An algorithm to find k-tight optimal double-loop networks *

Jianqin Zhou
Telecommunication School
Hangzhou Dianzi University, Hangzhou 310018, China
Computer Science School
Anhui University of Technology, Ma'anshan 243002, China
zhou9@yahoo.com

Abstract

A double-loop network(DLN) $G(N;1,s)$ with $1 < s < N$, is a digraph with the vertex set $V = \{0, 1, \ldots, N-1\}$ and the edge set $E = \{u \to v | v - u \equiv 1, s(\mod N), u, v \in V\}$. Let $D(N;1,s)$ be the diameter of G and let us define $D(N) = \min\{D(N;1,s) | 1 < s < N\}$ and $lb(N) = \lceil \sqrt{3N} \rceil - 2$. A given DLN $G(N;1,s)$ is called k-tight if $D(N;1,s) = lb(N) + k(k \geq 0)$. A k-tight DLN is called optimal if $D(N) = lb(N) + k(k \geq 0)$.

It is known that finding k-tight optimal DLN is a difficult task as the value k increases. In this work, a practical algorithm is derived for finding k-tight optimal double-loop networks($k \geq 0$), and it is proved that the average complexity to judge whether there exists a k-tight L-shaped tile with N nodes is $O(k^2)$. As application examples, we give some 9-tight optimal DLN and their infinite families.
Keywords: Double-loop network, k-tight optimal, L-shaped tile, infinite family

1 Introduction

Double-loop digraphs $G = G(N;1,s)$, with $1 < s < N$, have the vertex set $V = \{0, 1, \ldots, N-1\}$ and the adjacencies are defined by $v \to v + 1($

*The research was supported by Zhejiang Natural Science Foundation of Outstanding Youth Team Project (No. R1090138) and Chinese Natural Science Foundation (No. 60473142).

mod N) and $v \rightarrow v + s($ mod $N)$ for $v \in V$. The hops 1 and s between vertices are called steps. These kinds of digraphs have been widely studied as architecture for local area networks, known as double-loop networks (DLN). For surveys about these networks, refer to [3,7].

From the metric point of view, the minimization of the diameter of G corresponds to a faster transmission of messages in the network. The diameter of G is denoted by $D(N;1,s)$. As G is vertex symmetric, its diameter can be computed from the expression $\max\{d(0;i)|i \in V\}$, where $d(u;v)$ is the distance from u to v in G. For a fixed integer $N > 0$, the optimal value of the diameter is denoted by $D(N) = \min\{D(N;1,s)|1 < s < N\}$.

Since the work of Wong and Coppersmith [10], a sharp lower bound is known for $D(N)$:

$$D(N) \geq \lceil \sqrt{3N} \rceil - 2 = lb(N)$$

A given DLN $G(N;1,s)$ is called k-tight if $D(N;1,s) = lb(N) + k(k \geq 0)$. A k-tight DLN is called optimal if $D(N) = lb(N) + k(k \geq 0)$, where integer N is called k-tight optimal. The 0-tight DLN are known as tight ones and they are also optimal.

The metrical properties of $G(N;1,s)$ are fully contained in its related L-shaped tile $L(N;l,h,x,y)$ where $N = lh - xy, l > y$ *and* $h \geq x$. In Figure 1, we illustrate generic dimensions of an L-shaped tile.

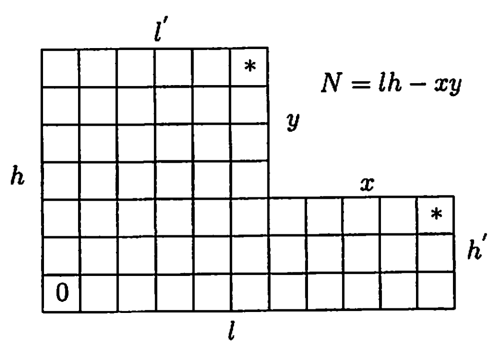

Figure 1: Generic dimensions of an L-shaped tile

Let $D(L) = D(L(N;l,h,x,y)) = \max\{l + h - x - 2, l + h - y - 2\}$. For obvious reasons, the value $D(L)$ is called the diameter of the tile L. It is known that an L-shaped tile $L(N;l,h,x,y)$ can be assigned to a $G(N;1,s)$ without any confusion. However, we can not find double-loop network $G(N;1,s)$ from some L-shaped tiles. When an L-shaped tile $L(N;l,h,x,y)$

has diameter $lb(N)+k$, we say it is k-tight.

Esqué, Aguiló and Fiol [6] characterized a complete set of families of 0-tight double-loop networks. Xu and Liu [11] gave an infinite family of 4-tight optimal double-loop networks. It is known that finding k-tight optimal DLN is a difficult task as the value k increases.

For general positive integer N, Aguiló and Fiol [1] gave an algorithm to search an L-shaped tile with diameter $\lceil\sqrt{3N}\,\rceil-2+k$ in the order $k=0,1,2,\ldots$. The first-found L-shaped tile must have minimum diameter. They estimated the time complexity of this algorithm to be $O(k^3)O(logN)$ for a k-tight L-shaped tile.

In section 3, a simple algorithm is derived for finding k-tight optimal double-loop networks($k\geq 0$), and it is proved that the average complexity to judge whether there exists a k-tight L-shaped tile with N nodes is $O(k^2)$. Experiments show that the algorithm is fast and easy to realize. As application examples, section 4 presents some k-tight optimal $DLN(6\leq k\leq 9)$ and their infinite families.

2 Preliminary

We introduce some lemmas, which will be used in the following sections. The following Lemma 1, 2, 3 and 4 can be found in [6 or 8 or 9].

Lemma 1[6, 9]. Let t be a nonnegative integer. We define $I_1(t)=[3t^2+1,3t^2+2t]$, $I_2(t)=[3t^2+2t+1,3t^2+4t+1]$ and $I_3(t)=[3t^2+4t+2,3(t+1)^2]$. Then we have $[4,3T^2+6T+3]=\bigcup_{t=1}^{T}\bigcup_{i=1}^{3}I_i(t)$, where $T>1$, and $lb(N)=3t+i-2$ if $N\in I_i(t)$ for $i=1,2,3$.

Lemma 2[8]. Let $L(N;l,h,x,y)$ be an L-shaped tile, $N=lh-xy$. Then, there exists $G(N;1,s)$ realizing the L-shaped tile iff $l>y$, $h\geq x$ and $gcd(h,y)=1$, where $s\equiv \alpha l-\beta(l-x)(\mod N)$ for some integral values α and β satisfying $\alpha y+\beta(h-y)=1$.

Lemma 3[9]. Let $L(N;l,h,x,y)$ be an L-shaped tile, $N=lh-xy$. Then
(a) If $L(N;l,h,x,y)$ is realizable, then $|y-x|<\sqrt{N}$;
(b) If $x>0$ and $|y-x|<\sqrt{N}$, then
$D(L(N;l,h,x,y))\geq\sqrt{3N-\frac{3}{4}(y-x)^2}+\frac{1}{2}|y-x|-2$;
(c) Let $f(z)=\sqrt{3N-\frac{3}{4}z^2}+\frac{1}{2}z$. Then $f(z)$ is strictly increasing when $0\leq z\leq\sqrt{N}$.

Lemma 4[9]. Let $N(t) = 3t^2 + At + B \in I_i(t)$ and L be the L-shaped tile $L(N(t); l, h, x, y)$, where A and B are integral values; $l = 2t + a$, $h = 2t + b$, $z = |y - x|$, a, b, x, y are all integral polynomials of variable t, and $j = i + k (k \geq 0)$. Then L is k-tight iff the following identity holds

$$(a + b - j)(a + b - j + z) - ab + (A + z - 2j)t + B = 0. \qquad (1)$$

The following Lemma 5 is the generalization of Theorem 2 in [11], and can be found in [12].
Lemma 5. Let $H(z, j) = (2j - z)^2 - 3[j(j - z) + (A + z - 2j)t + B]$, and the identity (1) be an equation of a and b. A necessary condition for the equation (1) to have integral solution is that $4H(z, j) = s^2 + 3m^2$, where s and m are integers.

It is easy to show that the following Lemma 6 is equivalent to Theorem 1 in [11]. Lemma 6 can be found in [12].
Lemma 6. Let n, s and m be integers, $n = s^2 + 3m^2$. If n has a prime factor p, here $p \equiv 2(\mod 3)$, then there exists an even integer q, such that n is divisible by p^q , but not divisible by p^{q+1}.

Lemma 7[12]. Let $N = N(t) = 3t^2 + At + B \in I_i(t)$ and L-shaped tile $L(N; l, h, x, y)$ be k-tight$(k \geq 0)$ and realizable. Let $z = |y - x|$. Then the following hold

Case 1. If $A = 0$ or $A = 2$(if $i = 2$) or $A = 4$(if $i = 3$), and $3N - \frac{3}{4}(2k + 3)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k + 2$.

Case 2. If $A = 1$ or $A = 3$ or $A = 5$, and $3N - \frac{3}{4}(2k + 2)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k + 1$.

Case 3. If $A = 2$(if $i = 1$) or $A = 4$(if $i = 2$) or $A = 6$, and $3N - \frac{3}{4}(2k + 1)^2 > (3t + \frac{A-1}{2})^2$, then $0 \leq z \leq 2k$.

3 An algorithm to find k-tight optimal double-loop networks

We first introduce the following algorithm.
Algorithm 1. To judge whether N is k-tight$(k \geq 0)$ optimal. $kmax$ is a suitable constant, such as $kmax = 10$.

Step 1. Calculate: $t = \lceil \sqrt{N/3}\ \rceil - 1$; $A = \lfloor (N - 3t^2)/t \rfloor + 1$; $B = N - (3t^2 + At) \leq 0$; if($A = 3$ and $B = -t$), then{$A = 2$ and $B = 0$};
if($A = 5$ and $-t \leq B \leq -t + 1$), then{$A = 4$ and $B = B + t$};
if($A = 7$), then{$A = 6$ and $B = B + t$}.

Step 2. Determine $j0$: if $(A <= 2)j0=1$; if $(2 < A <= 4)j0=2$; if $(A > 4)j0=3$.

Determine $z0$: if(A=1 or A=3 or A=5)$z0$=2; else $z0$=1.

flag=-1;

Step 3. for($j = j0; j < j0 + kmax; j$++)

```
{
    for(z = 0; z < 2(j - j0)+z0; z++)
    {
        h = (2j - z)^2 - 3[j(j - z) + (A + z - 2j)t + B];
        if 4h = s^2 + 3m^2, where s and m are integers,
        then {flag=j - j0; break;}
    } //The end of loop for z.
    if(flag>-1)break;
}//The end of loop for j.
```

Step 4. Let e=flag. If the following holds

$$3N - \frac{3}{4}(2e + z0 - 2)^2 > (3t + \frac{A-1}{2})^2 ,$$

then N is k-tight($k \geq e$) optimal.

From Lemma 7, it is easy to show the validity of Algorithm 1. From Lemma 1, we know that Step 1 and the determination of $j0$ are correct.

Suppose that N is $(e-1)$-tight optimal. Since $3N - \frac{3}{4}(2e + z0 - 2)^2 > (3t + \frac{A-1}{2})^2$ and Lemma 7, thus $0 \leq z < 2e + z0 - 2$.

From Step 3, $4H(z,j)$ has no the form of $s^2 + 3m^2$. By lemma 5, the equation (1) has no integral solutions of a and b. By lemma 4, there is no k-tight L-shaped tile $L(N(t); l, h, x, y)$ for $0 \leq z < 2e + z0 - 2$. This is a contradiction.

Similarly, we can show that N is not k-tight optimal for $0 \leq k < (e-1)$. Therefore, N is k-tight($k \geq e$) optimal.

Now we come to the complexity of Algorithm 1. Suppose the number of nodes is less than 10^{10}. From $t = \lceil \sqrt{N/3} \rceil - 1$, we know $t < 57736$.

Let $kmax$=9. Since $h = (2j - z)^2 - 3[j(j - z) + (A + z - 2j)t + B]$ and $|B| < t$, so $h < 6(j0 + kmax)t$, thus $h \in (0, 5000000)$. We judged whether that $4h$ has the form of $s^2 + 3m^2$ for every integer $h \in (0, 5000000)$, and save these results in a file for the algorithm retrieving.

In the Step 3 of Algorithm 1, the number of loop time is at most $k(2k+2)$ if there exists a k-tight L-shaped tile with N nodes. Therefore, the complexity of Algorithm 1 for every integer $N \in (4, 10^{10})$ is less than a constant $O(kmax^2)$.

For general positive integer N, we know that $t = O(N^{\frac{1}{2}})$. Since $h = (2j-z)^2-3[j(j-z)+(A+z-2j)t+B]$ and $|B| < t$, thus $h \in (0, O(kmaxN^{\frac{1}{2}}))$. At the preparation stage, we judge whether that $4h$ has the form of s^2+3m^2, thus the running time complexity of the preparation stage is $O(kmaxN^{\frac{1}{2}}) \cdot O((kmaxN^{\frac{1}{2}})^{\frac{1}{2}}) = O(kmax^{\frac{3}{2}}N^{\frac{3}{4}})$. Note that the running time complexity in the Step 3 of Algorithm 1 to judge n with a k-tight L-shaped tile is $O(k^2)$, thus the average complexity of Algorithm 1 for every integer $n \in (4, N)$ is $O(k^2) + \frac{1}{N} \cdot O(kmax^{\frac{3}{2}}N^{\frac{3}{4}}) = O(k^2)$.

We remark that although the condition in Step 4 holds for most values of N, there are also another very small set of integers for which the condition in Step 4 does not hold.

We have made an experiment on the integers in (4, 1500000) with the algorithm. The following integers have the result: flag=5,

417289, 464533, 526429, 858157, 1302637, 1368379, 1498333.

All of them (except 464533) are 5-tight optimal.

For 464533, $t = 393, A = 4, B = -386, k = 2 < 5$, the following does not hold.

$3N - \frac{3}{4}(2k+1)^2 > (3t + \frac{A-1}{2})^2$.

In fact, $3N - \frac{3}{4}(2k+1)^2 = (3t + \frac{A-1}{2})^2 = 1393580.25$. When $z = 2k+1 = 5$, there exists a realizable 2-tight L-shaped tile $L(464533; 787, 787, 391, 396)$.

It must be known that even though flag=e and $3N - \frac{3}{4}(2e + z0 - 2)^2 > (3t + \frac{A-1}{2})^2$, N may not be e-tight optimal, as some e-tight L-shaped tiles may not be realizable.

4 Some application examples

With Algorithm 1, we found that

7243747 is the smallest integer with 6-tight optimal DLN;

81190689 is the smallest integer with 7-tight optimal DLN;

2530527211 is the smallest integer with 8-tight optimal DLN.

Let $N(t) = 3t^2 + 6t - 582187$. Then $N(1500498)$ is 9-tight optimal.

Let $N(t) = 3t^2 + 6t - 1496791$. Then $N(2000174)$ is 9-tight optimal.

Let $N(t) = 3t^2 + 6t - 461159$. Then $N(2500044)$ is 9-tight optimal.

Let $N(t) = 3t^2 + 4t - 2222698$. Then $N(2500139)$ is 9-tight optimal.

Example 1. We now prove that $N(2500139)$ is 9-tight optimal, and give an infinite family of 9-tight optimal integers $N(t) = 3t^2 + 4t - 2222698$, which including $N(2500139)$ as a starting element.

We first prove that $N(t)$ is not k-tight$(0 \le k \le 8)$ optimal, where

$t = g \times e + 2500139 (e \geq 0)$ and g is a nonnegative integer .

Let $A = 4, B = -2222698$ and $k = 8$. The following inequality
$3N - \frac{3}{4}(2k+1)^2 > (3t + \frac{A-1}{2})^2$,
is equivalent to
$3(t + B) > \frac{3}{4}(2k+1)^2 + (\frac{A-1}{2})^2$,
which is true. Let L-shaped tile $L(N(2500139); l, h, x, y)$ be k-tight$(0 \leq k \leq 8)$ and realizable. Let $z = |y - x|$. By Lemma 7, $0 \leq z \leq 2k$.

For $2 \leq j \leq 10, 0 \leq z \leq 2(j-2), t = 2500139$, we have verified that $H(z,j) = (2j-z)^2 - 3[j(j-z) + (A+z-2j)t + B]$ has a prime factor $p(z,j)$, where $p(z,j) \equiv 2(\mod 3)$, with an odd power $q(z,j)$. We only show the case of $H(z,10)$ in Table 1.

Table 1

z	$H(z,10)$	p	$power$
0	126674866	2	1
1	119174440	5	1
2	111674016	2	5
3	104173594	2	1
4	96673174	2	1
5	89172756	971	1
6	81672340	5	1
7	74171926	2	1
8	66671514	2	1
9	59171104	2	5
10	51670696	2	3
11	44170290	2	1
12	36669886	2	1
13	29169484	17	1
14	21669084	17	1
15	14168686	2	1
16	6668290	2	1

Let $g = $lcm $\{p(z,j)^{q(z,j)+1} | 2 \leq j \leq 10, 0 \leq z \leq 2(j-2)\}$, where lcm stands for Lowest Common Multiple.

Let $t = g \times e + 2500139 (e \geq 0)$. For $2 \leq j \leq 10, 0 \leq z \leq 2(j-2)$, it is easy to show that $H(z,j) = (2j-z)^2 - 3[j(j-z) + (A+z-2j)t + B]$ has a prime factor $p(z,j)$, here $p(z,j) \equiv 2(\mod 3)$, with an odd power $q(z,j)$. By Lemma 6, $H(z,j)$ has no the form of $s^2 + 3m^2$. By Lemma 5, the equation (1) has no integral solutions of a and b. By Lemma 4, there is no $(j-2)$-tight L-shaped tile $L(N(t); l, h, x, y)$ for (z,j).

As a conclusion, $N(t)$ is not k-tight$(0 \leq k \leq 8)$ optimal, $t = g \times e + 2500139 (e \geq 0)$.

Secondly, we prove that $N(t)$ is 9-tight optimal, where $t = f(e) + 2500139 (e \geq 0)$ and $f(e)$ is a polynomial of order 1 or 2.

For $z = 18, j = 2+9, A = 4, B = -2222698, A+z-2j = 0$, the equation (1) of a and b has a solution (a, b)=(-1035,-671).

From Lemma 4, if the identity (1) holds, then the L-shaped tile $L(N(t); l(f), h(f), x(f), y(f))$ is 9-tight. Hence let

$l(f) = 2t + a = 2t - 1035, \;\; h(f) = 2t + b = 2t - 671,$

$x(f) = t + a + b - j = t - 1717, \;\; y(f) = x(f) + z = t - 1699,$

$l'(f) = l(f) - x(f) = t - b + j, \;\; h'(f) = h(f) - y(f) = t - a + j - z,$

$h'(f) - y(f) = -2a - b + 2j - 2z = 2727.$

Let $t = 2727f + 2500139$. Then, $h'(f) = 2727f + 2501167$.

Since 1852(2501167)-1698629(2727)=1, thus,

$1852(h'(f) - f(h'(f) - y(f))) - 1698629(h'(f) - y(f)) = 1.$

That is, $(1852f + 1698629)y(f) + (-1852f - 1696777)h'(f) = 1.$

Hence, $gcd(y(f), h'(f)) = 1.$

From Lemma 2, let $s(f) = (1852f+1698629)l(f)+(1852f+1696777)l'(f)$. Then $G(N(t); 1, s(f))$ is a 9-tight optimal double-loop network, where $t = 2727g \times e + 2500139, f = g \times e (e \geq 0)$.

Suppose $e = 0$, then $f = 0, t = 2500139, s(0) = 1698629 \times 4999243 + 1696777 \times 2500821 = 12735194691764$. Thus, $G(N(2500139); 1, s(0))$ is a 9-tight optimal double-loop network.

In fact, for $z = 18, A + z - 2j = 0$, this is a special case. We now consider a more general case, where $A + z - 2j \neq 0$.

For $z = 2, j = 2 + 9, A = 4, B = -2222698, t = 2500139, A + z - 2j = -16$, the equation (1) of a and b has a solution (a, b)=(-6441,-93). Thus $l_0 = 2t + a = 4993837, h_0 = 2t + b = 5000185, x_0 = t + a + b - j = 2493594, y_0 = t + a + b - j + z = 2493596$. So, $h_0 - 2y_0 = 12993$.

Let $a(f) = 16 \times 12993f - 6441, \quad b(f) = -32 \times 12993f - 93$. By replacing a and b of the equation (1) with $a(f)$ and $b(f)$, we have $t(f) = 48 \times 12993^2 f^2 + 12993f(3 \times 93 + 22 - 2) + 2500139 = 48 \times 12993^2 f^2 + 299 \times 12993f + 2500139$.

Let

$l(f) = 2t(f) + a(f), \;\; h(f) = 2t(f) + b(f),$

$x(f) = t(f) + a(f) + b(f) - j,$

$y(f) = x(f) + z = t(f) - 16 \times 12993f - 6543 = 48 \times 12993^2 f^2 + 283 \times$

$12993f + 2493596,$
$l'(f) = l(f) - x(f), \; h'(f) = h(f) - y(f),$
$h'(f) - y(f) = 12993.$

Since 6092(2493596)-1169167(12993)=1, thus,
$6092(y(f) - f(h'(f) - y(f))(48 \times 12993f + 283)) - 1169167(h'(f) - y(f)) = 1.$

That is,
$[6092f(48 \times 12993f + 283) + 1175259]y(f) + [-6092f(48 \times 12993f + 283) - 1169167]h'(f) = 1.$

Hence, $\gcd(y(f), h'(f)) = 1$.

From Lemma 2, let $s(f) = [6092f(48 \times 12993f + 283) + 1175259]l(f) + [6092f(48 \times 12993f + 283) + 1169167]l'(f)$. Then $G(N(t); 1, s(f))$ is a 9-tight optimal double-loop network, where $t = 48 \times 12993^2 f^2 + 299 \times 12993f + 2500139, f = g \times e (e \geq 0)$.

Suppose $e = 0$, then $f = 0, t = 2500139, s(0) = 1175259 \times 4993837 + 1169167 \times 2500243 = 8792253486364$. Thus, $G(N(2500139); 1, s(0))$ is a 9-tight optimal double-loop network.

With a similar argument as in the case of $N(2500139)$, we can derive infinite families of k-tight optimal integers from other starting integers. Here only the key parameters are shown in Table 2, where $N(1553) = 7243747, N(5202) = 81190689, N(29043) = 2530527211$.

Table 2

A	B	t	j	k	z	a	b
6	-998	1553	9	6	11	13	47
2	-2127	5202	8	7	14	5	45
2	-18422	29043	9	8	16	-125	-18
6	-582187	1500498	12	9	16	-2089	499
6	-1496791	2000174	12	9	17	-2157	1081
6	-461159	2500044	12	9	16	-1685	-977

Acknowledgment

The author thanks the anonymous referee for many constructive comments that significantly improved the presentation of this paper.

References

[1] F. Aguiló, M.A. Fiol, An efficient algorithm to find optimal double loop networks, *Discrete Mathematics*,138(1995) 15-29.

[2] F. Aguiló, E. Simó, M. Zaragozá, Optimal double-loop networks with non-unit steps, *The Electronic Journal of Combinatorics*, 10 (2003), ♯R2.

[3] J.-C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer networks: a survey, *J. Parallel Distribut. Comput.*, 24 (1995) 2-10.

[4] C.F. Chan, C. Chen and Z.X. Hong, A simple algorithm to find the steps of double-loop networks, *Discrete Applied Mathematics*, Vol.121 (2002) 61-72.

[5] P. Erdös, D.F. Hsu, Distributed loop networks with minimum transmission delay, *Theoret. Comput. Sci.*, 100 (1992) 223-241.

[6] P.Esqué, F. Aguiló and M.A. Fiol, Double commutative-step digraphs with minimum diameters, *Discrete Mathematics*,114 (1993)147-157.

[7] F.K. Hwang, A complementary survey on double-loop networks, *Theoret. Comput. Sci.*, 263(2001) 211-229.

[8] M.A. Fiol, J.L.A. Yebra, I. Alegre and M. Valero, A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.*,C-36 (1987) 702-713

[9] Li Q., Xu J. and Zhang Z., The infinite families of optimal double loop networks, *Discrete Applied Mathematics*, Vol.46(1993)179-183.

[10] C.K. Wong and D. Coppersmith, A combinatorial problem related to multimode memory organizations, *J. Ass. Comput. Mach.*, 21 (1974) 392-402.

[11] Xu J. and Liu Q., An infinite family of 4-tight optimal double loop networks, *Science in China, Series A*, Vol. A-46 num. 1 (2003) 139-143.

[12] Zhou J. and Xu X., On infinite families of optimal double-loop networks with non-unit steps, *Ars Combinatoria*, accepted.