On the sum of reciprocal Tribonacci numbers

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Abstract

The Tribonacci Zeta functions are defined by $\zeta_T(s) = \sum_{k=1}^\infty T_k^{-s}$. We discuss the partial infinite sum $\sum_{k=n}^\infty T_k^{-s}$ for some positive integer n. We also consider the continued fraction expansion including Tribonacci numbers.

1 Introduction.

Consider the *Tribonacci Zeta functions*, defined by

$$\zeta_T(s) = \sum_{n=1}^\infty \frac{1}{T_n^s},$$

where T_n is the n-th Tribonacci number ([6, Ch.46], [8, A000073]) defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (n \geq 3), \quad T_0 = 0, \quad T_1 = T_2 = 1.$$

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In [7] the partial infinite sums of reciprocal Fibonacci numbers were studied. In [2, 5] their results were generalized. In this paper we shall show the following. Here, $(\!(\cdot)\!)$ denotes the nearest integer. Namely, $(\!(x)\!) = \lfloor x + 1/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part.

Theorem 1

$$\left(\!\!\left(\left(\sum_{k=n}^{\infty} \frac{1}{T_k} \right)^{-1} \right)\!\!\right) = T_n - T_{n-1} \quad (n \geq 1).$$

2 Proof of Theorem 1

Let α, β, and γ be the roots of the equation $x^3 - x^2 - x - 1 = 0$. It is known (e.g., see [3]) that for any integer n

$$\begin{aligned} T_n &= c_1 \alpha^{n+1} + c_2 \beta^{n+1} + c_3 \gamma^{n+1} \\ &= c_4 \alpha^n + c_5 \beta^n + c_6 \gamma^n , \end{aligned}$$

where

$$c_1 = \frac{1}{(\alpha - \beta)(\alpha - \gamma)}, \quad c_2 = \frac{1}{(\beta - \alpha)(\beta - \gamma)}, \quad c_3 = \frac{1}{(\gamma - \alpha)(\gamma - \beta)}$$
$$c_4 = \frac{1}{-\alpha^2 + 4\alpha - 1}, \quad c_5 = \frac{1}{-\beta^2 + 4\beta - 1}, \quad c_6 = \frac{1}{-\gamma^2 + 4\gamma - 1}.$$

Assume that α is the real root of the equation $x^3 - x^2 - x - 1 = 0$, given by

$$\alpha = \frac{\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1}{3} = 1.839286755 .$$

On the other hands,

$$\begin{aligned} \beta, \gamma &= \frac{2 - (1 \pm \sqrt{-3}) \sqrt[3]{19 - 3\sqrt{33}} - (1 \mp \sqrt{-3}) \sqrt[3]{19 + 3\sqrt{33}}}{6} \\ &= -0.4196433776 \pm 0.6062907292\sqrt{-1} . \end{aligned}$$

Then, for any positive integer n

$$T_n = ((c_4\alpha^n)) \quad (c_4 = 0.33622811699).$$

Precisely speaking, we have the following.

Lemma 1 *For any positive integer n*

$$|T_n - c_4\alpha^n| < c_7 d^n,$$

where $c_7 = 0.51998$ *and* $d = 0.7373527$.

Proof. Put $\beta = -a + bi$ ($i = \sqrt{-1}$)with $a = 0.4196433776$ and $b = 0.6062907292$. Then

$$\begin{aligned} c_5\beta^n + c_6\gamma^n &= \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n \\ &= \frac{1}{2bi}\left(\frac{-a+bi}{-a-\alpha+bi}(a^2+b^2)^{n/2}(\cos\theta + i\sin\theta)^n \right. \\ &\qquad \left. - \frac{-a-bi}{-a-\alpha-bi}(a^2+b^2)^{n/2}(\cos\theta - i\sin\theta)^n\right), \end{aligned}$$

where

$$\cos\theta = -\frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Thus,

$$\begin{aligned} & c_5\beta^n + c_6\gamma^n \\ &= \frac{(a^2+b^2)^{n/2}((a-bi)(a+\alpha+bi)e^{in\theta} - (a+bi)(a+\alpha-bi)e^{-in\theta})}{2bi(a+\alpha-bi)(a+\alpha+bi)} \\ &= \frac{(a^2+b^2)^{n/2}}{b((a+\alpha)^2+b^2)}((a(a+\alpha)+b^2)\sin n\theta - b\alpha\cos n\theta) \\ &= \frac{\sqrt{(a(a+\alpha)+b^2)^2+(b\alpha)^2}\sin(n\theta-\phi)}{b((a+\alpha)^2+b^2)}(a^2+b^2)^{n/2}, \end{aligned}$$

where

$$\cos\phi = \frac{a(a+\alpha)+b^2}{\sqrt{(a(a+\alpha)+b^2)^2+(b\alpha)^2}}$$

and

$$\sin\phi = \frac{b\alpha}{\sqrt{(a(a+\alpha)+b^2)^2+(b\alpha)^2}}\,.$$

We set

$$c_7 = \frac{\sqrt{(a(a+\alpha)+b^2)^2+(b\alpha)^2}}{b\big((a+\alpha)^2+b^2\big)} = 0.51998$$

and

$$d = \sqrt{a^2+b^2} = 0.7373527\,.$$

∎

Put

$$\epsilon_k = \frac{T_k - c_4\alpha^k}{c_4\alpha^k} = \frac{c_5}{c_4}\left(\frac{\beta}{\alpha}\right)^k + \frac{c_6}{c_4}\left(\frac{\gamma}{\alpha}\right)^k \quad (k=1,2,\dots)\,,$$

so that $T_k = (1+\epsilon_k)c_4\alpha^k$. Since $|\beta/\alpha| = |\alpha/\beta|$,

$$c_4 = 0.33622811699$$

and

$$c_5, c_6 = -0.16811405849 \mp 0.19832414008\sqrt{-1}\,,$$

we have $\epsilon_k \to 0$ $(k\to\infty)$. In special, for $k \geq 1$

$$|\epsilon_k| \leq \epsilon_1 = 0.617024232\,.$$

Using the expansion formula

$$\frac{1}{1\pm\epsilon} = 1 \mp \epsilon + O(\epsilon^2) \quad (|\epsilon|<1)\,,$$

we obtain

$$\frac{1}{T_k} = \frac{1}{c_4\alpha^k(1+\epsilon_k)}$$
$$= \frac{1}{c_4\alpha^k}(1-\epsilon_k+O(\epsilon_k^2)).$$

Taking the summation,

$$\sum_{k\ge n}\frac{1}{T_k}$$
$$= \frac{1}{c_4}\sum_{k\ge n}\frac{1}{\alpha^k} - \frac{c_5}{c_4^2}\sum_{k\ge n}\left(\frac{\beta}{\alpha^2}\right)^k - \frac{c_6}{c_4^2}\sum_{k\ge n}\left(\frac{\gamma}{\alpha^2}\right)^k + \frac{1}{c_4}O\left(\sum_{k\ge n}\frac{\epsilon_k^2}{\alpha^k}\right)$$
$$= \frac{\alpha}{c_4\alpha^n(\alpha-1)} - \frac{c_5}{c_4^2}\frac{\beta^n}{\alpha^{2n}}\frac{\alpha^2}{\alpha^2-\beta} - \frac{c_6}{c_4^2}\frac{\gamma^n}{\alpha^{2n}}\frac{\alpha^2}{\alpha^2-\gamma} + O\left(\frac{\epsilon_n^2}{\alpha^n}\right).$$

Taking its reciprocal,

$$\left(\sum_{k\ge n}\frac{1}{T_k}\right)^{-1}$$
$$= \left(\frac{\alpha}{c_4\alpha^n(\alpha-1)}\right.$$
$$\left.\times\left(1-\frac{c_5}{c_4}\left(\frac{\beta}{\alpha}\right)^n\frac{(\alpha-1)\alpha}{\alpha^2-\beta} - \frac{c_6}{c_4}\left(\frac{\gamma}{\alpha}\right)^n\frac{(\alpha-1)\alpha}{\alpha^2-\gamma}\right)+O(\epsilon_n^2)\right)^{-1}$$
$$= \frac{c_4\alpha^n(\alpha-1)}{\alpha}$$
$$\times\left(1+\frac{c_5}{c_4}\left(\frac{\beta}{\alpha}\right)^n\frac{(\alpha-1)\alpha}{\alpha^2-\beta} + \frac{c_6}{c_4}\left(\frac{\gamma}{\alpha}\right)^n\frac{(\alpha-1)\alpha}{\alpha^2-\gamma}+O(\epsilon_n^2)\right)$$
$$= c_4\alpha^n - c_4\alpha^{n-1} + \delta_n + O(\alpha^n\epsilon_n^2)$$
$$= T_n - T_{n-1} - c_5\beta^{n-1}(\beta-1) - c_6\gamma^{n-1}(\gamma-1) + \delta_n + O(\alpha^n\epsilon_n^2),$$

where

$$\delta_n = \frac{c_5(\alpha-1)^2}{\alpha^2-\beta}\beta^n + \frac{c_6(\alpha-1)^2}{\alpha^2-\gamma}\gamma^n \to 0 \quad (n\to\infty)$$

and $|\delta_n| \le \delta_1 = 0.07$ $(n \ge 1)$. Notice that

$$\alpha^n \epsilon_n^2 = \frac{(c_5\beta^n + c_6\gamma^n)^2}{c_4^2\alpha^n} \to 0 \quad (n \to \infty)$$

and $\alpha^n\epsilon_n^2 \le \alpha\epsilon_1^2 = 0.700251$ $(n \ge 1)$. Therefore, the error term

$$-c_5\beta^{n-1}(\beta - 1) - c_6\gamma^{n-1}(\gamma - 1) + \delta_n + O(\alpha^n\epsilon_n^2)$$

is less than $1/2$ if n is large enough. A precise argument is that the absolute value of

$$O(\alpha^n\epsilon_n^2) = \frac{c_4\alpha^n(\alpha - 1)}{\alpha}\frac{A_n^2}{1 - A_n}$$

is less than 0.002 for any small positive integer n, where

$$A_n = \delta_n - \frac{\alpha - 1}{\alpha}\sum_{k \ge n}\frac{\epsilon_k^2}{\alpha^{k-n}(1 + \epsilon_k)}\,.$$

In addition,

$$\begin{aligned}|c_5\beta^{n-1}(\beta - 1) + c_6\gamma^{n-1}(\gamma - 1)| &\le |c_5\beta^3(\beta - 1) + c_6\gamma^3(\gamma - 1)| \\ &= 0.244129 \quad (n \ge 3)\,.\end{aligned}$$

Therefore, the error term above is less than $1/2$ for $n \ge 3$. The cases for $n = 1, 2$ are manually checked.

3 Related results

The following analogous results are similarly obtained as in that of Theorem 1 .

Theorem 2

$$\left(\left(\left(\sum_{k=n}^{\infty}\frac{1}{T_{2k}}\right)^{-1}\right)\right)=T_{2n}-T_{2n-2}\quad (n\geq 1)\,. \tag{1}$$

$$\left(\left(\left(\sum_{k=n}^{\infty}\frac{1}{T_{2k-1}}\right)^{-1}\right)\right)=T_{2n-1}-T_{2n-3}\quad (n\geq 2)\,. \tag{2}$$

$$\left(\left(\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{T_{k}}\right)^{-1}\right)\right)=(-1)^n(T_{n}+T_{n-1})\quad (n\geq 2)\,. \tag{3}$$

$$\left(\left(\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{T_{2k}}\right)^{-1}\right)\right)=(-1)^n(T_{2n}+T_{2n-2})\quad (n\geq 1)\,. \tag{4}$$

$$\left(\left(\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{T_{2k-1}}\right)^{-1}\right)\right)=(-1)^n(T_{2n-1}+T_{2n-3})\quad (n\geq 2)\,. \tag{5}$$

Proof. We shall sketch the proof by showing that the identities hold for any large positive integer n. By numerical calculations or more precise arguments, we can see that they also hold for smaller positive integers n.

First, we shall prove the identity (1). The identity (2) is similarly proved.

By Lemma 1

$$\frac{1}{T_{2k}}=\frac{1}{c_4\alpha^{2k}+O(d^{2k})}=\frac{1}{c_4\alpha^{2k}\left(1+O\left(\left(\frac{d}{\alpha}\right)^{2k}\right)\right)}$$

$$=\frac{1}{c_4\alpha^{2k}}\left(1+O\left(\left(\frac{d}{\alpha}\right)^{2k}\right)\right)$$

$$=\frac{1}{c_4\alpha^{2k}}+O\left(\left(\frac{d}{\alpha^2}\right)^{2k}\right)\,.$$

By taking the summation,

$$\sum_{k\ge n}\frac{1}{T_{2k}}=\frac{1}{c_4}\sum_{k\ge n}\frac{1}{\alpha^{2k}}+O\left(\sum_{k\ge n}\left(\frac{d}{\alpha^2}\right)^{2k}\right)$$
$$=\frac{\alpha^2}{c_4\alpha^{2n}(\alpha^2-1)}+O\left(\left(\frac{d}{\alpha^2}\right)^{2n}\right).$$

Taking its reciprocal, by Lemma 1

$$\left(\sum_{k\ge n}\frac{1}{T_{2k}}\right)^{-1}=\left(\frac{\alpha^2}{c_4\alpha^{2n}(\alpha^2-1)}\left(1+O\left(\left(\frac{d}{\alpha}\right)^{2n}\right)\right)\right)^{-1}$$
$$=\frac{c_4\alpha^{2n}(\alpha^2-1)}{\alpha^2}\left(1+O\left(\left(\frac{d}{\alpha}\right)^{2n}\right)\right)$$
$$=c_4\alpha^{2n}-c_4\alpha^{2n-2}+O(d^{2n})$$
$$=T_{2n}-T_{2n-2}+O(d^{2n}).$$

If n is sufficiently large, the error term $O(d^{2n})$ becomes less than 1/2.

Next, we shall prove the identity (3). The identities (4) and (5) are similarly proved.
Taking the summation,

$$\sum_{k\ge n}\frac{(-1)^k}{T_k}=\frac{1}{c_4}\sum_{k\ge n}\left(-\frac{1}{\alpha}\right)^k+O\left(\sum_{k\ge n}\left(-\frac{d}{\alpha^2}\right)^k\right)$$
$$=\frac{\alpha}{c_4(-\alpha)^n(\alpha+1)}+O\left(\left(-\frac{d}{\alpha^2}\right)^n\right).$$

Taking its reciprocal, by Lemma 1

$$\begin{aligned}\left(\sum_{k\geq n}\frac{(-1)^k}{T_k}\right)^{-1} &= \left(\frac{\alpha}{c_4(-\alpha)^n(\alpha+1)}\left(1+O\left(\left(\frac{d}{\alpha}\right)^n\right)\right)\right)^{-1}\\ &= \frac{c_4(-\alpha)^n(\alpha+1)}{\alpha}\left(1+O\left(\left(\frac{d}{\alpha}\right)^n\right)\right)\\ &= (-1)^n(c_4\alpha^n+c_4\alpha^{n-1})+O(-d^n)\\ &= (-1)^n(T_n+T_{n-1})+O(d^n).\end{aligned}$$

If n is sufficiently large, the error term $O(d^n)$ becomes less than 1/2. ∎

4 Continued fraction expansion of Tribonacci Zeta functions

The author studied several continued fraction expansions of some types of Fibonacci zeta functions and Lucas zeta functions in [4].

A continued fraction expansion of $\zeta_T(s)$ is given by

$$\zeta_T(s) = \cfrac{1}{T_1^s - \cfrac{T_1^{2s}}{T_1^s+T_2^s - \cfrac{T_2^{2s}}{T_2^s+T_3^s - \cfrac{T_3^{2s}}{T_3^s+T_4^s - \ddots - \cfrac{T_{n-1}^{2s}}{T_{n-1}^s+T_n^s - \dots}}}}}.$$

Now A_n (respectively B_n) are defined as the numerator (respectively denominator) convergent of the continued fraction expansion given for $\zeta_T(s)$:

$$\frac{A_n}{B_n}=\cfrac{1}{T_1^s-\cfrac{T_1^{2s}}{T_1^s+T_2^s-\cfrac{T_2^{2s}}{T_2^s+T_3^s-\cfrac{T_3^{2s}}{T_3^s+T_4^s-\ddots-\cfrac{T_{n-1}^{2s}}{T_{n-1}^s+T_n^s}}}}}.$$

Hence $\{A_\nu\}_{\nu\geq 0}$ and $\{B_\nu\}_{\nu\geq 0}$ satisfy the following recurrence formulas.

$$A_\nu=(T_{\nu-1}^s+T_\nu^s)A_{\nu-1}-T_{\nu-1}^{2s}A_{\nu-2}\quad(\nu\geq 2),\quad A_0=0,\quad A_1=1;$$
$$B_\nu=(T_{\nu-1}^s+T_\nu^s)B_{\nu-1}-T_{\nu-1}^{2s}B_{\nu-2}\quad(\nu\geq 2),\quad B_0=1,\quad B_1=T_1^s$$

In fact, A_ν and B_ν can be expressed explicitly as follows.

Lemma 2 *For* $n=1,2,\dots$

$$A_n=(T_1T_2\dots T_n)^s\sum_{\nu=1}^n\frac{1}{T_\nu^s},\qquad B_n=(T_1T_2\dots T_n)^s.$$

Proof. By induction we have $B_n=(T_1T_2\dots T_n)^s$. Thus,

$$A_n=B_n\sum_{\nu=1}^n\frac{1}{T_\nu^s}=(T_1T_2\dots T_n)^s\sum_{\nu=1}^n\frac{1}{T_\nu^s}.$$

5 Another possible approach

There is the possibility to lead Theorem 1 by a different approach. Such a method is originally used in [7] and developed in [2].

Theorem 1 is equivalent to

$$\frac{1}{T_n-T_{n-1}+1/2}<\sum_{k=n}^\infty\frac{1}{T_k}<\frac{1}{T_n-T_{n-1}-1/2}\qquad(n\geq 1).$$

Instead of the approach in the above sections, we would like to show that for $n \geq 4$

$$\frac{1}{T_n - T_{n-1} - 1/2} > \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2} - T_{n+1} - 1/2} \tag{6}$$

and

$$\frac{1}{T_n - T_{n-1} + 1/2} < \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2} - T_{n+1} + 1/2}. \tag{7}$$

For, if the inequality (6) holds for $n \geq 1$, then

$$\begin{aligned}
&\frac{1}{T_n - T_{n-1} - 1/2} \\
&> \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2} - T_{n+1} - 1/2} \\
&> \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2}} + \frac{1}{T_{n+3}} + \frac{1}{T_{n+4} - T_{n+3} - 1/2} \\
&> \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2}} + \frac{1}{T_{n+3}} + \frac{1}{T_{n+4}} + \frac{1}{T_{n+5}} + \cdots .
\end{aligned}$$

It is similar for (7).

Our attempt is to rewrite (6) as

$$\begin{aligned}
&\frac{1}{T_{n-2} + T_{n-3} - 1/2} > \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_n + T_{n+1} - 1/2} \\
&\iff \frac{1}{T_{n-2} + T_{n-3} - 1/2} - \frac{1}{T_n + T_{n+1} - 1/2} > \frac{1}{T_n}\left(1 + \frac{T_n}{T_{n+1}}\right) \\
&\iff \frac{2T_{n-1}}{\left(T_{n-2} + T_{n-3} - \frac{1}{2}\right)\left(T_n + T_{n+1} - \frac{1}{2}\right)} \\
&\quad = \frac{T_n + T_{n-1} - T_{n-2} - T_{n-3}}{\left(T_{n-2} + T_{n-3} - \frac{1}{2}\right)\left(T_n + T_{n+1} - \frac{1}{2}\right)} > \frac{1}{T_n}\left(1 + \frac{T_n}{T_{n+1}}\right) \\
&\iff 2 > U(n),
\end{aligned}$$

where

$$U(n) = \left(1 + \frac{T_n}{T_{n+1}}\right)\left(1 + \frac{2T_{n-1} - 1}{2T_n}\right)\left(\frac{2T_{n-2} - 1}{2T_{n-1}} + \frac{T_{n-3}}{T_{n-1}}\right).$$

Similarly, (7) is equivalent to

$$2 < P(n)\,,$$

where

$$P(n) = \left(1 + \frac{T_n}{T_{n+1}}\right)\left(1 + \frac{2T_{n-1} + 1}{2T_n}\right)\left(\frac{2T_{n-2} + 1}{2T_{n-1}} + \frac{T_{n-3}}{T_{n-1}}\right).$$

However, to prove (6) and/or (7) may not be so easy, because numerical evidences imply

$$U(n) \nearrow 1.999\ldots \quad \text{and} \quad P(n) \searrow 2.000\ldots (n \to \infty)\,.$$

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