Values of a Class of Generalized Euler and Bernoulli Numbers

Jin-Hua Yang and Feng-Zhen Zhao

(Dalian University of Technology, Dalian 116024, China)

Abstract

In this paper, the authors discuss the values of a class of generalized Euler numbers and generalized Bernoulli numbers at rational points.

1. Introduction

We first give some notations. Let Z be the set of integers. Let N_0 be the set of nonnegative integers. For $n \in N_0$, the Euler polynomials $E_n(u)$ and the Bernoulli polynomials $B_n(u)$ are defined by

$$\sum_{n=0}^{\infty} E_n(u)\frac{x^n}{n!} = \frac{2e^{ux}}{e^x+1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n(u)\frac{x^n}{n!} = \frac{xe^{ux}}{e^x-1}.$$

Obviously, $2^nE_n(1/2) = E_n$ and $B_n(0) = B_n$ are the Euler numbers and Bernoulli numbers, respectively. Euler numbers, Euler polynomials, Bernoulli numbers, and Bernoulli polynomials have important application in many subjects, especially in function theory and analytic number theory. The values of Euler polynomials and Bernoulli polynomials at rational points have been receiving much attention (see [1-2] or [5-8]). For example, for Euler polynomials $E_n(u)$, Fox [2] proved that for a rational number r/s, $s^n(E_n(r/s) + (-1)^{rs-1}E_n(0)) \in Z$ for every $n \geq 0$, and Sury [6] extended this result: $s^nE_n(r/s) \in Z$ if s is even and $s^n(E_n(r/s)+(-1)^{r-1}E_n(0)) \in Z$

if s is odd. For Bernoulli polynomials $B_n(x)$, Almkvist and Meurman [1] showed that $s^n(B_n(r/s)-B_n(0)) \in Z$ for every $n \geq 0$, and a simple proof of this was given in [5]. Inspired by the above conclusions, we want to discuss the values of a class of generalized Euler numbers $\varepsilon_n(u)$ and generalized Bernoulli numbers $\beta_n(u)$ at rational points. For convenience, we introduce the definition of $\varepsilon_n(u)$ (see[3]). For $u \in [1, 1+\delta]$,

$$\frac{2ue^x}{e^{2x}+u} = \sum_{n=0}^{\infty} \varepsilon_n(u)\frac{x^n}{n!}. \tag{1}$$

For the application of $\varepsilon_n(u)$, see [3-4]. Similarly, $\beta_n(u)$ is defined by

$$\frac{uxe^{(u-1)x}}{e^x - u} = \sum_{n=0}^{\infty} \beta_n(u)\frac{x^n}{n!}. \tag{2}$$

It is clear that $\varepsilon_n(1) = E_n$ and $\beta_n(1) = B_n$. Now we give the generalizations of $\varepsilon_n(u)$ and $\beta_n(u)$, respectively. For each positive integer k, we define

$$\frac{2^k u^k e^{kx}}{(e^{2x}+u)^k} = \sum_{n=0}^{\infty} \varepsilon_n^{(k)}(u)\frac{x^n}{n!} \tag{3}$$

and

$$\frac{u^k x^k e^{k(u-1)x}}{(e^x - u)^k} = \sum_{n=0}^{\infty} \beta_n^{(k)}(u)\frac{x^n}{n!}.$$

The aim of this paper is to investigate values of $\varepsilon_n(u)$, $\beta_n(u)$, $\varepsilon_n^{(k)}(u)$ and $\beta_n^{(k)}(u)$ at rational points.

2. Main Results

We first prove a lemma before we give the main results of this paper.

Lemma: For $\varepsilon_n(u)$ and $\beta_n(u)(n \geq 0)$,

$$\varepsilon_0(u) = \frac{2u}{1+u}, \quad \varepsilon_n(u) = \frac{2u}{1+u} - \sum_{p=0}^{n-1} \binom{n}{p} \frac{2^{n-p}\varepsilon_p(u)}{1+u}, \quad n \geq 1, \tag{4}$$

$$\beta_0(u)=0,\quad \beta_1(u)=\frac{u}{1-u},\quad \beta_n(u)=-u(u-1)^{n-2}n+\sum_{p=0}^{n-1}\binom{n}{p}\frac{\beta_p(u)}{u-1},$$
$$\text{for}\quad u\in(1,1+\delta],\quad n\geq 2. \tag{5}$$

Proof: It follows from (1) that

$$2ue^x=\left(\sum_{m=0}^{\infty}\frac{2^m x^m}{m!}+u\right)\sum_{n=0}^{\infty}\varepsilon_n(u)\frac{x^n}{n!}.$$

Then

$$2u\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{p=0}^{n}\varepsilon_p(u)\binom{n}{p}2^{n-p}\right)\frac{x^n}{n!}+u\sum_{n=0}^{\infty}\varepsilon_n(u)\frac{x^n}{n!}. \tag{6}$$

Comparing the coefficients of $\frac{x^n}{n!}$ on both sides of (6), we have

$$2u=\sum_{p=0}^{n}\varepsilon_p(u)\binom{n}{p}2^{n-p}+u\varepsilon_n(u).$$

Naturally, (4) holds.

Similarly, from (2) we can prove that (5) holds. This completes the proof.

Theorem 1: For any arbitrary rational number r/s with $r/s\geq 1$ and $s>0$, we have for each n that

(i) $(r+s)^{n+1}\varepsilon_n(r/s)$ is even and $(r+s)^{n+1}\varepsilon_n(r/s)\equiv 0(mod\ \ (2r))$;

(ii) $(r+s)^{n+1}(\varepsilon_n(r/s)-E_n)\in Z$, $(r+s)^{n+1}(\varepsilon_n(r/s)-E_n)\equiv 0(mod\ \ (r-s))$, and $(r+s)^{n+1}(\varepsilon_n(r/s)+E_n)\in Z$;

(iii) $(r+s)^{n+1}(\varepsilon_n(r/s)-E_n)$ and $(r+s)^{n+1}(\varepsilon_n(r/s)+E_n)$ are even when r and s are odd.

Proof: We only give the proofs of (i) and (ii) and the proof of (iii) is omitted.

(i) Let $A_n = \varepsilon_n(r/s)$. We prove that $(r+s)^{n+1}A_n$ is even and $(r+s)^{n+1}A_n \equiv 0(mod\ \ (2r))$ by induction.

Clearly, $A_0 = 2r/(r+s)$, $(r+s)A_0 = 2r$ is even, and $(r+s)A_0 \equiv 0(mod\ \ (2r))$. Assume that $(r+s)^n A_{n-1}$ is even and $(r+s)^n A_{n-1} \equiv 0(mod\ \ (2r))$ when $n \geq 2$. It follows from (4) that

$$A_n = \frac{2r}{r+s} - \frac{s}{r+s}\sum_{t=0}^{n-1} 2^{n-t}\binom{n}{t}A_t, \quad n \geq 1.$$

Then

$$(s+r)A_n + s\sum_{t=0}^{n-1} 2^{n-t}A_t\binom{n}{t} = 2r,$$

$$(s+r)^{n+1}A_n = 2r(s+r)^n - s\sum_{t=0}^{n-1} 2^{n-t}(s+r)^{t+1}A_t\binom{n}{t}(s+r)^{n-t-1}.$$

Taking into account the fact that $2r(r+s)^n$ and $(r+s)^{t+1}A_t$ are even and $(r+s)^{t+1}A_t \equiv 0(mod\ \ (2r))$ when $0 \leq t \leq n-1$, we see that $(s+r)^{n+1}A_n$ is even and $(s+r)^{n+1}A_n \equiv 0(mod\ \ (2r))$.

(ii) Let $\bar{B}_n = \varepsilon_n(r/s) - E_n$. Evidently, $\bar{B}_0 = (r-s)/(r+s)$, $(r+s)\bar{B}_0 = r-s \in Z$, and $(r+s)\bar{B}_0 \equiv 0(mod\ \ (r-s))$. Suppose that $(r+s)^n\bar{B}_{n-1} \in Z$ and $(r+s)^n\bar{B}_{n-1} \equiv 0(mod\ \ (r-s))$ when $n \geq 2$. It follows from (1) that

$$\sum_{n=0}^{\infty}\bar{B}_n\frac{x^n}{n!} = \frac{2(r-s)e^{3x}}{se^{4x}+(r+s)e^{2x}+r}.$$

Then

$$\left(s\sum_{n=0}^{\infty}\frac{4^n x^n}{n!} + (r+s)\sum_{n=0}^{\infty}\frac{2^n x^n}{n!} + r\right)\sum_{n=0}^{\infty}\bar{B}_n\frac{x^n}{n!} = 2(r-s)\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}. \quad (7)$$

Comparing the coefficients of $\frac{x^n}{n!}$ on both sides of (7), we obtain

$$2(r+s)\bar{B}_n + s\sum_{t=0}^{n-1}2^{2n-2t}\binom{n}{t}\bar{B}_t + (r+s)\sum_{t=0}^{n-1}2^{n-t}\binom{n}{t}\bar{B}_t$$
$$= 2(r-s)\times 3^n, \quad n \geq 1.$$

Hence

$$(r+s)^{n+1}\bar{B}_n = -s\sum_{t=0}^{n-1} 2^{2n-2t-1}(r+s)^n\bar{B}_t\binom{n}{t} - \sum_{t=0}^{n-1} 2^{n-1-t}(r+s)^{n+1}\bar{B}_t\binom{n}{t} + 3^n(r-s)(r+s)^n.$$

By assumption $(r+s)^{t+1}\bar{B}_t \in Z$ and $(r+s)^{t+1}\bar{B}_t \equiv 0(mod\ \ (r-s))$ when $0 \le t \le n-1$, we have $(r+s)^{n+1}\bar{B}_n \in Z$ and $(r+s)^{n+1}\bar{B}_n \equiv 0(mod\ \ (r-s))$.

Using the same method, we can show that $\varepsilon_n(r/s) + E_n \in Z$. This completes the proof.

Now, for Theorem 1, we give some special cases of (i-ii). Putting $r = 3$, $s = 2$, $G_n = 5^{n+1}\varepsilon_n(3/2)$, and $H_n = 5^{n+1}(\varepsilon_n(3/2) - E_n)$ we have

$$G_0 = 6,\ \ G_1 = 6,\ \ G_2 = -138,\ \ G_3 = -714,$$
$$G_n = 5^n \times 6 - 4\sum_{t=0}^{n-1} 10^{n-1-t}\binom{n}{t}G_t,\quad n \ge 4.$$

$$H_0 = 1,\ \ H_1 = 6,$$
$$H_n = 15^n - 4\sum_{t=0}^{n-1} 20^{n-1-t}\binom{n}{t}H_t - 5\sum_{t=0}^{n-1} 10^{n-1-t}\binom{n}{t}H_t,\quad n \ge 2.$$

Corollary 1: Let k be a positive integer with $k \ge 1$. Then $(r+s)^{n+k}\varepsilon_n^{(k)}(r/s)$ is even and $(r+s)^{n+k}\varepsilon_n^{(k)}(r/s) \equiv 0(mod\ \ (2r)^k)$.

Proof: Let $C_n = (r+s)^{n+k}\varepsilon_n^{(k)}(r/s)$. It follows from (3) that

$$\sum_{n=0}^{\infty} C_n\frac{x^n}{n!} = \frac{(2r)^k(r+s)^k e^{k(r+s)x}}{[se^{2(r+s)x}+r]^k}.$$

Noting that

$$\frac{(2r)^k(r+s)^k e^{k(r+s)x}}{[se^{2(r+s)x}+r]^k} = \left(\sum_{n=0}^{\infty}(r+s)^{n+1}\varepsilon_n^{(k)}(r/s)\frac{x^n}{n!}\right)^k,$$

we have that

$$C_n = \sum_{i_1+\cdots+i_k=n} \frac{n!(r+s)^{i_1+1}\varepsilon_{i_1}(r/s)\cdots(r+s)^{i_k+1}\varepsilon_{i_k}(r/s)}{i_1!\cdots i_k!}.$$

We have shown that $(r+s)^{n+1}\varepsilon_n(r/s)$ is even and $(r+s)^{n+1}\varepsilon_n(r/s) \equiv 0(mod\ \ (2r))$. Hence C_n is even and $C_n \equiv 0(mod\ \ (2r)^k)$. This completes the proof.

We note that Corollary 1 generalizes (i) of Theorem 1.

Theorem 2: For any arbitrary rational number r/s with $r/s > 1$ and $s > 0$, we have for each n that

(i) $s^n\beta_n(r/s) \le 0$;

(ii) $(s-r)^n s^n \beta_n(r/s)$ is even when r or s is even;

(iii) $(s-r)^n s^n \beta_n(r/s) \in Z$ and $(s-r)^n s^n \beta_n(r/s) \equiv 0(mod\ \ (rs))$.

Proof: Let $D_n = s^n\beta_n(r/s)$. It is not difficult to verify that

$$D_0 = 0 \text{ and } D_1 = \frac{rs}{s-r} < 0.$$

Assume that $D_{n-1} \le 0 (n \ge 3)$. Now we show that $D_n \le 0$. From (5) we have

$$(s-r)D_n = rs(r-s)^{n-1}n - s\sum_{p=0}^{n-1} D_p \binom{n}{p} s^{n-p}, \quad n \ge 2,$$

$$D_n = -rs(r-s)^{n-2}n + \frac{s}{r-s}\sum_{n=0}^{n-1} D_p \binom{n}{p} s^{n-p}, \quad n \ge 2. \qquad (8)$$

Since $D_p \le 0 (0 \le p \le n-1)$ and $\frac{s}{r-s} > 0$, $D_n \le 0$. This proves by induction on n that $D_n \le 0$ for all n.

From the above proof, we know that $(s-r)^n G_n(n = 0,1)$ is even. Assume that $(s-r)^{n-1}D_{n-1}$ is even when $n \ge 3$. By means of (8), we have

$$(s-r)^n D_n = -(-1)^n rs(r-s)^{2n-2} + (-1)^n s\sum_{p=0}^{n-1}(r-s)^{n-1} D_p \binom{n}{p} s^{n-p}. \quad (9$$

Due to (9) and $(s-r)^p D_p (0 \le p \le n-1)$ is even, $(s-r)^n D_n$ is even.

The proof of (iii) is omitted. This completes the proof.

Corollary 2: Let k be a positive integer with $k \geq 1$. Then

(i) $s^n \beta_n^{(k)}(r/s) \leq 0$ if k is odd and $s^n \beta_n^{(k)}(r/s) \geq 0$ if k is even;

(ii) $(s-r)^n s^n \beta_n^{(k)}(r/s)$ is even if r or s is even;

(iii) $(s-r)^n s^n \beta_n^{(k)}(r/s) \in Z$ and $(s-r)^n s^n \beta_n^{(k)}(r/s) \equiv 0 (mod \ (rs)^k)$.

The proof of Corollary 2 is omitted and we leave it to the readers.

ACKNOWLEDGEMENT

The authors wish to thank the anonymous referee for his /her valuable suggestions for this paper.

References

[1] G. Almkvist and A. Meurman, Values of Bernoulli polynomials and Hurwitz's zeta function at rational points, C. R. Math. Rep. Acad. Sci. Canada, 13 (1991): 104–108.

[2] G. J. Fox, Euler polynomials at rational numbers, C. R. Math. Rep. Acad. Sci. Canada, 21.3 (1999): 87–90.

[3] H. Tsumura, Multiple harmonic series related to multiple Euler numbers, J. Number Theory, 106 (2004): 155-168.

[4] H. Tsumura, Certain functional relations for the double harmonic series related to the double Euler numbers, J. Aust. Math. Soc., 79 (2005): 319–333.

[5] B. Sury, The Values of Bernoulli polynomials at rational numbers, Bull. London Math. Soc., 25 (1993): 327–329.

[6] B. Sury, Values of Euler polynomials, C. R. Math. Rep. Acad. Sci. Canada, 23.1 (2001): 12–15.

[7] Feng-Zhen Zhao and Tianming Wang, Values of Nörlund Euler polynomials and Nörlund Bernoulli polynomials, C. R. Math. Rep. Acad. Sci. Canada, 26.4 (2004): 97-101.

[8] Feng-Zhen Zhao and Tianming Wang, Values of certain polynomials, Utilitas Mathematica, 68 (2005): 63–70.

AMS Classification Number: 11B37, 11B39, 11B68.