The Crossing Numbers of Cartesian Product of Cone Graph $C_m + \overline{K_l}$ with Path P_n *

Zheng Wenping[1,2], Lin Xiaohui[1], Yang Yuansheng[1], Yang Xiwu[1]
1. Department of Computer Science, Dalian University of Technology, Dalian, 116024, P. R. China, yangys@dlut.edu.cn
2. School of Computer and Information Technology, Shanxi University, Taiyuan, 030006, P. R. China, wpzheng@sxu.edu.cn

Abstract. Crossing numbers of graphs are in general very difficult to compute. There are several known exact results on the crossing numbers of Cartesian products of paths, cycles or stars with small graphs. In this paper we study $cr(W_{l,m} \square P_n)$, the crossing number of Cartesian product $W_{l,m} \square P_n$, where $W_{l,m}$ be the cone graph $C_m + \overline{K_l}$. Klešč showed that $cr(W_{1,3} \square P_n) = 2n$(Journal of Graph Theory, 6(1994), 605-614), $cr(W_{1,4} \square P_n) = 3n-1$ and $cr(W_{2,3} \square P_n) = 4n$(Discrete Mathematics, 233(2001), 353–359). Huang et al. showed that $cr(W_{1,m} \square P_n) = (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n + 1$ for $n \le 3$(Journal of Natural Science of Hunan Normal University, 28(2005), 14–16). We extend these results and prove $cr(W_{1,m} \square P_n) = (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n + 1$ and $cr(W_{2,m} \square P_n) = 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$.
Keywords. crossing number, Cartesian product, cone graph, path, wheel

1 Introduction

We consider only finite undirected graphs without loops or multiple edges.

Let G be a graph with vertex set V and edge set E. We consider only *good drawings* of a graph, i.e., a drawing satisfies (i) no edge crosses itself; (ii) adjacent edges do not cross; (iii) crossing edges do so only once; (iv) edges do not cross vertices and (v) no more than two edges cross at a common point. We denote the crossing number of G for the plane by $cr(G)$. If $D(G)$ is a good drawing of G, then $\nu(D(G))$ denotes the number of the crossings in $D(G)$. It is clear that $cr(G) \le \nu(D(G))$. The *Cartesian*

*Research supported by NFSC (60373096, 60573022) and SRFDP (20030141003).
Correspondence author: yangys@dlut.edu.cn (Yang Yuansheng)

Product $G \,\square\, H$ of graphs G and H has vertex set $V(G) \times V(H)$ and edge set $E(G \,\square\, H) = \{\{(x_1, y_1), (x_2, y_2)\} \mid x_1 = x_2 \text{ and } y_1y_2 \in E(H) \text{ or } y_1 = y_2 \text{ and } x_1x_2 \in E(G)\}$. (In the references of this paper, the authors also use $G \times H$ to represent the Cartesian Product of graphs G and H.) P_n is a path with length n. The *cone graph* $C_m + \overline{K_l}$ is obtained by adding l new independent vertices to the $m-$ cycle C_m and joining each one of them to every vertex of C_m. We use $W_{l,m}$ to represent the graph $C_m + \overline{K_l}$. The graph $W_{1,m}$ is known as a *wheel* W_m and $W_{2,m}$ is a *double cone*.

Zarankiewicz[14] studied the crossing number of $K_{m,n}$ and proved that $cr(K_{m,n}) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. The equality holds for $\min(m, n) \leq 6$[6] and for the the special cases $7 \leq m \leq 8$, $7 \leq n \leq 10$[12].

Harary et al. conjectured that $cr(C_m \,\square\, C_n) = (m-2)n$, for $3 \leq m \leq n$. This has been verified for $m \leq 7$[1]. Glebsky and Salazar[11] also showed that the conjecture holds for $n \geq m(m+1)$ and $m \geq 3$.

Beineke et al.[2] and Jendroĭ et al.[5] determined the crossing numbers of products of all 4-vertex graphs with cycles. Klešč[7] determined the crossing numbers of products of all 4-vertex graphs with paths and stars. Klešč[8, 9, 10] showed the crossing numbers of products of all 5-vertex graphs with paths.

Klešč showed that $cr(W_3 \,\square\, P_n) = 2n$ [7], $cr(W_4 \,\square\, P_n) = 3n - 1$[9] and $cr(W_{2,3} \,\square\, P_n) = 4n$[9], where W_3 is isomorphic to K_4, W_4 and $W_{2,3}$ are isomorphic to the graphs G_{19} and G_{20} in [9] respectively.

In [13], Huang Yuanqiu et al. proved,

Lemma 1.1. For $1 \leq n \leq 3$, $cr(W_m \,\square\, P_n) = (n-1) \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + n + 1$.□

In [15], Yang Yuansheng et al. proved,

Lemma 1.2. $cr(K_{2,m} \,\square\, P_n) = 2n \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. □

In this paper we extend the results of $cr(W_{l,m} \,\square\, P_n)$ and prove that $cr(W_m \square P_n) = (n-1) \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + n + 1$ and $cr(W_{2,m} \square P_n) = 2n \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 2n$.

2 Basic Lemmas

Let

$$V(W_{l,m} \,\square\, P_n) = \{u_j^i \mid 0 \leq j \leq m + l - 1, 0 \leq i \leq n\},$$

$$E(W_{l,m} \,\square\, P_n) = (\bigcup_{i=0}^{n} \{u_j^i u_k^i \mid 0 \leq j \leq m-1,\ m \leq k \leq m + l - 1\}) \cup$$
$$(\bigcup_{i=0}^{n} \{u_0^i u_1^i, \cdots, u_{m-2}^i u_{m-1}^i, u_{m-1}^i u_0^i\}) \cup$$

$$(\bigcup_{i=1}^{n} \{u_j^{i-1}u_j^i \mid 0 \le j \le m+l-1\}).$$

Let $V^i = \{u_j^i \mid 0 \le j \le m+l-1\}$, $E^i = \{u_j^i u_k^i \mid 0 \le j \le m-1, m \le k \le m+l-1\} \cup \{u_0^i u_1^i, \cdots, u_{m-2}^i u_{m-1}^i, u_{m-1}^i u_0^i\}$, and $W_{l,m}^i = (V^i, E^i)$ for $0 \le i \le n$. Let $P^i = \{u_j^{i-1}u_j^i \mid 0 \le j \le m+l-1\}$ for $1 \le i \le n$. Then, we have

$$\begin{array}{llll} E^i \cap E^j & = & \emptyset, & 0 \le i < j \le n, \\ P^i \cap P^j & = & \emptyset, & 1 \le i < j \le n, \\ E^i \cap P^j & = & \emptyset, & 0 \le i \le n, 1 \le j \le n, \\ E(W_{l,m} \square P_n) & = & (\bigcup_{i=0}^{n} E^i) \cup (\bigcup_{i=1}^{n} P^i). & \end{array}$$

Let A, B be two disjoint subsets of $E(G)$. In a drawing D, the number of the crossings between an edge in A and another edge in B is denoted by $\nu_D(A, B)$. The number of the crossings that involve a pair of edges in A is denoted by $\nu_D(A)$. So $\nu(D) = \nu_D(E(G))$. If an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D.

Let X be a subset of $V(G)$ or of $E(G)$ for a graph G. Then $\langle X \rangle$ denotes the subgraph of G induced by X.

Lemma 2.1. Let A, B, C be mutually disjoint subsets of $E(G)$. Then,

$$\begin{aligned} \nu_D(C, A \cup B) &= \nu_D(C, A) + \nu_D(C, B), \\ \nu_D(A \cup B) &= \nu_D(A) + \nu_D(B) + \nu_D(A, B). \end{aligned} \qquad \square$$

Lemma 2.2. Let A be a subset of $E(G)$. If there exist x crossings on edges of A in a drawing D and deleting all edges in A results in a new drawing D^*, then $\nu(D) \ge \nu(D^*) + x$. $\square$

Lemma 2.3. Let $V(W_{2,m}) = \{v_0, v_1, \cdots, v_{m-1}, v_m, v_{m+1}\}$, $C = v_0v_1 \cdots v_{m-1}v_0$, $E(W_{2,m}) = E(C) \cup \{v_iv_j \mid 0 \le i \le m-1, m \le j \le m+1\}$. Let D be a good drawing of $W_{2,m}$, in which C is not crossed by any edge not in $E(C)$. If v_m and v_{m+1} lie in the same region of C, then $\nu_D(\{v_iv_m \mid 0 \le i \le m-1\}, \{v_iv_{m+1} \mid 0 \le i \le m-1\}) \ge \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. $\square$

Recently Bokal developed an operation on general graph drawings called "zip product" [4]. Let $N_{G_i}(v_i)$ represent the neighborhood of v_i in G_i for $i = 1, 2$. He defined that the *zip product* $G_1 \odot_\sigma G_2$ of the graphs G_1 and G_2 according σ is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edge $u\sigma(u)$ for every u in $N_{G_1}(v_1)$, where $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, v_1 and v_2 have the same degree, and the *zip funtion* σ is a bijection $\sigma : N_{G_1}(v_1) \to N_{G_2}(v_2)$.

Let D_1 be a drawing of G_1 and D_2 be a drawing of G_2. The *zip product* $D_1 \odot_\sigma D_2$ of the drawings D_1 and D_2 is obtained from D_1 by adding a mirrored copy of D_2 that has v_2 on the infinite region disjointly into some region of D_1 that contains v_1, removing the vertices v_1 and v_2 together with small disks around them from D_1 and D_2, and then joining the edges according to the function σ.

In the following discussion, we need two notations, $\widetilde{G}$ and $G\widehat{\Box}P_n$, denoted by Bokal[4].

With $\widetilde{G}$, Bokal denotes a double suspension of G, that is the graph obtained from G by adding two vertices v_1 and v_2 and the edges v_iv for $i = 1, 2$ and each $v \in V(G)$.

Let $P_n = u_0e_1u_1e_2 \dots e_nu_n$, where $e_i = u_{i-1}u_i, 1 \leq i \leq n$. With $G\widehat{\Box}P_n$, he denotes the *capped Cartesian product* of G and P_n, i.e. the graph, obtained from $G \Box P_n$ by adding two new vertices v_0 and v_n and connecting v_0 with all the vertices of $G \Box \{u_0\}$ and v_n with all the vertices of $G \Box \{u_n\}$.

Bokal proved the following lemma,

Lemma 2.4 (Bokal [4]). For $i = 1, 2$, let D_i be an optimal drawing of G_i, $v_i \in G_i$, v_1 and v_2 have same degree, and σ be a zip function of D_1 and D_2 according to v_1 and v_2. Then $cr(G_1 \odot_\sigma G_2) \leq cr(G_1) + cr(G_2)$. □

One can easily see that a zip product of $(n-1)$ copies of a drawing of $\widetilde{W}_{l,m}$ yields a drawing D of a capped Cartesian product of $W_{l,m}\widehat{\Box}P_{n-2}$ with two special vertices v and u, and then a zip product of two copies of a drawing of $W_{l,m} + K_1$ with D at the vertices v and u yields a drawing of $W_{l,m} \Box P_n$. By Lemma 2.4, the sum of crossing numbers of these graphs is an upper bound to the crossing number of $W_{l,m} \Box P_n$, i.e., we have

Corollary 2.5. $cr(W_{l,m} \Box P_n) \leq (n-1)\ cr(\widetilde{W}_{l,m}) + 2\ cr(W_{l,m} + K_1)$. □

In a graph G, if there is a vertex $v \in V(G)$ with $N[v] = V(G)$, then v is called a domination vertex of G. For a graph with a domination vertex, Bokal proved,

Lemma 2.6 (Bokal [4]). Let G be a graph with a domination vertex. Then for $n \geq 0$, $cr(G\widehat{\Box}P_n) = (n+1)\ cr(\widetilde{G})$. □

3 The crossing number of $W_m \Box P_n$

Lemma 3.1. $cr(\widetilde{W}_m) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$ for $m \geq 3$.

Proof. Let $V(\widetilde{W}_m) = \{x_1, x_2, v_0, v_1, \cdots, v_m\}$, $E_0 = \{v_mv_i \mid 0 \leq i \leq m-1\}, E_1 = \{x_1v_i \mid 0 \leq i \leq m-1\}, E_2 = \{x_2v_i \mid 0 \leq i \leq m-1\}$ and $E(\widetilde{W}_m) = E_0 \cup E_1 \cup E_2 \cup \{v_0v_1, \cdots, v_{m-2}v_{m-1}, v_{m-1}v_0\} \cup \{v_mx_1, v_mx_2\}$. Let the m-cycle $C = v_0 \cdots v_{m-1}v_0$. Let $H = \langle E_0 \cup E_1 \cup E_2 \rangle$, then $H \cong K_{3,m}$.

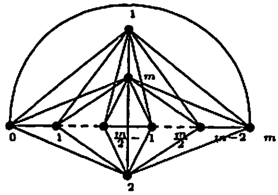


Figure 3.1. A good drawing of $\widetilde{W}_m$

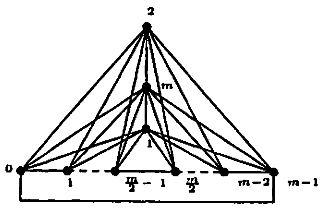


Figure 3.2. C is clean

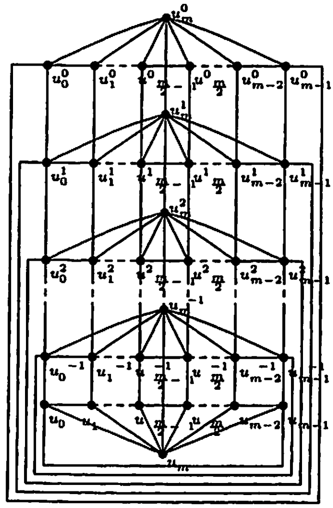


Figure 3.3. A good drawing of $W_m \,\square\, P_k$

Figure 3.1 shows a good drawing of $\widetilde{W}_m$. In this drawing, C is crossed and the corresponding drawing of H is an optimal drawing of $K_{3,m}$. Hence, this drawing has $cr(K_{3,m}) + 1$ crossings, so, $cr(\widetilde{W}_m) \leq cr(K_{3,m}) + 1 = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$. We need only to prove $cr(\widetilde{W}_m) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$.

Let D be an arbitrary drawing of $\widetilde{W}_m$. If C is not clean in D, then since $H \cong K_{3,m}$, by Lemma 2.1, $\nu(D) \geq cr(K_{3,m}) + 1 = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$. If C is clean in D, then C divides the plane into two regions, *int* C and *ext* C. Without loss of generality, we may assume that v_m lies in *ext* C, then since both x_1 and x_2 are adjacent to v_m, they have to lie in *ext* C(see Figure 3.2). Since $\langle E(C) \cup E_0 \cup E_1 \rangle \cong W_{2,m}$, by Lemma 2.3 $\nu_D(E_0, E_1) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. Similarly, we have $\nu_D(E_0, E_2) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ and $\nu_D(E_1, E_2) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. Hence for $m \geq 3$, $\nu(D) \geq \nu_D(E_0, E_1) + \nu_D(E_0, E_2) + \nu_D(E_1, E_2) \geq 3 \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$, i.e., $\nu(D) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$. □

Lemma 3.2. In a drawing D of $W_m \,\square\, P_n (n \geq 2)$, if there is an E^i on which there are at least $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$ crossings, then $\nu(D) \geq cr(W_m \,\square\, P_{n-1}) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$.

Proof. By deleting all the edges in E^i, we get a drawing D^*. By Lemma 2.2, $\nu(D) \geq \nu(D^*) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$. The graph corresponding to D^* is homeomorphic to $W_m \,\square\, P_{n-1}$, which implies that $\nu(D) \geq \nu(D^*) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1 \geq cr(W_m \,\square\, P_{n-1}) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$. □

Theorem 3.3. $cr(W_m \,\square\, P_n) = (n-1) \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + n + 1$.

Proof. Klešč proved that $cr(W_3 \square P_n) = 2n$[7] and $cr(W_4 \square P_n) = 3n-1$[9].

We need only to prove the case for $m \geq 5$. The proof is by induction on n.
(i) For $n \leq 3$, $cr(W_m \square P_n) = (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n + 1$ by Lemma 1.1.
(ii) Suppose that for $n = k-1 (k \geq 4)$, $cr(W_m \square P_n) = (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n+1$. Consider $W_m \square P_k$. Let D be an arbitrary good drawing of $W_m \square P_k$. Huang[13] showed that $cr(W_m \square P_k) \leq (k-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + k+1$ (see Figure 3.3). We need only prove that $cr(W_m \square P_k) \geq (k-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + k + 1$.
Case 1. Suppose that there is an $E^i (0 \leq i \leq k)$ on which there are at least $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$ crossings, then, by Lemma 3.2, $\nu(D) \geq cr(W_m \square P_{k-1}) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1 = (k-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + k + 1$.
Case 2. Suppose that there exist at most $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ crossings on each E^i in D.

Since W_m is a graph with a domination vertex, by Lemma 2.6, $cr(W_m \widehat{\square} P_{k-2}) = (k-1)\, cr(\widetilde{W}_m)$. By Lemma 3.1, $cr(W_m \widehat{\square} P_{k-2}) = (k-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + k - 1$.

Let $C^i = u_0^i u_1^i \cdots u_{m-1}^i u_0^i$ for $0 \leq i \leq k$. Let G' be the graph obtained by deleting all the edges of C^0 and C^k from $W_m \square P_k$. Let D' be the corresponding drawing of G' in D. Since G' is homeomorphic to $W_m \widehat{\square} P_{k-2}$, $\nu(D') \geq cr(W_m \widehat{\square} P_{k-2}) = (k-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + k - 1$.

Case 2.1. Suppose that both C^0 and C^k are not clean. If C^0 is crossed by C^k, there are even crossings between them. If C^0 is not crossed by C^k, then there are at least two more crossings than that in D'. Hence, $\nu(D) \geq \nu(D') + 2 \geq (k-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + k + 1$.

Case 2.2. Suppose that at least one of C^0 and C^k is clean. Without loss of generality, we may assume that C^0 is clean.

Note that the crossings in $\nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\} \cup P^1)$ are either self-crossings of edges in the corresponding drawing of $W_m \widehat{\square} P_{k-2}$ or they appear on the edges emanating from the same vertex, then we have

$$\nu(D') \geq (k-1)\lfloor\tfrac{m}{2}\rfloor\lfloor\tfrac{m-1}{2}\rfloor + k - 1 + \nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\} \cup P^1). \quad (3.1)$$

If $\nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\}, P^1) \geq 2$, from (3.1) we have

$$\nu(D) \geq \nu(D') \geq (k-1)\lfloor\tfrac{m}{2}\rfloor\lfloor\tfrac{m-1}{2}\rfloor + k + 1.$$

Now assume that $\nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\}, P^1) \leq 1$. Consider the subgraph $\langle E^0 \cup P^1 \cup \{u_m^1 u_i^1 \mid 0 \leq i \leq m-1\}\rangle$. C^0 divides the plane into two regions, *int* C^0 and *ext* C^0. u_m^0 and u_m^1 have to lie in the same region, say *ext* C^0. Since $\langle E^0 \cup \{u_m^1 u_i^1 \mid 0 \leq i \leq m-1\} \cup P^1\rangle$ contains a subgraph homeomorphic to $W_{2,m}$, by Lemma 2.3, we have $\nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\}, \{u_m^1 u_i^1 \mid 0 \leq i \leq m-1\} \cup P^1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. By Lemma 2.1,

$$\lfloor\tfrac{m}{2}\rfloor\lfloor\tfrac{m-1}{2}\rfloor \leq \nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\}, \{u_m^1 u_i^1 \mid 0 \leq i \leq m-1\}) + \nu_D(\{u_m^0 u_i^0 \mid 0 \leq i \leq m-1\}, P^1). \quad (3.2)$$

Similarly, for the subgraph $\langle E^0 \cup P^1 \cup P^2 \cup \{u_m^2 u_i^2 \mid 0 \le i \le m-1\}\rangle$, we have $\nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, \{u_m^2 u_i^2 \mid 0 \le i \le m-1\} \cup P^1 \cup P^2) \ge \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. By Lemma 2.1,

$$\begin{aligned}\lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor \le\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, \{u_m^2 u_i^2 \mid 0 \le i \le m-1\}) + \\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, P^1) + \\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, P^2).\end{aligned}$$

Since $\nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, P^1) \le 1$,

$$\begin{aligned}\lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor - 1 \le\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, \{u_m^2 u_i^2 \mid 0 \le i \le m-1\}) + \\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, P^2).\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned}2 \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor - 1 \le\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, \{u_m^1 u_i^1 \mid 0 \le i \le m-1\}) + \\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, \{u_m^2 u_i^2 \mid 0 \le i \le m-1\}) + \\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, P^1) + \\ & \nu_D(\{u_m^0 u_i^0 \mid 0 \le i \le m-1\}, P^2).\end{aligned}$$

Hence, for $m \ge 5$, there would be at least $2 \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor - 1 \ge \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$ crossings on E^0. A contradiction to the assumption that there are at most $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ crossings on each E^i in D.

Since for any drawing D, $\nu(D) \ge (k-1) \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + k + 1$, we have $cr(W_m \square P_k) \ge (k-1) \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + k + 1$.

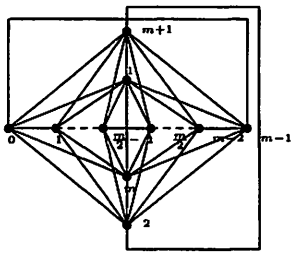
From (i), (ii), $cr(W_m \square P_n) \ge (n-1) \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + n + 1$ for $n \ge 1$. □

4 The crossing number of $W_{2,m} \square P_n$

For the graph $W_{2,m} \square P_n$, let $C^i = u_0^i u_1^i \cdots u_{m-1}^i u_0^i$, $E_j^i = \{u_j^i u_k^i \mid 0 \le k \le m-1\}$, $P_1^i = \{u_k^{i-1} u_k^i \mid 0 \le k \le m-1\}$ and $P_2^i = \{u_k^{i-1} u_k^i \mid m \le k \le m+1\}$ for $0 \le i \le n$ and $m \le j \le m+1$. Then $E^i = E(C^i) \cup E_m^i \cup E_{m+1}^i$ and $P^i = P_1^i \cup P_2^i$. For each C^i, we define function f_D counting the number of crossings on C^i in D as follows:

$$\begin{aligned}f_D(i) = \nu_D(E(C^i)) + \textstyle\sum_{k=0}^{n} \nu_D(E(C^i), E^k \setminus E(C^k)) + \\ \textstyle\sum_{k=1}^{n} \nu_D(E(C^i), P^k) + \sum_{0 \le k \ne i \le n} \nu_D(E(C^i), E(C^k))/2.\end{aligned}$$

Note that $\nu_D(E(C^i), E(C^k))$ is even, so $f_D(i)$ is a non negative integer.



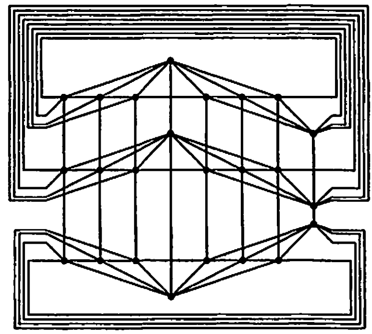


Figure 4.1. A good drawing of $\widetilde{W}_{2,m}$ Figure 4.2. A good drawing of $W_{2,6}\Box P_2$

Lemma 4.1. $cr(\widetilde{W}_{2,m}) = 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ for $m \geq 3$.

Proof. Let $V(\widetilde{W}_{2,m}) = \{x_1, x_2, v_0, v_1, \cdots, v_{m+1}\}$, $E_0 = \{v_m v_i \mid 0 \leq i \leq m-1\}$, $E_1 = \{v_{m+1}v_i \mid 0 \leq i \leq m-1\}$, $E_2 = \{x_1 v_i \mid 0 \leq i \leq m-1\}$, $E_3 = \{x_2 v_i \mid 0 \leq i \leq m-1\}$, $C = v_0 v_1 \cdots v_{m-1} v_0$, $C' = x_1 v_m x_2 v_{m+1} x_1$ and $E(\widetilde{W}_{2,m}) = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E(C) \cup E(C')$. Let $H = \langle E_0 \cup E_1 \cup E_2 \cup E_3\rangle$, then $H \cong K_{4,m}$.

Figure 4.1 shows a good drawing of $\widetilde{W}_{2,m}$. In this drawing, C crosses C' twice, and the corresponding drawing of H is an optimal drawing of $K_{4,m}$. Hence, this drawing has $cr(K_{4,m})+2$ crossings, so, $cr(\widetilde{W}_{2,m}) \leq cr(K_{4,m}) + 2 = 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$. We need only prove $cr(\widetilde{W}_{2,m}) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$.

Let D be an arbitrary drawing of $\widetilde{W}_{2,m}$. Since $H \cong K_{4,m}$, $\nu_D(E(H)) \geq cr(K_{4,m}) = 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. If there are at least 2 crossings on $C \cup C'$, then by Lemma 2.1 $\nu(D) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$. If there exists at most one crossing on $C \cup C'$, then since $\nu_D(C, C')$ is even, we can obtain that $\nu_D(C, C') = 0$ and at least one of C and C' has to be clean.

If C is clean, then C divides the plane into two regions, *int* C and *ext* C. All vertices of C' have to lie in the same region, say *int* C. Since $\langle E(C) \cup E_0 \cup E_1\rangle \cong W_{2,m}$ and v_m and v_{m+1} both lie in *int* C, by Lemma 2.3, we have $\nu_D(E_0, E_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Similarly, we have $\nu_D(E_i, E_j) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ for $0 \leq i < j \leq 3$. Thus, $\nu(D) \geq \nu_D(E(H)) \geq \binom{4}{2}\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ for $m \geq 3$.

If C' is clean, then C' divides the plane into two regions, *int* C' and *ext* C'. All vertices of C has to lie in the same region, say *int* C'. For $0 \leq i < j \leq m-1$, since $\langle\{v_i x_1, v_i x_2, v_i v_m, v_i v_{m+1}\} \cup \{v_j x_1, v_j x_2, v_j v_m, v_j v_{m+1}\} \cup E(C')\rangle \cong W_{2,4}$ and v_i and v_j both lie in *int* C', by Lemma 2.3, $\nu_D(\{v_i x_1, v_i x_2, v_i v_m, v_i v_{m+1}\}, \{v_j x_1, v_j x_2, v_j v_m, v_j v_{m+1}\}) \geq 2$. Thus $\nu(D) \geq \nu_D(E(H)) \geq 2\binom{m}{2} = m(m-1) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ for $m \geq 3$.

Since $\nu(D) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ for any drawing D of $\widetilde{W}_{2,m}$, we have

$cr(\widetilde{W}_{2,m}) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+2$. □

Lemma 4.2. $cr(W_{2,m} \square P_n) \leq 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$.

Proof. Since $W_{2,m} + K_1 \cong \widetilde{W}_m$, by Lemma 3.1, $cr(W_{2,m} + K_1) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+1$. By Corollary 2.5 and Lemma 4.1, we have $cr(W_{2,m} \square P_n) \leq (n-1)\, cr(\widetilde{W}_{2,m})+2\, cr(W_{2,m}+K_1) = 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+2n$. Figure 4.2 shows a drawing of $W_{2,6} \square P_2$ with 28 crossings. This drawing can be extended to a drawing of $W_{2,m} \square P_n$ with $2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$ crossings. □

Lemma 4.3. For a drawing D of $W_{2,m} \square P_n$, $\nu(D) \geq 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + \sum_{i=0}^{n} f_D(i)$.

Proof. By deleting the edges of each C^i, we get a subgraph H of $W_{2,m}\square P_n$, such that $H \cong K_{2,m} \square P_n$. By Lemma 1.2, $\nu_D(E(H)) \geq 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. By Lemma 2.1, we have $\nu(D) = \nu_D(E(H)) + \sum_{i=0}^{n} f_D(i)$. Hence, $\nu(D) \geq 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + \sum_{i=0}^{n} f_D(i)$. □

Lemma 4.4. In a drawing D of $W_{2,m} \square P_n (n \geq 1, m \geq 4)$, if there are at most $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$ crossings on E^i for all i satisfying $0 \leq i \leq n$, then $f_D(j) \geq 1 (0 \leq j \leq n)$.

Proof. By contradiction. Suppose that there is a j with $f_D(j) < 1$. By symmetry, we need only consider the cases for $j \leq \lfloor\frac{n}{2}\rfloor$. Since $f_D(j)$ is a non negative integer, we have $f_D(j) = 0$. Hence, C^j is clean. It divides the plane into two regions, *int* C^j and *ext* C^j. Without loss of generality, we may assume u^j_m lies in *int* C^j. Since there is a path $P_{u^j_m u^{j+1}_m u^{j+1}_0 u^{j+1}_{m+1} u^j_{m+1}}$, both u^{j+1}_m and u^j_{m+1} have to lie in *int* C^j. Since the subgraph $\langle E^j_m \cup E^j_{m+1} \cup E(C^j)\rangle$ is homeomorphic to $W_{2,m}$, we have $\nu_D(E^j_m, E^j_{m+1}) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ by Lemma 2.3. Similarly, we have $\nu_D(E^j_m, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\nu_D(E^j_{m+1}, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Thus for $m \geq 4$ there would be at least $3\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings on E^j, a contradiction. □

Lemma 4.5. $cr(W_{2,m} \square P_1) = 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ for $m \geq 4$.

Proof. By Lemma 4.2, we have $cr(W_{2,m} \square P_1) \leq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$. Let D be an arbitrary drawing of $W_{2,m} \square P_1$. We need only prove that $\nu(D) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$. By contradiction, suppose that $\nu(D) \leq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$, then there are at most $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$ crossings on E^i for $i = 0, 1$. By Lemma 4.4, $f_D(0) \geq 1$ and $f_D(1) \geq 1$. By Lemma 4.3, $\nu(D) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+f(0)+f(1) \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+2$, a contradiction. □

Lemma 4.6. In a drawing D of $W_{2,m} \square P_n (n \geq 2)$, if there is an E^i on which there are at least $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings, then $\nu(D) \geq cr(W_{2,m} \square P_{n-1}) + 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$.

Proof. By deleting all the edges in E^i in D, we get a drawing D^*. By Lemma 2.2, we have $\nu(D) \geq \nu(D^*) + 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$. The graph corresponding to D^* is homomorphic to $W_{2,m} \square P_{n-1}$ or contains $W_{2,m} \square P_{n-1}$

as a subgraph, which implies that $\nu(D) \geq \nu(D^*) + 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2 \geq cr(W_{2,m}\Box P_{n-1}) + 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$. □

Lemma 4.7. In a drawing D of $W_{2,m} \Box P_n (n \geq 2, m \geq 4)$, if there are at most $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$ crossings on E^i for all i satisfying $0 \leq i \leq n$, then $f_D(j) \geq 2$ $(1 \leq j \leq n-1)$.

Proof. By contradiction. Suppose that there is a j $(1 \leq j \leq n-1)$ with $f_D(j) < 2$. By Lemma 4.4, $f_D(j) \geq 1$. Since $f_D(j)$ is an integer, we have $f_D(j) = \nu_D(C^j) + \sum_{k=0}^{n} \nu_D(C^j, E^k \setminus C^k) + \sum_{k=1}^{n} \nu_D(C^j, P^k) + \sum_{0 \leq k \neq j \leq n} \nu_D(C^j, C^k)/2 = 1$.

Case 1. Suppose that there exists a $C^i (i \neq j)$ such that $\nu_D(C^i, C^j) > 0$. Since $f_D(j) = 1$ and $\nu_D(C^i, C^j)$ is even, we have $\nu_D(C^j) = 0$ and C^j is not crossed by any edge in $E^j_m \cup E^j_{m+1} \cup (E^{j+1}_m \cup P^{j+1}_1)$. C^j divides the plane into two regions, *int* C^j and *ext* C^j. Since there is a path $P_{u^j_m u^{j+1}_m u^{j+1}_0 u^{j+1}_{m+1} u^j_{m+1}}$, all the vertices u^j_m, u^j_{m+1} and u^{j+1}_m have to lie the same region, say *int* C^j. Since the subgraph $\langle E^j_m \cup E^j_{m+1} \cup E(C^j)\rangle$ is homeomorphic to $W_{2,m}$, by Lemma 2.3, we have $\nu_D(E^j_m, E^j_{m+1}) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Similarly, we have $\nu_D(E^j_m, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\nu_D(E^j_{m+1}, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Thus, for $m \geq 4$, there would be at least $3\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings on E^j, a contradiction.

Case 2. Suppose that $\nu_D(C^j, C^i) = 0$ for each i, $0 \leq i \neq j \leq n$.

Case 2.1. Suppose that $\nu_D(C^j) = 0$, then C^j divides the plane into two regions, *int* C^j and *ext* C^j. Without loss of generality, we may assume that u^j_m lies in *int* C^j. Then since $f_D(j) = 1$ and there are paths $P_{u^j_m u^{j+1}_m u^{j+1}_0 u^{j+1}_{m+1} u^j_{m+1}}$ and $P_{u^j_m u^{j-1}_m u^{j-1}_0 u^{j-1}_{m+1} u^j_{m+1}}$, u^j_{m+1} has to lie in *int* C^j. Hence, u^{j+1}_m and u^{j-1}_m have to lie in *int* C^j.

If C^j is not crossed by any edge in $E^j_m \cup E^j_{m+1} \cup (E^{j+1}_m \cup P^{j+1}_1)$, then by Lemma 2.3, we have $\nu_D(E^j_m, E^j_{m+1}) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, $\nu_D(E^j_m, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\nu_D(E^j_{m+1}, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Thus, for $m \geq 4$, there would be at least $3\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings on E^j, a contradiction.

If C^j is crossed by some edge $e \in E^j_m \cup E^j_{m+1}$, say $e \in E^j_{m+1}$, then since $f_D(j) = 1$, C^j is not crossed by any edge in $E^j_m \cup (E^j_{m+1} \setminus \{e\}) \cup (E^{j+1}_m \cup P^{j+1}_1) \cup (E^{j-1}_m \cup P^j_1)$. By Lemma 2.3, since $\langle E(C^j) \cup E^j_m \cup (E^j_{m+1} \setminus \{e\})\rangle$ contains a subgraph homeomorphic to $W_{2,m-1}$, we have $\nu_D(E^j_m, E^j_{m+1} \setminus \{e\}) \geq \lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m-2}{2}\rfloor$. Similarly, we have $\nu_D(E^j_m, E^{j+1}_m \cup P^{j+1}_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\nu_D(E^j_m, E^{j-1}_m \cup P^j_1) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. For $m \geq 4$, there would be at least $\nu_D(E^j_{m+1}, C^j) + \nu_D(E^j_m, E^j_{m+1} \setminus \{e\}) + \nu_D(E^j_m, E^{j+1}_m \cup P^{j+1}_1) + \nu_D(E^j_m, E^{j-1}_m \cup P^j_1) \geq 1 + \lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m-2}{2}\rfloor + 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings on E^j, a contradiction.

If C^j is crossed by some edge $e \in E_m^{j+1} \cup P_1^{j+1}$, then since $f_D(j) = 1$, C^j is not crossed by any edge in $E_m^j \cup E_{m+1}^j \cup (E_m^{j-1} \cup P_1^j)$. By Lemma 2.3, we have $\nu_D(E_m^j, E_{m+1}^j) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, $\nu_D(E_m^j, E_m^{j-1} \cup P_1^j) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ and $\nu_D(E_{m+1}^j, E_m^{j-1} \cup P_1^j) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Thus for $m \geq 4$, there would be at least $\nu_D(E_m^j, E_{m+1}^j) + \nu_D(E_m^j, E_m^{j-1} \cup P_1^j) + \nu_D(E_{m+1}^j, E_m^{j-1} \cup P_1^j) \geq 3\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings on E^j, a contradiction.

Case 2.2. Suppose that $\nu_D(C^j) > 0$, then since $f_D(j) = 1$, C^j is only crossed by itself, i.e., $\nu_D(E(C^j), E \setminus E(C^j)) = 0$. C^j divides the plane into three regions. All the vertices u_m^j, u_{m+1}^j and u_m^{j-1} have to lie in the same region of C^j. For $m \geq 4$, by an argument similar to the one above, there would be at least $\nu_D(E_m^j, E_{m+1}^j) + \nu_D(E_m^j, E_m^{j-1} \cup P_1^j) + \nu_D(E_{m+1}^j, E_m^{j-1} \cup P_1^j) \geq 3\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor \geq 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings on E^j, a contradiction. □

Theorem 4.8. $cr(W_{2,m} \,\square\, P_n) = 2n \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$.

Proof. Klešč proved that $cr(W_{2,3} \,\square\, P_n) = 4n$[9]. We need only prove the case for $m \geq 4$. The proof is by induction on n.

(i) For $n = 1$, $cr(W_{2,m} \,\square\, P_n) = 2n \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$ by Lemma 4.5.

(ii) Suppose that for $n = k-1 (k \geq 2)$, $cr(W_{2,m} \square P_n) = 2n \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$. Consider $W_{2,m} \,\square\, P_k$. By Lemma 4.2, $cr(W_{2,m} \,\square\, P_k) \leq 2k \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2k$. We need only prove that $cr(W_{2,m} \,\square\, P_k) \geq 2k \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2k$. Let D be an arbitrary drawing of $W_{2,m} \,\square\, P_k$.

Case 1. Suppose that there is an $E^i (0 \leq i \leq k)$ on which there are at least $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2$ crossings, then, by Lemma 4.6, $\nu(D) \geq cr(W_{2,m} \,\square\, P_{k-1}) + 2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2 = 2k \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2k$.

Case 2. Suppose that there exist at most $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$ crossings on each E^i in D.

By Lemma 4.4, $f_D(0) \geq 1$ and $f_D(k) \geq 1$. By Lemma 4.7, $f_D(i) \geq 2$ for $1 \leq i \leq k-1$. By Lemma 4.3, $\nu(D) \geq 2k\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + \sum_{i=0}^{k} f_D(i) \geq 2k\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2k$.

Since for any drawing D of $W_{2,m} \,\square\, P_k$, $\nu(D) \geq 2k\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2k$, we have $cr(W_{2,m} \,\square\, P_k) \geq 2k\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2k$.

From (i), (ii), $cr(W_{2,m} \square P_n) \geq 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$ for $n \geq 1$ and $m \geq 4$.□

5 Conclusion

Computing the exact crossing number of $W_{l,m} \,\square\, P_n$ is a very difficult task.

Huang[13] proved that $cr(W_m \square P_n) = (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n + 1$ for $n \leq 3$. Klešč showed that $cr(W_3 \,\square\, P_n) = 2n$ [7] and $cr(W_4 \,\square\, P_n) = 3n-1$[9]. Klešč showed that $cr(W_{2,3} \,\square\, P_n) = 4n$[9]. We prove that $cr(W_m \,\square\, P_n) = (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n + 1$ for $n \geq 4$ and that $cr(W_{2,m} \,\square\, P_n) \geq 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n$

for $n \geq 1$ in this paper. Next we will give a general upper bound to $cr(W_{l,m} \square P_n)$.

Lemma 5.1. $cr(W_{l,m} \square P_n) \leq (n-1)F_2(l,m) + 2F_1(l,m)$, where

$$F_2(l,m) = \lfloor \tfrac{l+2}{2} \rfloor \lfloor \tfrac{l+1}{2} \rfloor \lfloor \tfrac{m+2}{2} \rfloor \lfloor \tfrac{m+1}{2} \rfloor - lm + l$$

$$\text{and} \quad F_1(l,m) = \lfloor \tfrac{l+1}{2} \rfloor \lfloor \tfrac{l}{2} \rfloor \lfloor \tfrac{m+1}{2} \rfloor \lfloor \tfrac{m}{2} \rfloor - \lfloor \tfrac{l}{2} \rfloor \lfloor \tfrac{m}{2} \rfloor + \lfloor \tfrac{l+1}{2} \rfloor.$$

Proof.

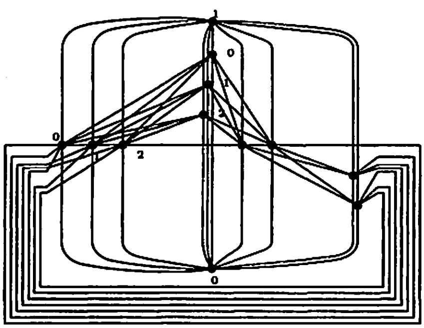


Figure 5.1. $D(\widetilde{W}_{5,5})$

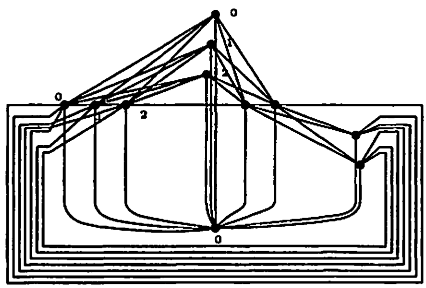


Figure 5.2. A good drawing of $D(W_{5,5} + K_1)$

A drawing $D(\widetilde{W}_{5,5})$ with $F_2(5,5)$ crossings is exhibited in Figure 5.1. A drawing $D(W_{5,5} + K_1)$ with $F_1(5,5)$ crossings can be obtained by deleting z_1 and all related edges in $D(\widetilde{W}_{5,5})$ (see Figure 5.2). These two drawings can be extended to produce a drawing $D(\widetilde{W}_{l,m})$ with $F_2(l,m)$ crossings and a drawing $D(W_{l,m} + K_1)$ with $F_1(l,m)$ crossings. By Corollary 2.5, we have $cr(W_{l,m} \square P_n) \leq (n-1)\ F_2(l,m) + 2F_1(l,m)$. □

From Lemma 5.1, we have the following conjecture,

Conjecture 5.1. $cr(W_{l,m} \square P_n) = (n-1)F_2(l,m) + 2F_1(l,m)$, where

$$F_2(l,m) = \lfloor \tfrac{l+2}{2} \rfloor \lfloor \tfrac{l+1}{2} \rfloor \lfloor \tfrac{m+2}{2} \rfloor \lfloor \tfrac{m+1}{2} \rfloor - lm + l$$

$$\text{and} \quad F_1(l,m) = \lfloor \tfrac{l+1}{2} \rfloor \lfloor \tfrac{l}{2} \rfloor \lfloor \tfrac{m+1}{2} \rfloor \lfloor \tfrac{m}{2} \rfloor - \lfloor \tfrac{l}{2} \rfloor \lfloor \tfrac{m}{2} \rfloor + \lfloor \tfrac{l+1}{2} \rfloor.$$

By Theorems 3.3 and 4.8, the Conjecture 5.1 holds for the cases $l \leq 2$.

References

[1] J. Adamsson and R. B. Richter, Arrangements, circular arrangements and the crossing number of $C_7 \times C_n$, *J. Combinatorial Theory, Series B,* 90(2004), 21–39.

[2] L. W. Beineke and R. D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four, *J. Graph Theory,* 4(1980), 145–155.

[3] L. Beineke and R. Wilson, Selected topics in graph theory, *Academic Press* 1978, 68–72.

[4] D. Bokal, On the crossing number of Cartesian products with paths, *J. Combinatorial Theory, Series B,* to appear.

[5] S. Jendroľ and M. Ščerbová, On the crossing numbers of $S_m \times P_n$ and $S_m \times C_n$, *Časopis pro pěstování matematiky,* 107(1982), 225–230.

[6] D. J. Kleitman, The crossing number of $K_{5,n}$, *J. Combin. Theory* 9(1971) 315–323.

[7] M. Klešč, The crossing numbers of products of paths and stars with 4-vertex graphs, *J. Graph Theory,* 6(1994), 605–614.

[8] M. Klešč, The crossing numbers of $K_5 \times P_n$, *Tatra Mt. Math. Publ.,* 18(1999), 63–68.

[9] M. Klešč, The crossing numbers of Cartesian products of paths with 5-vertex graphs, *Discrete Mathematics,* 233(2001), 353–359.

[10] M. Klešč, On the Crossing Numbers of Products of Stars and Graphs of Order Five, *Graphs and Combinatorics,* 17(2001), 289–294.

[11] Lev Yu. Glebsky and Gelasio Salazar, The crossing number of $C_m \times C_n$ is as conjectured for $n \geq m(m+1)$, *J. Graph Theory,* 47(2004), 53–72.

[12] D. R. Woodall, Cyclic-Order graphs and Zarankiewicz's crossing-number conjecture, *J. Graph Theory* 17(1993) 657–671.

[13] Yu Ping, Huang Yuanqiu, On the crossing numbers of $P_m \times W_n$, *Journal of Natural Science of Hunan Normal University,* 28(2005),14–16.

[14] K. Zarankiewicz, On a problem of P. Turán concerning graphs, *Fund. Math* 41(1954) 137–145.

[15] Zheng Wenping, Lin Xiaohui, Yang Yuansheng, The Crossing Number of $K_{m,n} \square P_l$, *submitted* 2005.