, and we write $d(x)$ to replace $d(x, G)$. The minimum degree of G will be denoted by $\delta(G)$. If there is no fear of confusion, we often identify a subgraph H of G with its vertex set $V(H)$. For a subset U of $V(G)$, we denote by $G[U]$ the subgraph of G induced by

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U and write $d_H(U) = \sum_{x \in U} d_H(x)$ for a subgraph H of G. Let C be a cycle. We use $l(C)$ to denote the length of C, then $l(C) = |C|$. A Hamiltonian cycle of G is a cycle which contains all vertices of G, and a Hamiltonian path of G is a path of G which contains every vertex in G. A cycle of length 4 is called a quadrilateral. A *chorded cycle* in G is a cycle with at least one chord. For a graph G, we define

$$\sigma_2(G) = \min\{d(x) + d(y) | xy \notin E(G), x \neq y \text{ and } x, y \in V(G)\}.$$

When G is a complete graph, we define $\sigma_2(G) = \infty$.

One of the basic results on paths and cycles is Dirac's theorem [4] that every graph of order $n \geq 3$ and minimum degree $\geq n/2$ is Hamiltonian. In 1963, Corrádi and Hajnal [3] proved Erdös's conjecture in the early 1960s which concerns independent cycles in a graph.

Theorem 1.1 *(Corrádi and Hajnal [3]) Suppose $n \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Enomoto and Wang proved a stronger result than Theorem 1.1, independently.

Theorem 1.2 *(Enomoto [6]; Wang [9]) Suppose $n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.*

Theorem 1.1 is in a sense a natural generalization of the well know fact that every graph G with $\delta(G) \geq 2$ contains a cycle. Pósa posed the same question for chorded cycles [10] and he proved that any graph G with $\delta(G) \geq 3$ contains a chorded cycle. In view of this, Bialostocki et al [1] propose the following natural common generalization of the previous result.

Conjecture 1.3 *Let r, s be two nonnegative integers and let G be a graph with $|V(G)| \geq 3r + 4s$ and minimum degree $\delta(G) \geq 2r + 3s$. Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.*

Note that the complete bipartite graph $K_{2r+3s-1,n-2r-3s+1}$ shows that the minimum degree is sharp if $n \geq 4r + 6s - 2$. With respect to Conjecture 1.3, Bialostocki et al verified the case for $r = 0$, $s = 2$ and for $s = 1$. Finkel [8] proved that this conjecture is true if $r = 0$ (only chorded cycles).

Theorem 1.4 *Let G be a graph with $|V(G)| \geq 4k$ and $\delta(G) \geq 3k$. Then G contains k disjoint chorded cycles.*

Very recently, we prove that Conjecture 1.3 is true for any nonnegative integers r and s. In this paper, we consider a similar generalization likewise Theorem 1.1 to Theorem 1.2. Our main result is as follows.

Theorem 1.5 *Let G be a graph with $|V(G)| \geq 4$ and $\sigma_2(G) \geq 5$. Then G contains a chorded cycle.*

Theorem 1.6 *Let G be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq 6k-1$. Then G contains k disjoint chorded cycles.*

Note that the degree conditions in Theorems 1.5 and 1.6 are also sharp by previous example. Theorem 1.6 generalizes Theorem 1.4.

2 Lemmas

In the following, G is a graph of order $n \geq 3$.

Lemma 2.1 *[8] Let C be a chorded cycle and w be a vertex not on C. Suppose $l(C) \geq 5$ and $d(w, C) \geq 4$. Then there is a chorded cycle C' on a subset of $V(C) \cup \{w\}$ with $l(C') < l(C)$.*

Lemma 2.2 *Let $P_1 = x_1x_2 \dots x_p$ and $P_2 = w_1w_2 \dots w_l$ be two paths and C a quadrilateral in G such that they are disjoint. Suppose $e(\{x_1, x_p, w_1, w_l\}, C) \geq 13$, then $G[v(P_1 \cup P_2 \cup C]$ contains two disjoint chorded cycles.*

Proof Label $C = y_1y_2y_3y_4y_1$. By symmetry, we may assume that $e(\{x_1, x_p\}, C) \geq e(\{w_1, w_l\}, C)$. As $e(P, C) \geq 13$, then $e(\{x_1, x_p\}, C) \geq 7$ and $e(\{w_1, w_l\}, C) \geq 5$. Without loss of generality, say $e(\{w_1, w_l\}, y_3y_4) \geq 3$. Then $G[V(P_2) \cup y_3y_4]$ contains a chorded cycle C', which disjoints from the chorded cycle C'' in $G[V(P_1) \cup y_1y_2]$. This proves the lemma. □

3 Proof of Theorem 1.5

Proof By contradiction. Suppose that G does not contain a chorded cycle. Now, we choose a maximal path P in G. Clearly, $|V(P)| \geq 3$. Label $P = u_1u_2u_3 \dots u_l$. We may assume that $u_1u_l \in E(G)$. Otherwise, $u_1u_l \notin E(G)$. Since $d(u_1, P) + d(u_l, P) \geq 5$, by the maximality of P, it is easy to see that $G[V(P)]$ contains a chorded cycle, a contradiction.

Since $u_1u_l \in E(G)$, $V(P)$ contains a cycle $u_1u_2 \dots u_lu_1$. Furthermore, there is no vertex of P may have a neighbor outside $V(P)$, else the maximality of P will be violated. We can assume that there exists a pair of nonadjacent vertices z and $w \in V(P)$, otherwise, we immediately have a chorded cycle. However, note that $d(z, P) + d(w, P) \geq 5$, without loss of generality, say $d(z, P) \geq 3$. Then it is easy to see that $G[V(P)]$ contains a chorded cycle, a contradiction. □

4 Proof of Theorem 1.6

Proof. By induction on k. For $k = 1$, Theorem 1.5 gives the required result. Hence, we may assume that $k \geq 2$. Suppose the theorem is true for all $s \leq k-1$, and take a graph G with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq 6k-1$. By induction on k we obtain that G contains $k-1$ disjoint chorded cycles $C_1, \ldots, C_{k-1}$. We choose $C_1, \ldots, C_{k-1}$ such that

$$\sum_{i=1}^{k-1} l(C_i) \text{ is minimized.} \tag{1}$$

Let $D = G - V(\bigcup_{i=1}^{k-1} C_i)$. Subject to (1), we choose $C_1, \ldots, C_{k-1}$ such that

$$\text{The length of a longest path in } D \text{ is maximized.} \tag{2}$$

Let $P = x_1 \ldots x_p$ be a longest path in D. Let $H = \bigcup_{i=1}^{k-1} C_i$ and $|D| = d$. Since $|G| \geq 4k$ and $\sigma_2(G) \geq 6k-1$, we can remove any three vertices from $V(G)$, and the graph induced by what remains still contains $k-1$ disjoint chorded cycles by induction hypothesis, so $d \geq 3$. We may assume that D does not contain a chorded cycle.

Claim 1. We can properly choose $C_1, \ldots, C_{k-1}$ such that D contains at least one edge.

Proof Otherwise, D is an independent vertex set. Take any pair of $u, v \in V(D)$. Then $d(u,H) + d(v,H) \geq 6k-1 = 6(k-1)+5$. This implies that there exists $C_i \in H$ such that $d(u,C_i) + d(v,C_i) \geq 7$. By Lemma 2.1 and (1), C_i is a chorded quadrilateral. Without loss of generality, say $d(u,C_i) = 4$ and label $C_i = w_1w_2w_3w_4w_1$ such that $\{w_1, w_2, w_3\} \subseteq N(v, C_i)$. Then $G[V(C_i) \cup \{u, v\}]$ contains a chorded quadrilateral $vw_1w_2w_3v$, which disjoints from an edge uw_4, contradicting (2). □

Claim 2. We can properly choose $C_1, \ldots, C_{k-1}$ such that P is a Hamiltonian path in D.

Proof Otherwise, suppose $p < d$. If $\delta(D-P) \geq 3$, by Pósa' theorem [7], $D-P$ contains a chorded cycle, a contradiction. Hence, $\delta(D-P) \leq 2$. We chose $u \in V(D-P)$ such that $d(u, D-P)$ is minimum. Then $d(u, D-P) \leq 2$ and so $d(u,D) \leq 4$.

Furthermore, we may assume that $d(u,D) \leq 3$. Otherwise, suppose $d(u,D) \geq 4$. This gives $d(u, D-P) = 2$ and $d(u,P) = 2$. By the choice of u, we see that $G[V(D-P)]$ contains a cycle. We choose a maximal cycle in $G[V(D-P)]$, denoted by Q. For each pair of $z_1, z_2 \in V(Q)$. It is easy to see that $d(z_i, P) \leq 1$ and $d(z_i, D-P) = 2$ for some $i \in \{1,2\}$, otherwise, $G[V(D)]$ contains a chorded cycle, a contradiction. Without loss of generality, say $i = 1$, then replace u with z_1, we have $d(z_1, D) \leq 3$.

We claim that $x_1x_p \notin E(G)$. Otherwise, assume $x_1x_p \in E(G)$. By the maximality of P, $d(u, P) = 0$ and so $d(u, D) \leq 2$. As $ux_1 \notin E(G)$ and $ux_p \notin E(G)$, it follows that

$$2d(u, H) + d(x_1, H) + d(x_p, H) \geq 2(6k-1) - 8 = 12(k-1) + 2.$$

This implies that there exists $C_i \in H$, say C_1, such that $2d(u, C_1) + d(x_1, C_1) + d(x_p, C_1) \geq 13$. Clearly, $d(u, C_1) \geq 3$ as $d(x_1, C_1) + d(x_p, C_1) \leq 8$. If $d(x_1, C_1) + d(x_p, C_1) \leq 6$, then $d(u, C_1) = 4$ and so $d(x_1, C_1) + d(x_p, C_1) \geq 5$. By Lemma 2.1 and (1), C_1 is a chorded quadrilateral. By symmetry, we may assume $d(x_1, C_1) \geq 3$. Label $C_1 = u_1u_2u_3u_4u_1$ such that $u_1 \in N(x_1, C_1)$. Then $G[V(C_1) \cup \{u, x_1\}]$ contains a chorded quadrilateral $uu_2u_3u_4u$ and a longer path $P + u_1$, which contradicts (2). Hence, we may assume that $d(x_1, C_1) + d(x_p, C_1) \geq 7$. By Lemma 2.1 and (1) again, C_1 is a chorded quadrilateral. Without loss of generality, say $d(x_1, C_1) = 4$ and label $C_1 = w_1w_2w_3w_4w_1$ such that $\{w_1, w_2, w_3\} \subseteq N(x_p, C_1)$ and $w_3u \in E(G)$. Then $G[V(C_1) \cup \{u, x_1\}]$ contains a chorded quadrilateral $x_1w_4w_1w_2x_1$ and a longer path $P + w_3u$, which contradicts (2).

Now let $S = \{x_1, x_p, u\}$, S is a independent set. It is easy to check that $\sum_{x \in S} d(x) \geq \frac{3}{2} \times (6k-1)$. Hence, we obtain

$$\sum_{x \in S} d(x, H) \geq \frac{3}{2} \times (6k-1) - 7 \geq 9(k-1) + 0.5. \tag{3}$$

It follows from the fact that the sum is an integer that there exists $C_i \in H$ such that $\sum_{x \in S} d(x, C_i) \geq 10$. By Lemma 2.1 and (1), C_i is a chorded quadrilateral. Suppose $d(x_1, C_i) + d(x_p, C_i) \leq 6$, then $d(u, C_i) = 4$ and $d(x_1, C_i) + d(x_p, C_i) = 6$. By symmetry, we may assume that $x_1z \in E(G)$ with $z \in V(C_i)$. Then $G[V(C_i \cup P)]$ contains a chorded quadrilateral $C_i' = C_i - z + u$. Replace C_i with C_i', we obtain a longer path $P + z$ than P, contradicting (2). Hence, we must have $d(x_1, C_i) + d(x_p, C_i) \geq 7$ and $d(u, C_1) \geq 2$. By Lemma 2.1 and (1) again, C_i is a chorded quadrilateral. Without loss of generality, say $d(x_1, C_i) = 4$ and label $C_i = w_1w_2w_3w_4w_1$ such that $\{w_1, w_2, w_3\} \subseteq N(x_p, C_i)$ and $w_3u \in E(G)$. Then $G[V(C_i) \cup \{u, x_1\}]$ contains a chorded quadrilateral $x_1w_4w_1w_2x_1$ and a longer path $P + w_3u$, which contradicts (2) again. □

Claim 3. $d = 3$.

Proof By contradiction. Suppose $d \geq 4$. By Claim 2, $P = x_1x_2 \ldots x_d$ is a Hamiltonian path in D. We want to show that there exists a subset $X = \{x_1, w, w', x_d\}$ in this order in P such that $\sum_{x \in X} d(x, P) \leq 9$. As $G[V(P)]$ contains no chorded cycles, $d(x_1, P) \leq 2$, $d(x_d, P) \leq 2$, $d(w, P) \leq 3$ and

$d(w') \leq 3$. Hence, it is sufficient to prove that P contains some vertex w besides the endpoints satisfying $d(w, P) = 2$. Otherwise, we assume $d(u, P) = 3$ for each $u \in P - \{x_1, x_d\}$. Note that we may assume that $d \geq 5$, otherwise, it is easy to check that $G[V(P)]$ contains a chorded cycle, a contradiction. In particular, $d(x_2, P) = d(x_3, P) = 3$. Say $N(x_2, P) = \{x_1, x_3, x_m\}$, $4 \leq m \leq d$. Denote the adjacent vertex of x_3 other than x_2 and x_4 by x_l. If $l \leq m$, then $x_2 \ldots x_m x_2$ or $x_1 x_3 \ldots x_m x_2 x_1$ is a chorded cycle, a contradiction. Thus, $l > m$ and then we must have $d(x_4, P) = 2$. For otherwise, denote the neighbor of x_4 other than x_3 and x_5 by x_q. Clearly, $q \geq l$ and then $x_4 x_q \ldots x_m x_2 x_3 x_4$ is a cycle with chord $x_3 x_l$, a contradiction.

Now, we will show that $G[X]$ contains two pair of nonadjacent vertices. If $x_1 x_d \in E(G)$, then $x_1 w' \notin E(G)$ and $w x_d \notin E(G)$ since $G[X]$ contains no chorded cycle. Hence, we may assume that $x_1 x_d \notin E(G)$ and so $ww' \in E(G)$. when $d \geq 5$, again, we see that $x_1 w' \notin E(G)$ and $x_d w \notin E(G)$. Hence, it remains the case $d = 4$. Clearly, exactly one of $x_1 x_3$ and $x_2 x_4$ exists. Without loss of generality, say $x_1 x_3 \notin E(G)$ and $x_2 x_4 \in E(G)$. Then

$$2d(x_1, H) + d(x_3, H) + d(x_4, H) \geq 2(6k - 1) - 6 = 12(k - 1) + 4.$$

Without loss of generality, we may assume that $C_1 \in H$ such that $2d(x_1, C_1) + d(x_3, C_1) + d(x_4, C_1) \geq 13$. By Lemma 2.1 and (1) again, C_1 is a chorded quadrilateral. Label $C_1 = w_1 w_2 w_3 w_4 w_1$. If $d(x_1, C_1) = 4$, without loss of generality, say $w_1 \in N(x_3, C_1) \cap N(x_4, C_1)$. Then $G[V(C_1 \cup P)]$ contains two disjoint chorded cycles $x_1 w_2 w_3 w_4 x_1$ and $x_3 w_1 x_4 x_2 x_3$, a contradiction. Therefore, we may assume $d(x_1, C_1) = 3$ and so $d(x_3, C_1) + d(x_4, C_1) \geq 7$. Without loss of generality, say $\{w_1, w_2, w_3\} = N(x_1, C_1)$. Suppose $d(x_3, C_1) = 4$. If $w_4 \in N(x_4, C_1)$, then as above, $G[V(C_1 \cup P)]$ contains two disjoint chorded quadrilaterals, a contradiction. So, we have $\{w_1, w_2, w_3\} = N(x_4, C_1)$. Since C_1 is a chorded quadrilateral, then $G[V(C_1 \cup P)]$ contains two disjoint chorded quadrilaterals $x_1 w_1 w_4 w_3 x_1$ and $x_3 w_2 x_4 x_2 x_3$ if $w_1 w_3 \in E(G)$, a contradiction. Hence, we may assume that $w_1 w_3 \notin E(G)$ and so $w_2 w_4 \in E(G)$. Then $G[V(C_1 \cup P)]$ contains two disjoint chorded quadrilaterals $x_1 w_2 w_4 w_3 x_1$ and $x_3 w_1 x_4 x_2 x_3$, a contradiction.

Consequently, it follows from the degree condition that

$$\sum_{x \in X} d(x, H) \geq 2(6k - 1) - 9 = 12(k - 1) + 1.$$

This implies that there exists $C_i \in H$ such that $\sum_{x \in X} d(x, C_i) \geq 13$. By Lemma 2.1 and (1), C_i is a chorded quadrilateral. Denote $P_1 = x_1 \ldots w$ and $P_2 = w' \ldots x_d$. Then it follows from Lemma 2.2 that $G[V(C_i \cup P_1 \cup P_2)]$ contains two disjoint chorded cycles, therefore, G contains k disjoint chorded cycles. □

Now we are in the position to complete the proof. By Claim 3, $P = x_1x_2x_3$ must be a hamiltonian path in D. We use the following iteration appeared in [8].

Let

$$T_1 = \{\text{chorded cycles } D \in H | d(y, P) = 3 \text{ for some } y \in D\},$$

and define iteratively

$$T_{i+1} = \left\{\text{chorded cycles } D \in H \setminus (\cup_{j=1}^{i} T_j) | d(y, E) = 4 \text{ for some } y \in D, E \in T_i\right\}. \tag{4}$$

Obviously, $T_i = \emptyset$ for some i since H contains only finitely many chorded cycles (Note that T_1 may be empty. In this case, we still continue the iteration of (4)). Say T_l is the last nonempty set obtained from the process above. Define $\overline{K} = P \cup (\bigcup_{i=1}^{l} T_i)$.

Claim 4(Lemma 3 in [8]). If $T_1 \neq \emptyset$, then every chorded cycle $D \in \bigcup_{i=1}^{l} T_i$ has exactly 4 vertices. If $T_1 = \emptyset$, then $D \in \bigcup_{i=2}^{l-1} T_i$ has exactly 4 vertices.

Now define $G' = G - \overline{K}$. Then $\overline{K}$ contains $s \leq k - 1$ disjoint chorded quadrilaterals and P, so $|\overline{K}| = 4s + 3$. It follows that $|G'| \geq 4k - (4s + 3) \geq 1$. This implies that there exists a chorded cycle $E \in H$ such that $w \in E \subset G'$. By our construction, $d(w, P) \leq 2$ and $d(w, D) \leq 3$ for each $D \in \overline{K} - P$, otherwise, w would be in $\overline{K}$. Therefore, $d(w, \overline{K}) \leq 3s + 2$. Consequently, $\sigma_2(G') \geq 6k - 1 - 2(3s + 2) \geq 6(k - s - 1) + 1$ and so $\sigma_2(G' - \{w\}) \geq 6(k - s - 1) - 1$. Note that $|G' - \{w\}| \geq 4k - (4s + 3) - 1 = 4(k - s - 1)$. Therefore, by the induction hypothesis, $G' - \{w\}$ contains $k - s - 1$ disjoint chorded cycles. It follows that $(G' - \{w\}) + (\overline{K} - P) \subset G - P - \{w\}$ contains $k - s - 1 + s = k - 1$ disjoint chorded cycles. But $w \in H$, this contradicts the minimality of H, a final contradiction.

Applying induction, we complete the proof of Theorem 1.6.

5 Concluding Remark

It is natural to consider whether the minimum degree condition can be replaced by the Ore-type condition. We will show that this is true in [5].

Theorem 5.1 *Let r, s be two nonnegative integers and let G be a graph with $|V(G)| \geq 3r + 4s$ and $d(x) + d(y) \geq 4r + 6s - 1$ for each pair of nonadjacent vertices x, $y \in V(G)$. Then G contains a collection of r cycles and s chorded cycles, all vertex disjoint.*

Note that the Ore-type degree condition is also sharp in Theorem 5.1. The proof of Theorem 5.1 heavily depends on the proof of Theorem 1.6.

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