Highly Fault-Tolerant Routings in Some Cartesian Product Digraphs*

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Abstract

Consider a communication network G in which a limited number of edge(arc) and/or vertex faults F might occur. A routing ρ, i.e. a fixed path between each pair of vertices, for the network must be chosen without knowing which components might become faulty. The diameter of the surviving route graph $R(G,\rho)/F$, where $R(G,\rho)/F$ is a digraph with the same vertices as $G-F$ and a vertex x being adjacent to another vertex y if and only if $\rho(x,y)$ avoids F, could be an important measurement for the routing ρ. In this paper, the authors consider the Cartesian product digraphs whose factors satisfy some given conditions and show that the diameter of the surviving route graph is bounded by three for any minimal routing ρ when the number of faults is less than some integer. This result is also useful for the Cartesian product graphs and generalizes some known results.

Keywords: Diameter, Fault-tolerant routing, Surviving route graph, Cartesian product digraph, Generalized hypercube.

AMS Classification: 05C12.

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1 Introduction

We consider the problem of fault-tolerant routing in a communication network. The communication network is often modeled by a graph or digraph $G = (V, E)$, where vertices correspond to switching offices or processors, and edges or arcs correspond to communication links.

A *routing* ρ assigns a fixed (directed) path $\rho(x, y)$ from x to y for each ordered pair of vertices (x, y) in a network. The (directed) paths specified by ρ are called *routes*. It is assumed that the routing table is computed only once for a given communication network configuration, and thus all messages must be sent by using these routes. When a vertex or an edge (arc) fails, the route that goes through it becomes unusable. However, communication is still possible through a sequence of surviving routes. Then given a graph or digraph G, a routing ρ, and a faulty component set F, we consider the *surviving route graph* $R(G, \rho)/F$ with the same vertices as $G - F$, and a vertex x being adjacent to another vertex y if and only if $\rho(x, y)$ avoids F, studied first by Dolev *et al.* [2].

In a communication network with a fixed routing, the time required to send a message along a route is often dominated by the message processing time at the two terminal vertices of the route. Under this assumption, the diameter of the surviving route graph is a good criterion of the network vulnerability. Namely, a good routing is one that minimizes this diameter. This new parameter can provide more accurate measure for fault tolerance of a large-scale parallel processing system and, thus, has received much attention of many researchers (see, for example, [2, 3, 6, 7, 8]). Among them, Dolev *et al.* [2] studied the well known Boolean cube and showed that the diameter of the surviving route graph is bounded by three for any minimal routing if the number of faults is less than the connectivity of the Boolean cube. Xu [7] generalized the result to Cartesian product digraph of directed cycles. Wada *et al.* [6] gained a similar result for n-dimensional generalized d-hypercubes, i.e. the Cartesian product graph of n identical complete graphs with d vertices.

It is well known that the Cartesian product method is an important method for designing large-scale communication networks. In this paper, we use some techniques from [2] and [6], and generalize the above results to Cartesian product digraphs whose factors satisfy some given conditions. Although we mainly discuss digraphs here, our result is useful for graphs, i.e. undirected graphs, since every graph can be thought of as a particular digraph, symmetric digraph, in which there is a pair of symmetric arcs corresponding to each edge.

Our results are given in Section 3. Some necessary terminology and notation are given in Section 2.

2 Terminology and Notation

In this paper, we only consider (strongly) connected, simple (without loops and parallel edges or arcs) (di)graph. We refer the reader to [1] or [8] for basic graph-theoretical terminology and notation not defined here.

Let G be a (di)graph with vertex set $V(G)$ and edge(or arc) set $E(G)$. A digraph G is said to be *strongly connected* if for each pair of vertices $x, y \in V(G)$, there are both directed (x, y)-path and (y, x)-path in G. If G is a digraph, two arcs $(x, y), (y, x) \in E(G)$ are called a pair of *symmetric* arcs. A graph can be thought of as a particular digraph, *symmetric digraph*, in which there is a pair of symmetric arcs corresponding to each edge. The symbol K_n denotes a complete graph of order n. Let $D(G; x, y)$ denote the *distance* from a vertex x to another vertex y in G. The *diameter* of G, denoted by $D(G)$, is the maximum of $D(G; x, y)$ over all pairs $x, y \in V(G)$. For two vertices x and y of G, the *interval* between x and y, denoted by $Sc(x, y)$, is the subgraph induced by all arcs in shortest directed paths from x to y.

A routing ρ is called to be *minimal* if all (directed) paths specified by ρ are shortest.

For a given integer k, we say a digraph G has the property $\mathcal{P}_k$ if G satisfies the following two conditions.

(1) For every pair (x, y) of two distinct vertices in G, there are k internally-disjoint directed (x, y)-paths such that one of them is a shortest directed (x, y)-path and each of others can be a concatenation of at most three shortest directed paths.

(2) For every vertex x in G, there are k directed cycles that contain x and are vertex-disjoint except for x such that each of them can be a concatenation of at most three shortest directed paths.

Let us illustrate the property $\mathcal{P}_k$ above introduced with the following example.

Example 1. It is easy to check that a directed cycle, an undirected tree and every strongly connected digraph have the property $\mathcal{P}_1$, and a complete graph K_{k+1} has the property $\mathcal{P}_k$.

Let $F \subset V(G) \cup E(G)$ be a set of *faults*, and let

$$||F|| = \max\{|F'| : F' \subseteq F \text{ and } F' \text{ contains no pair of symmetric arcs}\}.$$

Note that from the definition of $||F||$ we can see that each pair of symmetric arcs in F is calculated only once. Thus if G is a graph, viewed as a symmetric digraph, then $|F| = ||F||$.

An object such as route, path and subgraph *avoids* F if no element of F is contained in the object. For a routing ρ and a fault-set F such that $G - F$ is (strongly) connected, the *surviving route graph*, denoted by

$R(G,\rho)/F$, is a digraph with the same vertex set as $G-F$, and a vertex x being adjacent to another vertex y if and only if $\rho(x,y)$ avoids F.

An ordered pair of vertices x and y is said to *be safe with respect to F* if every shortest (directed) path from x to y avoids F. A sequence of vertices $x_1, x_2, \ldots, x_k$ is safe with respect to F if each consecutive ordered pair of vertices in the sequence is safe with respect to F.

The *Cartesian product* (di)graph G of n (di)graphs $G_1, G_2, \ldots, G_n$, denoted by $G = G_1 \Box G_2 \Box \ldots \Box G_n$, is the (di)graph with the vertex-set $V(G) = V(G_1) \times V(G_2) \times \ldots \times V(G_n)$, and an (arc) edge from a vertex $x = x_1x_2\ldots x_n$ to another vertex $y = y_1y_2\ldots y_n (x_i, y_i \in V(G_i), i = 1,2,\ldots,n)$ if and only if they differ in exactly one coordinate, and for this coordinate, say j^{th}, there is an (arc) edge from the vertex x_j to the vertex y_j in G_j. $G_1, G_2, \ldots, G_n$ are called factors of $G_1 \Box G_2 \Box \ldots \Box G_n$. For more desirable properties of the Cartesian product (di)graphs, the reader can be referred to [8], [4] and [5].

The *generalized hypercube (or Hamming graph)* $Q(d_1, d_2, \ldots, d_n)$ is defined as $K_{d_1} \Box K_{d_2} \Box \ldots \Box K_{d_n}$, where $d_i \geq 2$ is an integer for each $i = 1,2,\ldots,n$. In particular, $Q(\overbrace{2,2,\ldots,2}^{n})$ and $Q(\overbrace{d,d,\ldots,d}^{n})$ are the Boolean cube and the n-dimensional d-hypercube, respectively.

Let $P_{d_i} = (1,2,\ldots,d_i)$ be a directed path, then every vertex of $P_{d_1} \Box P_{d_2} \Box \ldots \Box P_{d_n}$ can be expressed as a string $x_1x_2\ldots x_n$ of length n, where $x_i \in \{1,2,\ldots,d_i\}$ for each $i = 1,2,\ldots,n$. If $x = x_1x_2\ldots x_n$, $y = y_1y_2\ldots y_n$ and $(x,y) \in E(P_{d_1} \Box P_{d_2} \Box \ldots \Box P_{d_n})$, then there is a coordinate, say i^{th}, such that $x_j = y_j$ for all $j \neq i$ but $y_i = x_i + 1$. For convenience of our statement, we express the arc (x,y) as the string $x_1 \ldots x_{i-1}(x_i + 0.5)x_{i+1}\ldots x_n$. We write a string $x = 1^n$ for $x = 11\ldots1$.

By dropping the i^{th} coordinate, any n-dimensional object can be *projected* along the i^{th} coordinate onto an $(n-1)$-dimensional object. Let R_i be the *operator* for projecting along the i^{th} coordinate. Note that an arc of $P_{d_1} \Box P_{d_2} \Box \ldots \Box P_{d_n}$ may be projected to a vertex. For example, $R_4(x_1x_2x_3(x_4 + 0.5)) = x_1x_2x_3 = R_4(x_1x_2x_3x_4)$.

We define the *weight* of a string $x = x_1x_2\ldots x_n$, denoted by $|x|$, as the sum of its coordinates, i.e. $|x| = x_1 + x_2 + \ldots + x_n$. We write $x \leq y$ if $x_i \leq y_i$ for each $i = 1,2,\ldots,n$, and $x < y$ if $x \leq y$ and $x_i < y_i$ for some i.

3 Our Results

Theorem 1. *$D(R(G_1 \Box G_2 \Box \ldots \Box G_n, \rho)/F) \leq 3$ for any minimal routing ρ and $||F|| < \sum_{i=1}^{n} k_i$ if G_i is a strongly connected digraph of order at least two and has a unique minimal routing and the property $\mathcal{P}_{k_i}$ for each $i = 1,2,\ldots,n$.*

Let $G = G_1 \Box G_2 \Box \ldots \Box G_n$. We first prove two lemmas.

Lemma 2. Suppose G_i is a strongly connected digraph of order at least two and has a unique minimal routing for each $i = 1, 2, \ldots, n$. Let $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ be two vertices of G, then we have the following assertions.
(a) $Sc(x, y) = P^1(x_1, y_1) \Box P^2(x_2, y_2) \Box \ldots \Box P^n(x_n, y_n)$ and $Sc(x, y)$ is isomorphic to $P_{d_1} \Box P_{d_2} \Box \ldots \Box P_{d_n}$, where $P^i(x_i, y_i)$ denotes the shortest directed (x_i, y_i)-path in G_i and $d_i = D(G_i; x_i, y_i) + 1$ for each $i = 1, 2, \ldots, n$.
(b) After relabeling the vertices in $Sc(x, y)$ with the labels of counterparts in $P_{d_1} \Box P_{d_2} \Box \ldots \Box P_{d_n}$ such that $x = 1^n$ and $y = d_1 d_2 \ldots d_n$, if $u, v \in V(Sc(x, y))$ with $x < u < v \leq y$, then $Sc(x, u) \cup Sc(u, v) \cup Sc(v, y) \subseteq Sc(x, y)$.

Proof. Note that for each $i = 1, 2, \ldots, n$, all shortest directed paths in G_i are unique, since G_i has the unique minimal routing. Let $b_1 \ldots b_{i-1} P^i(u_i, v_i) b_{i+1} \ldots b_n = \{b_1\} \Box \ldots \Box \{b_{i-1}\} \Box P^i(u_i, v_i) \Box \{b_{i+1}\} \Box \ldots \Box \{b_n\}$, where $b_j \in V(G_j)$ for each $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$. Let $Q = (x_1 x_2 \ldots x_n, u_1 u_2 \ldots u_n, \ldots, v_1 v_2 \ldots v_n, y_1 y_2 \ldots y_n)$ be a shortest directed (x, y)-path in G.

We claim that the directed path P^i determined by the i^{th} coordinates $x_i, u_i, \ldots, v_i, y_i$ of vertices of Q in the original order is a shortest directed (x_i, y_i)-path of G_i.

In fact, obviously $D(G; x, y) \geq \sum_{i=1}^n D(G_i; x_i, y_i)$ since $D(G; x, y)$ is equal to the length of Q, which is equal to the sum of that of P^i, and P^i is a directed (x_i, y_i)-path in G_i for each $i = 1, 2, \ldots, n$. On the other hand, let $Q' = P^1(x_1, y_1) x_2 \ldots x_{n-1} x_n \cup y_1 P^2(x_2, y_2) \ldots x_{n-1} x_n \cup \ldots \cup y_1 y_2 \ldots y_{n-1} P^n(x_n, y_n)$. Then it is easy to see that Q' is a directed (x, y)-path of length $\sum_{i=1}^n D(G_i; x_i, y_i)$ in G. It follows that $D(G; x, y) \leq \sum_{i=1}^n D(G_i; x_i, y_i)$. So $D(G; x, y) = \sum_{i=1}^n D(G_i; x_i, y_i)$. Then P^i is of length $D(G_i; x_i, y_i)$ for each $i = 1, 2, \ldots, n$. Thus the claim is true.

By the above claim and the uniqueness of shortest directed paths in G_i, $P^i = P^i(x_i, y_i)$ for each $i = 1, 2, \ldots, n$ and Q is of length $\sum_{i=1}^n D(G_i; x_i, y_i)$. To complete the proof of the assertion (a), it suffices to show that $E(Sc(x, y)) = E(P^1(x_1, y_1) \Box P^2(x_2, y_2) \Box \ldots \Box P^n(x_n, y_n))$. If $(z, w) = (z_1 z_2 \ldots z_n, w_1 w_2 \ldots w_n) \in E(P^1(x_1, y_1) \Box P^2(x_2, y_2) \Box \ldots \Box P^n(x_n, y_n))$, then there is a coordinate, say j^{th}, such that $z_i = w_i \in V(P^i(x_i, y_i))$ for each $i \neq j$ but $(z_j, w_j) \in E(P^j(x_j, y_j))$. By the uniqueness of shortest directed paths in G_i, $P^i(x_i, z_i) \cup P^i(z_i, y_i) = P^i(x_i, y_i)$ for each $i \neq j$ and $P^j(x_j, z_j) \cup P^j(z_j, w_j) \cup P^j(w_j, y_j) = P^j(x_j, z_j) \cup (z_j, w_j) \cup P^j(w_j, y_j) = P^j(x_j, y_j)$.

$$\begin{aligned} \text{Let } P(x, y) = \; & P^1(x_1, z_1) x_2 \ldots x_{n-1} x_n \cup z_1 P^2(x_2, z_2) \ldots x_{n-1} x_n \cup \ldots \\ & \cup z_1 z_2 \ldots z_{n-1} P^n(x_n, z_n) \cup z_1 \ldots z_{j-1} (z_j, w_j) z_{j+1} \ldots z_n \\ & \cup P^1(w_1, y_1) w_2 \ldots w_{n-1} w_n \cup y_1 P^2(w_2, y_2) \ldots w_{n-1} w_n \\ & \cup \ldots \cup y_1 y_2 \ldots y_{n-1} P^n(w_n, y_n). \end{aligned}$$

Obviously, $P(x,y)$ is a directed (x,y)-path of length $\sum_{i=1}^{n} D(G_i; x_i, y_i)$ in G. So $P(x,y)$ is a shortest directed (x,y)-path in G. Then, by the definition of $Sc(x,y)$, $(z,w) \in E(Sc(x,y))$, since $(z,w) \in E(P(x,y))$. Thus $E(P^1(x_1,y_1)\Box P^2(x_2,y_2)\Box \ldots \Box P^n(x_n,y_n)) \subseteq E(Sc(x,y))$. Conversely, if $(z,w) = (z_1z_2\ldots z_n, w_1w_2\ldots w_n) \in E(Sc(x,y))$, without loss of generality, assume $(z,w) \in E(Q)$. Then there is a coordinate, say j^{th}, such that $z_i = w_i \in V(P^i) = V(P^i(x_i,y_i))$ for each $i \neq j$ but $(z_j,w_j) \in E(P^j) = E(P^j(x_j,y_j))$. Then $(z,w) \in E(P^1(x_1,y_1)\Box P^2(x_2,y_2)\Box \ldots \Box P^n(x_n,y_n))$. So $E(Sc(x,y)) \subseteq E(P^1(x_1,y_1)\Box P^2(x_2,y_2)\Box \ldots \Box P^n(x_n,y_n))$. Therefore, $E(Sc(x,y)) = E(P^1(x_1,y_1)\Box P^2(x_2,y_2)\Box \ldots \Box P^n(x_n,y_n))$. Thus the assertion (a) holds.

By the assertion (a), we can relabel the vertices in $Sc(x,y)$ with the labels of counterparts in $P_{d_1}\Box P_{d_2}\Box \ldots \Box P_{d_n}$ such that $x = 1^n$ and $y = d_1d_2\ldots d_n$. Under this labeling, if $u = u_1u_2\ldots u_n, v = v_1v_2\ldots v_n \in V(Sc(x,y))$ with $x < u < v \leq y$, let

$$\begin{aligned} P = \; & P^1(x_1,u_1)x_2\ldots x_{n-1}x_n \cup u_1P^2(x_2,u_2)\ldots x_{n-1}x_n \cup \ldots \\ & \cup u_1u_2\ldots u_{n-1}P^n(x_n,u_n) \cup P^1(u_1,v_1)u_2\ldots u_{n-1}u_n \\ & \cup v_1P^2(u_2,v_2)\ldots u_{n-1}u_n \cup \ldots \cup v_1v_2\ldots v_{n-1}P^n(u_n,v_n) \\ & \cup P^1(v_1,y_1)v_2\ldots v_{n-1}v_n \cup y_1P^2(v_2,y_2)\ldots v_{n-1}v_n \cup \ldots \\ & \cup y_1y_2\ldots y_{n-1}P^n(v_n,y_n). \end{aligned}$$

It is easy to see, from the proof of Lemma 2 (a), that P is a shortest directed (x,y)-path and passes through u and v in G. Let $P(x,u)$, $P(u,v)$ and $P(v,y)$ be (x,u)-, (u,v)- and (v,y)-sections of P, respectively. Then $P(x,u)$, $P(u,v)$ and $P(v,y)$ are shortest directed (x,u)-,(u,v)- and (v,y)-paths in G, respectively. Assume $Q(u,v)$ is another shortest directed (u,v)-path in G. Then the directed path obtained by concatenating $P(x,u)$, $Q(u,v)$ and $P(v,y)$ is also a shortest directed (x,y)-path in G. Thus $Q(u,v) \subseteq Sc(x,y)$, and so $Sc(u,v) \subseteq Sc(x,y)$. Similarly, $Sc(x,u) \subseteq Sc(x,y)$ and $Sc(v,y) \subseteq Sc(x,y)$. So the assertion (b) holds. ∎

Lemma 3. If $||F|| < n$ and $x = 1^n$, $y = d_1d_2\ldots d_n \notin F$, then there are vertices u and v in $P_{d_1}\Box P_{d_2}\Box \ldots \Box P_{d_n}$, where $n \geq 2$ and $d_i > 1$ for each $i = 1, 2, \ldots, n$, such that $x < u < v \leq y$ is safe with respect to F.

Proof. Obviously $||F|| = |F|$, since $P_{d_1}\Box P_{d_2}\Box \ldots \Box P_{d_n}$ contains no pair of symmetric arcs. We proceed by induction on $n \geq 2$. If $n = 2$, without loss of generality, assume $F = \{f\}$. It is easy to see that there are at least two internally-disjoint shortest directed (x,y)-paths of length at least 2 and at least one of them, say $P(x,y)$, avoids f. If f is a vertex, then we can find a vertex, say u, on $P(x,y)$ with $|u| = |f|$. It is easy to see that $x < u < y \leq y$ is safe with respect to f. Suppose now f is an arc. Assume $f = (z,w)$, then $z < w$. If $z = x$, then we can choose a vertex, say u, on $P(x,y)$ with $|u| = |w|$. Thus $x < u < y \leq y$ is safe with respect to F.

Similarly, if $z \neq x$, we can find two vertices, say u and v, on $P(x,y)$ with $|u| = |z|$ and $|v| = |w|$. Then $x < u < v \leq y$ is safe with respect to F. Thus, the induction base holds. Assume the induction hypothesis for $n-1$ with $n > 2$. We proceed to the induction step by considering two cases, respectively.

Case 1. There is a positive integer i and an element $f \in F$ such that $R_i(f)$ is in $\{1^{n-1}, d_1 \dots d_{i-1}d_{i+1} \dots d_n\}$. Without loss of generality, assume $i = 1$. Let $F' = R_1(F) - \{1^{n-1}, d_2d_3 \dots d_n\}$. Then $R_1(P_{d_1} \Box P_{d_2} \Box \dots \Box P_{d_n}) = P_{d_2} \Box P_{d_3} \Box \dots \Box P_{d_n}$, $1^{n-1}, d_2d_3 \dots d_n \notin F'$ and $||F'|| < n-1$. By the induction hypothesis there are two vertices u, v in $P_{d_2} \Box P_{d_3} \Box \dots \Box P_{d_n}$ such that the sequence $1^{n-1} < u < v \leq d_2d_3 \dots d_n$ is safe with respect to F'. If $v < d_2d_3 \dots d_n$, then it is easy to check that $1^n < 1u < d_1v < d_1d_2 \dots d_n$ is safe with respect to F. And if $v = d_2d_3 \dots d_n$, then it is also easy to see that $1^n < 1u < d_1u < d_1d_2 \dots d_n$ is safe with respect to F.

Case 2. For each $i = 1, 2, \dots, n$, $R_i(F)$ contains neither 1^{n-1} nor $d_1 \dots d_{i-1}$ $d_{i+1} \dots d_n$.

We claim that there is an $i \in \{1, 2, \dots, n\}$ and an $f \in F$ such that $f_i > 1$ and $|R_i(f)| = \min\{|f'| : f' \in R_i(F)\}$.

Suppose to the contrary that, for each $i \in \{1, 2, \dots, n\}$ and every $f = f_1f_2 \dots f_n \in F$, if $|R_i(f)| = \min\{|f'| : f' \in R_i(F)\}$, then $f_i = 1$. Let $m_i = \min\{|f'| : f' \in R_i(F)\}$. Choose $g = g_1g_2 \dots g_n \in F$ with $|g| = \min\{|f| : f \in F\}$. Without loss of generality, assume $g_1 > 1$, since $g \neq 1^n$. Then $|R_1(g)| > m_1$ (otherwise, g is as required). Let $h = h_1h_2 \dots h_n \in F$ with $|R_1(h)| = m_1$, then

$$\begin{aligned} |h| &= h_1 + (h_2 + \dots + h_n) = h_1 + |R_1(h)| = 1 + m_1 \\ &< g_1 + |R_1(g)| = |g| = \min\{|f| : f \in F\}, \end{aligned}$$

which contradicts the choice of g. So the claim is true.

Without loss of generality, assume that there is an $f = f_1f_2 \dots f_n \in F$ such that $f_1 > 1$ and $|R_1(f)| = \min\{|f'| : f' \in R_1(F)\}$. Let $F'' = R_1(F - \{f\})$. Then $||F''|| < n-1$ and so, by the induction hypothesis, there exists at least one sequence safe with respect to F'' of the form $1^{n-1} < a < b \leq d_2d_3 \dots d_n$. Among all such sequences there must be one $1^{n-1} < u < v \leq d_2d_3 \dots d_n$ with $|u|$ maximal. We claim that $1^n < 1u < 1v < d_1d_2 \dots d_n$ is safe with respect to F. It is clearly safe with respect to $F - \{f\}$. So we must show only that it is safe for f. Since $f_1 > 1$, it suffices to show that $f \notin Sc(1v, d_1d_2 \dots d_n)$. But if $f \in Sc(1v, d_1d_2 \dots d_n)$, we must have $v < d_2d_3 \dots d_n$, since $R_1(f) \neq d_2d_3 \dots d_n$ and $|R_1(f)| \geq |v|$. It follows from the choice of f that $|R_1(f')| \geq |R_1(f)| \geq |v|$ for all $f' \in F - \{f\}$. But then $1^{n-1} < v < d_2d_3 \dots d_n \leq d_2d_3 \dots d_n$ is safe with respect to F'', contrary to the choice of u. ■

Proof of Theorem 1. To prove Theorem 1, it is sufficient to show that for each pair of vertices x and y in $G - F$ there are at most two vertices u and v such that the sequence x, u, v, y is safe with respect to F. For this purpose, let $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_n$ be arbitrary two vertices in $G - F$, where $x_i, y_i \in V(G_i)$ for each $i = 1, 2, \dots, n$. We now complete the proof of Theorem 1 by induction on n. For each $i = 1, 2, \dots, n$, we can assume $k_i \geq 1$, since G_i is strongly connected. The argument for $n = 1$ is straightforward, since G_1 satisfies the property $\mathcal{P}_{k_1}$ and has the unique minimal routing. Assume the induction hypothesis for $n - 1$ with $n \geq 2$. We proceed to the induction step by considering two cases, respectively.

Case 1. There is an $i \in \{1, 2, \dots, n\}$ such that $x_i = y_i$. Note that as an operation of graphs, the Cartesian products satisfy commutative law if we identify isomorphic graphs. Without loss of generality, we may assume $x_1 = y_1$. Define sets of faults F_0 and F' as $F_0 = \{f \in F : f \in \{x_1\}\square G_2 \square G_3 \square \dots \square G_n\}$ and $F' = F - F_0$.

If $||F_0|| < \sum_{i=2}^{n} k_i$, we can apply the induction hypothesis to $\{x_1\}\square G_2 \square G_3 \square \dots \square G_n$ and so the result follows.

We now assume $||F_0|| \geq \sum_{i=2}^{n} k_i$. Then $||F'|| < k_1$. By the property $\mathcal{P}_{k_1}$, for the vertex x_1 of G_1, there are k_1 directed cycles, denoted by $C_{1,j}(x_1)$, where $j = 1, 2, \dots, k_1$, which contain x_1 but are vertex-disjoint except for x_1, satisfying that for each $j = 1, 2, \dots, k_1$ there are vertices $u_j (\neq x_1)$ and $v_j (\neq x_1)$ on $C_{1,j}(x_1)$ such that (x_1, u_j)-, (u_j, v_j)- and (v_j, x_1)-sections of $C_{1,j}(x_1)$ are shortest directed (x_1, u_j)-, (u_j, v_j)- and (v_j, x_1)-paths in G_1, respectively. Let $u(u_j) = u_jx_2x_3 \dots x_n$, $v(v_j) = v_jy_2y_3 \dots y_n$ and $S_j = Sc(x, u(u_j)) \cup Sc(u(u_j), v(v_j)) \cup Sc(v(v_j), y)$, where $j = 1, 2, \dots, k_1$. By Lemma 2 (a), for each $j = 1, 2, \dots, k_1$,

$$\begin{aligned} S_j = \; & P^1(x_1, u_j)\square\{x_2\}\square\{x_3\}\square \dots \square\{x_n\} \\ & \cup P^1(u_j, v_j)\square P^2(x_2, y_2)\square P^3(x_3, y_3)\square \dots \square P^n(x_n, y_n) \\ & \cup P^1(v_j, x_1)\square\{y_2\}\square\{y_3\}\square \dots \square\{y_n\}. \end{aligned} \tag{1}$$

Then, by (1), it is easy to see that the k_1 subgraphs S_j of G are vertex-disjoint except for x and y, since $C_{1,j}(x_1) = P^1(x_1, u_j) \cup P^1(u_j, v_j) \cup P^1(v_j, x_1)$, where $j = 1, 2, \dots, k_1$, are vertex-disjoint except for x_1. So there is a $j \in \{1, 2, \dots, k_1\}$ such that S_j avoids F', since $||F'|| < k_1$. Then S_j avoids F. Thus the sequence $x, u(u_j), v(v_j), y$ is safe with respect to F.

Case 2. The vertices x and y are different on each coordinate. Define sets of faults F_S and F'' as $F_S = \{f \in F : f \in Sc(x, y)\} \cup \{(z, w) \in F \cap E(G) : (w, z) \in Sc(x, y)\}$ and $F'' = F - F_S$.

If $||F_S|| \geq n$, then $||F''|| < \sum_{i=1}^{n} k_i - n$. For each $i = 1, 2, \dots, n$, by the property $\mathcal{P}_{k_i}$ and the uniqueness of shortest directed paths in G_i, there are k_i internally-disjoint directed (x_i, y_i)-paths in G_i, denoted by $P_{i,j}(x_i, y_i)$, where $j = 1, 2, \dots, k_i$, satisfying that one of them, say $P_{i,k_i}(x_i, y_i)$, is the shortest directed (x_i, y_i)-path and for each $j = 1, 2, \ldots, k_i - 1$ there are vertices $u_{i,j}(\neq x_i)$ and $v_{i,j}(\neq y_i)$ on $P_{i,j}(x_i, y_i)$ such that $(x_i, u_{i,j})$-, $(u_{i,j}, v_{i,j})$- and $(v_{i,j}, y_i)$-sections of $P_{i,j}(x_i, y_i)$ are shortest directed $(x_i, u_{i,j})$-, $(u_{i,j}, v_{i,j})$- and $(v_{i,j}, y_i)$-paths in G_i, respectively. For each $i = 1, 2, \ldots, n$ and each $j = 1, 2, \ldots, k_i - 1$, let

$$u(u_{i,j}) = x_1 \ldots x_{i-1} u_{i,j} x_{i+1} \ldots x_n, \quad v(v_{i,j}) = y_1 \ldots y_{i-1} v_{i,j} y_{i+1} \ldots y_n,$$

and $S_{i,j} = Sc(x, u(u_{i,j})) \cup Sc(u(u_{i,j}), v(v_{i,j})) \cup Sc(v(v_{i,j}), y)$.

We claim that the $\sum_{i=1}^{n} k_i - n$ subgraphs $S_{i,j}$ of G are vertex-disjoint except for x and y.

In fact, by Lemma 2 (a),

$$\begin{aligned} S_{i,j} = \; & \{x_1\}\Box \ldots \Box\{x_{i-1}\}\Box P^i(x_i, u_{i,j})\Box\{x_{i+1}\}\Box \ldots \Box\{x_n\} \\ & \cup P^1(x_1, y_1)\Box \ldots \Box P^{i-1}(x_{i-1}, y_{i-1})\Box P^i(u_{i,j}, v_{i,j})\Box \\ & \quad P^{i+1}(x_{i+1}, y_{i+1})\Box \ldots \Box P^n(x_n, y_n) \\ & \cup\{y_1\}\Box \ldots \Box\{y_{i-1}\}\Box P^i(v_{i,j}, y_i)\Box\{y_{i+1}\}\Box \ldots \Box\{y_n\}. \end{aligned} \tag{2}$$

If $i, h \in \{1, 2, \ldots, n\}$ and $i \neq h$, let $j \in \{1, 2, \ldots, k_i - 1\}$ and $l \in \{1, 2, \ldots, k_h - 1\}$. By the uniqueness of shortest directed paths in G_i,

$$P^i(x_i, u_{i,j}) \cup P^i(u_{i,j}, v_{i,j}) \cup P^i(v_{i,j}, y_i) = P_{i,j}(x_i, y_i)$$

and

$$\{x_i\} \cup P^i(x_i, y_i) \cup \{y_i\} = P_{i,k_i}(x_i, y_i)$$

are vertex-disjoint except for x_i and y_i in G_i. Then, by (2), $S_{i,j}$ and $S_{h,l}$ are vertex-disjoint except for x and y in G. Similarly, $S_{i,j}$ and $S_{i,l}$, where $j, l \in \{1, 2, \ldots, k_i - 1\}$ and $j \neq l$, are vertex-disjoint except for x and y in G. So the claim is true.

Then, by the claim above, there is a pair (i, j) such that $S_{i,j}$ avoids F'', since $||F''|| < \sum_{i=1}^{n} k_i - n$. By Lemma 2 (a) and (2), it is easy to see that $Sc(x, y)$ and $S_{i,j}$ are vertex-disjoint except for x and y in G. Then $S_{i,j}$ avoids F. Thus the sequence $x, u(u_{i,j}), v(v_{i,j}), y$ is safe with respect to F.

If $||F_S|| < n$. By Lemma 2 (a), $Sc(x, y)$ is isomorphic to $P_{d_1}\Box P_{d_2}\Box \ldots \Box P_{d_n}$, where $d_i = D(G_i; x_i, y_i) + 1 > 1$ for each $i = 1, 2, \ldots, n$, since x and y are different on each coordinate. Then the theorem holds by Lemma 3 and Lemma 2 (b).

The proof of the theorem is complete. ■

By Theorem 1 and Example 1, we have the following corollaries.

Corollary 4 [2]. $D(R(Q(\overbrace{2, 2, \ldots, 2}^{n}), \rho)/F) \leq 3$ for any minimal routing ρ and $|F| < n$.

Corollary 5 [7]. $D(R(C_{d_1}\Box C_{d_2}\Box \ldots \Box C_{d_n}, \rho)/F) \leq 3$ for any minimal routing and $|F| < n$,where C_{d_i} is the directed cycle of order $d_i \geq 2$ for each $i = 1, 2, \ldots, n$.

Corollary 6. $D(R(Q(d_1, d_2, \dots, d_n), \rho)/F) \le 3$ for any minimal routing ρ and $|F| < \sum_{i=1}^{n} d_i - n$, where $d_i \ge 2$ for each $i = 1, 2, \dots, n$.

Corollary 7 [6]. $D(R(Q(\overbrace{d, d, \dots, d}^{n}), \rho)/F) \le 3$ for any minimal routing ρ and $|F| < n(d-1)$, where $d \ge 2$.

Corollary 8. $D(R(G_1 \square G_2 \square \dots \square G_n, \rho)/F) \le 3$ for any minimal routing ρ and $|F| < n$ if G_i is a connected graph of order at least two and has a unique minimal routing for each $i = 1, 2, \dots, n$.

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