On the Independence Number of Edge Chromatic Critical Graphs *

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Abstract

In 1968, Vizing conjectured that for any edge chromatic critical graph $G = (V, E)$ with maximum degree Δ and independence number $\alpha(G)$, $\alpha(G) \leq \frac{|V|}{2}$. This conjecture is still open. In this paper, we prove that $\alpha(G) \leq \frac{3\Delta-2}{5\Delta-2}|V|$ for $\Delta = 11, 12$ and $\alpha(G) \leq \frac{11\Delta-30}{17\Delta-30}|V|$ for $13 \leq \Delta \leq 29$. This improves the known bounds for $\Delta \in \{11, 12, \cdots, 29\}$.

Keywords: edge coloring, critical graphs, independence number.

1 Introduction

Throughout this paper, let $G = (V(G), E(G))$ be a simple graph with n vertices and m edges. A *k-vertex*, *$(\geq k)$-vertex* or *$(\leq k)$-vertex* is a vertex of degree k, at least k or at most k. We use $d_G(x)$ (or $d(x)$ if there is no confusion) to denote the degree of x for $x \in V(G)$. We call k-vertices adjacent to x *k-neighbors* of x and define $N_k(x)$ to be the set of k-neighbors of x and $d_k(x)$ to be the number of k-neighbors of x. Similarly, we define *$(\geq k)$-neighbors*, *$(\leq k)$-neighbors*, $N_{\geq k}(x)$ and $N_{\leq k}(x)$, $d_{\geq k}(x)$ and $d_{\leq k}(x)$. Let $\Delta(G)$, $\delta(G)$ (or Δ, δ) be the maximum degree and minimum degree of G, respectively.

An *edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors. A graph is *edge k-colorable*, if there is an edge coloring of the graph with colors from $\{1, \cdots, k\}$. In 1965, Vizing proved a theorem which states that if G is a graph of maximum degree Δ, then the edge

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chromatic number $\chi'(G)$ of G is either Δ or $\Delta+1$. A graph G is said to be of *class one* if $\chi'(G)=\Delta$, and it is said to be of *class two* if $\chi'(G)=\Delta+1$. G is said to be *critical* if it is connected, class two and $\chi'(G-e)<\chi'(G)$ for every edge $e\in E(G)$. A critical graph G of maximum degree Δ is called a Δ-*critical graph*. The following conjecture about Δ-critical graphs was proposed by Vizing in 1968.

Conjecture 1.1[9] Let G be a Δ-critical graph with n vertices, then

$$\alpha(G)\le\frac{n}{2}.$$

Conjecture 1.1 is still open so far. The following are some results towards this conjecture.

Theorem 1.2[1] Let G be a Δ-critical graph with n vertices, then

(1) $\alpha(G)\le\frac{(2k-1)\Delta-k(k-1))}{(3k-1)\Delta-k(k-1)}n$, $k=\lfloor\sqrt{\Delta(G)+\frac{1}{4}}+\frac{1}{2}\rfloor$.

(2) $\alpha(G)<\frac{2n}{3}$.

(3) $\alpha(G)\le\begin{cases}\frac{3\Delta-2}{5\Delta-2}n & \text{if } 3\le\Delta\le 6,\\ \frac{5\Delta-6}{8\Delta-6}n & \text{if } 7\le\Delta\le 10.\end{cases}$

In 2004, Grünewald and Steffen[2] verified this conjecture for critical graphs with many edges and in particular, they verified the conjecture for overfull critical graphs.

The following two results are due to Luo and Zhao.

Theorem 1.3[5] Let G be a Δ-critical graph with n vertices and $\Delta\ge\frac{n}{2}$, then $\alpha(G)\le\frac{n}{2}$.

Theorem 1.4[6] Let G be a Δ-critical graph with n vertices, then

$$\alpha(G)\le\begin{cases}\frac{3\Delta-2}{5\Delta-2}n & \text{if } 7\le\Delta\le 10,\\ \frac{9\Delta-20}{14\Delta-20}n & \text{if } 11\le\Delta\le 19.\end{cases}$$

In this note, we get the following better bounds for $\alpha(G)$ for $\Delta\in\{11,12,\cdots,29\}$.

Theorem 1.5 Let G be a Δ-critical graph with n vertices, then

$$\alpha(G)\le\begin{cases}\frac{3\Delta-2}{5\Delta-2}n & \text{if } 11\le\Delta\le 12,\\ \frac{11\Delta-30}{17\Delta-30}n & \text{if } 13\le\Delta\le 29.\end{cases}$$

2 Lemmas

Lemma 2.1(Vizing Adjacency Lemma, or VAL[9]) Let x be a vertex of a Δ-critical graph. Then

(*i*) if $d_k(x)\ge 1$, then $d_\Delta(x)\ge\Delta-k+1$;

(*ii*) $d_\Delta(x) \geq 2$.

Lemma 2.2[7,10] Let G be a Δ-critical graph, $xy \in E(G)$, and $d(x) + d(y) = \Delta + 2$. Then the following hold:

(1) every vertex of $N(x, y) \setminus \{x, y\}$ is a Δ-vertex,

(2) every vertex of $N(N(x, y)) \setminus \{x, y\}$ is of degree at least $\Delta - 1$,

(3) if $d(x), d(y) < \Delta$, then every vertex of $N(N(x, y)) \setminus \{x, y\}$ is a Δ-vertex.

Lemma 2.3[4] Let G be a Δ-critical graph with $\Delta \geq 5$ and x be a 3-vertex. Then there are at least two Δ-vertices in $N(x)$ which are not adjacent to any $(\leq \Delta - 2)$-vertices except x.

Lemma 2.4[4] Let G be a Δ-critical graph with $\Delta \geq 6$ and let x be a 4-vertex.

(1) If x is adjacent to a $(\Delta - 2)$-vertex, say y, then $N(N(x)) \setminus \{x, y\} \subseteq V_\Delta$,

(2) If x is not adjacent to any $(\Delta - 2)$-vertex and if one of the neighbors of x is adjacent to three $(\leq \Delta - 2)$-vertices, then each of the other three neighbors of x is adjacent to only one $(\leq \Delta - 2)$-vertex, which is x,

(3) If x is adjacent to two $(\Delta - 1)$-vertices, then each of the two Δ-neighbors is adjacent to exactly one $(\leq \Delta - 2)$-vertex, which is x.

Lemma 2.5[3] Let x be a 5-vertex in a Δ-critical graph G and suppose that x has a $(\Delta - 2)$-neighbor w.

(1) If w is adjacent to one $(\leq \Delta - 2)$-vertex, other than x, then all the remaining four neighbors of x are all Δ-vertices and each of them is adjacent to $(\geq \Delta - 1)$-vertices except x,

(2) If w is adjacent to only one $(\leq \Delta - 2)$-vertex which is x, then there are three $(\geq \Delta - 1)$-neighbors of x including at least two Δ-neighbors y satisfying the following situations: if it is a Δ-vertex, then it is adjacent to at most two $(\leq \Delta - 2)$-vertices; if it is a $(\Delta - 1)$-vertex, then it is adjacent to one $(\leq \Delta - 2)$-vertex which is x.

Lemma 2.6 Let G be a Δ-critical graph with $\Delta \geq 9$ and let x be a 5-vertex. If x is not adjacent to any $(\leq \Delta - 2)$-vertex and if one of the neighbors of x is adjacent to four $(\leq \Delta - 3)$-vertices, then each of the other four neighbors of x is adjacent to only one $(\leq \Delta - 3)$-vertex, which is x.

The proof of Lemma 2.6 is similar to that of Lemma 2.5 in [3]. We contain it for completeness and put it in Appendix.

3 The independence number of critical graphs

Proof of Theorem 1.5

Let G be a Δ-critical graph. Let $S \subset V$ be an independence set, and let $T = V \setminus S$. For $i \in \{2, 3, \cdots, \Delta\}$, let s_i denote the number of i-vertices in S. Let $A = \{v_t v_s \in E | v_t \in T, v_s \in S \text{ with } d(v_s) < \Delta\}$ and $A_i = \{v_t v_s \in E | v_t \in T, v_s \in S \text{ with } d(v_s) = i\}$. Clearly, $|A_i| = is_i$. We define $f(v_t v_s) : A \to R$ with $v_t \in T, v_s \in S$ as follows:

(i) if $d(v_s) \notin \{3, 4, 5\}$, then $f(v_t v_s) = \frac{1}{d(v_s)-1}$.

(ii) if $d(v_s) = 3$, then

$f(v_t v_s) = \frac{1}{2}$ if v_t is adjacent to exactly one $(\leq \Delta - 2)$-vertices in S distinct from v_s,

$f(v_t v_s) = \frac{\Delta-3}{\Delta-2}$ otherwise.

(iii) if $d(v_s) = 4$, then

$f(v_t v_s) = \frac{1}{3}$ if v_t is adjacent to exactly two $(\leq \Delta - 2)$-vertices in S distinct from v_s,

$f(v_t v_s) = \frac{\Delta-3}{2(\Delta-2)}$ if v_t is adjacent to exactly one $(\leq \Delta - 2)$-vertex in S distinct from v_s,

$f(v_t v_s) = \frac{\Delta-4}{\Delta-2}$ if v_t is adjacent to no other $(\leq \Delta - 2)$-vertices in S distinct from v_s.

(iv) if $d(v_s) = 5$, then

$f(v_t v_s) = \frac{1}{4}$ if v_t is adjacent to exactly three $(\leq \Delta - 3)$-vertices in S distinct from v_s,

$f(v_t v_s) = \frac{1}{d'_5(v_t)}[1 - \sum_{x \in N^-(v_t, v_s), d(x) \geq 6} \frac{1}{d(x)-1}]$ if v_t is adjacent to three $(\leq \Delta - 2)$-vertices (in which there is at least one $(\Delta - 2)$-vertex) in S distinct from v_s, where $d'_5(v_t)$ is the number of 5-neighbors of v_t in S and $N^-(v_t, v_s)$ is the set of $(\leq \Delta - 2)$-vertices in $N(v_t) \bigcap S \setminus \{v_s\}$, and it is easy to check that $f(v_t v_s) \geq \frac{\Delta-4}{3(\Delta-3)}$ in this case,

$f(v_t v_s) = \frac{\Delta-3}{3(\Delta-2)}$ if v_t is adjacent to exactly two $(\leq \Delta - 2)$-vertices (the two $(\leq \Delta - 2)$-vertices are all $(\leq \Delta - 3)$-vertices) in S distinct from v_s,

$f(v_t v_s) = \frac{\Delta-5}{2(\Delta-3)}$ if v_t is adjacent to exactly two $(\leq \Delta - 2)$-vertices (in which at most one of the two $(\leq \Delta - 2)$-vertices is a $(\leq \Delta - 3)$-vertex) in S distinct from v_s,

$f(v_t v_s) = \frac{\Delta-4}{2(\Delta-2)}$ if v_t is adjacent to exactly one $(\leq \Delta - 2)$-vertex in S distinct from v_s,

$f(v_t v_s) = \frac{\Delta-5}{\Delta-2}$ if v_t is adjacent to no other $(\leq \Delta - 2)$-vertex in S distinct from v_s.

(Remark: It is easy to check that $\frac{1}{4} \leq \frac{\Delta-4}{3(\Delta-3)} < \frac{\Delta-3}{3(\Delta-2)} < \frac{\Delta-5}{2(\Delta-3)} < \frac{\Delta-4}{2(\Delta-2)} < \frac{\Delta-5}{\Delta-2}$ when $\Delta \geq 8$.)

Let $v \in T$ such that v is not adjacent to a vertex u in S with $3 \leq d(u) \leq 5$, and let d be the minimum degree of a neighbor of v in S. Then by VAL, v is incident with at most $d-1$ edges in A. Hence we have that $\sum_{vv_s \in A} f(vv_s) \leq \frac{d-1}{d-1} = 1$.

Let $v \in T$ such that v is adjacent to a 3-vertex u in S. By VAL, v is adjacent to at most one ($\leq \Delta - 1$)-vertex in S distinct from u. If v is not adjacent to any ($\leq \Delta - 1$)-vertex in S distinct from u, then by (ii) $\sum_{vv_s \in A} f(vv_s) = f(vu) = \frac{\Delta-3}{\Delta-2} < 1$. If v is adjacent to a ($\leq \Delta - 1$)-vertex in S distinct from u, call it w. If $d(w) \notin \{3, 4, 5\}$, since $d(w) \neq 2$ we have that $\sum_{vv_s \in A} f(vv_s) = f(vu) + f(vw) = \frac{1}{2} + \frac{1}{d(w)-1} < 1$. If $d(w) = 3$, by (ii), we have that $\sum_{vv_s \in A} f(vv_s) = f(vu) + f(vw) = \frac{1}{2} + \frac{1}{2} = 1$. If $d(w) = 4$, by (iii) we have that $\sum_{vv_s \in A} f(vv_s) = f(vu) + f(vw) = \frac{1}{2} + \frac{\Delta-3}{2(\Delta-2)} < 1$. If $d(w) = 5$, by (iv) we have that $\sum_{vv_s \in A} f(vv_s) = f(vu) + f(vw) = \frac{1}{2} + \frac{\Delta-4}{2(\Delta-2)} < 1$.

Let $v \in T$ such that v is adjacent to a 4-vertex u in S. By VAL, v is adjacent to at most two ($\leq \Delta - 1$)-vertices in S distinct from u. If v is not adjacent to any ($\leq \Delta - 2$)-vertex in S distinct from u, then by (iii), $f(vu) = \frac{\Delta-4}{\Delta-2}$, and we have that $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-4}{\Delta-2} + \frac{1}{\Delta-2} + \frac{1}{\Delta-2} = 1$. If v is adjacent to exactly one ($\leq \Delta - 2$)-vertex in S distinct from u, call it w, then $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-3}{2(\Delta-2)} + \frac{\Delta-3}{2(\Delta-2)} + \frac{1}{\Delta-2} = 1$ or $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-3}{2(\Delta-2)} + \frac{\Delta-4}{2(\Delta-2)} + \frac{1}{\Delta-2} < 1$ or $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-3}{2(\Delta-2)} + \frac{1}{d(w)-1} + \frac{1}{\Delta-2} < 1$ according to wether $d(w) = 4$ or 5 or ≥ 6. If v is adjacent to exactly two ($\leq \Delta - 2$)-vertices in S distinct from u, call them w, z, then by (iii) $\sum_{vv_s \in A} f(vv_s) \leq \frac{1}{3} \times 3 = 1$ or $\leq \frac{1}{3} \times 2 + \frac{\Delta-3}{3(\Delta-2)} < 1$ or $\leq \frac{1}{3} + \frac{2(\Delta-3)}{3(\Delta-2)} < 1$ or $\leq \frac{1}{3} + \frac{\Delta-3}{3(\Delta-2)} + \frac{1}{5} < 1$ or $\leq \frac{1}{3} + \frac{\Delta-5}{2(\Delta-3)} + \frac{1}{8} < 1$ or $\leq \frac{1}{3} + 2 \times \frac{1}{5} < 1$ according to if $\{d(w), d(z)\} = \{4,4\}, \{4,5\}, \{5,5\}, \{5, i(6 \leq i \leq \Delta - 3)\}, \{5, j(j \geq \Delta - 2)\}$, or $\{k(k \geq 6), l(l \geq 6)\}$.

Let $v \in T$ such that v is adjacent to a 5-vertex u in S. By VAL, v is adjacent to at most three ($\leq \Delta - 1$)-vertices in S distinct from u. If v is not adjacent to any ($\leq \Delta - 2$)-vertex in S distinct from u, then by (iv), $f(vu) = \frac{\Delta-5}{\Delta-2}$, and we have that $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-5}{\Delta-2} + 3 \times \frac{1}{\Delta-2} = 1$. If v is adjacent to exactly one ($\leq \Delta - 2$)-vertex in S distinct from u, call it w, then $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-4}{2(\Delta-2)} + \frac{\Delta-4}{2(\Delta-2)} + \frac{2}{\Delta-2} = 1$ or $\sum_{vv_s \in A} f(vv_s) \leq \frac{\Delta-4}{2(\Delta-2)} + \frac{1}{d(w)-1} + \frac{2}{\Delta-2} < \frac{\Delta-4}{2(\Delta-2)} + \frac{\Delta-4}{2(\Delta-2)} + \frac{2}{\Delta-2} = 1$ (since $\frac{1}{d(w)-1} < \frac{\Delta-4}{2(\Delta-2)}$ when $d(w) \geq 6$ and $\Delta \geq 6$) according to wether $d(w) = 5$ or $d(w) \geq 6$. If v is adjacent to exactly two ($\leq \Delta - 2$)-vertices in S distinct from u, call them w, z, then by (iv), we have $\sum_{vv_s \in A} f(vv_s) \leq 3 \cdot \frac{\Delta-3}{3(\Delta-2)} + \frac{1}{\Delta-2} = 1$

or $\leq 2 \cdot \frac{\Delta-3}{3(\Delta-2)} + \frac{1}{5} + \frac{1}{\Delta-2} < 1$ or $\leq 2 \cdot \frac{\Delta-5}{2(\Delta-3)} + \frac{1}{\Delta-3} + \frac{1}{\Delta-2} < 1$ or $\leq \frac{\Delta-5}{2(\Delta-3)} + 2 \cdot \frac{1}{5} + \frac{1}{\Delta-2} < 1$ according to if $\{d(w), d(z)\} = \{5,5\}, \{5, i(6 \leq i \leq \Delta-3)\}, \{5, j(j \geq \Delta-2)\}$, or $\{k(k \geq 6), l(l \geq 6)\}$. If v is adjacent to three ($\leq \Delta-3$)-vertices in S distinct from u, call them w, y, z, then by (iv), $f(vu) \leq \frac{1}{4}, f(vw) \leq \frac{1}{4}, f(vy) \leq \frac{1}{4}$ and f($vz) \leq \frac{1}{4}$, so $\sum_{vv_s \in A} f(vv_s) \leq 1$. If v is adjacent to three ($\leq \Delta-2$)-vertices (in which there is at least one ($\Delta-2$)-vertex)in S distinct from u, then by (iv), we have that $\sum_{vv_s \in A} f(vv_s) \leq d'_5(v_t) \cdot \frac{1}{d'_5(v_t)}[1 - \sum_{x \in N^-(v_t,v_s), d(x) \geq 6} \frac{1}{d(x)-1}] + \sum_{x \in N^-(v_t,v_s), d(x) \geq 6} \frac{1}{d(x)-1} = 1$. Hence ,

$$|T| \geq \sum_{v \in T, vv_s \in A} f(vv_s) = \sum_{e \in A} f(e) = \sum_{i=2}^{\Delta-1} \sum_{e \in A_i} f(e).$$

Clearly, for $i \notin \{3,4,5,\Delta\}$, we have that $\sum_{e \in A_i} f(e) = \frac{is_i}{i-1}$. We need to estimate $\sum_{e \in A_i} f(e)$ for $i \in \{3,4,5\}$. First we consider $\sum_{e \in A_3} f(e)$. By Lemma 2.3, for each 3-vertex $v_s \in S$, it is adjacent to at least two Δ-vertices in T that are not adjacent to any ($\leq \Delta-2$)-vertices except v_s. Thus by (ii), each 3-vertex in S is incident with at least two edges $e \in A_3$ with $f(e) = \frac{\Delta-3}{\Delta-2}$ and we have that $\sum_{e \in A_3} f(e) \geq \frac{s_3}{2} + \frac{2(\Delta-3)s_3}{\Delta-2}$.

Now we consider $\sum_{e \in A_4} f(e)$. By Lemma 2.4, for each 4-vertex $v_s \in S$, either it has one neighbor in T that is adjacent to three ($\leq \Delta-2$)-vertices and each of the other three neighbors is adjacent to only one ($\leq \Delta-2$)-vertex in S, that is v_s or each of its four neighbors is adjacent to at most two ($\leq \Delta-2$)-vertices in S. Thus by (iii), each 4-vertex in S is either incident with one edge $e \in A_4$ with $f(e) = \frac{1}{3}$ and three edges $e' \in A$ with $f(e') = \frac{\Delta-4}{\Delta-2}$ or incident with four edges $e \in A_4$ with $f(e) \geq \frac{\Delta-3}{2(\Delta-2)}$. Since $\frac{1}{3} + \frac{3(\Delta-4)}{\Delta-2} > \frac{4(\Delta-3)}{2(\Delta-2)}$ for $\Delta \geq 7$, we have that $\sum_{e \in A_4} f(e) \geq \frac{4(\Delta-3)s_3}{2(\Delta-2)}$.

Then we consider $\sum_{e \in A_5} f(e)$. For each 5-vertex $v_s \in S$, if v_s is adjacent to a ($\Delta-3$)-vertex, then by Lemma 2.2 all the neighbors of v_s are adjacent only to Δ-vertices in S (if any) except v_s, by (iv), v_s is incident with five edges $e \in A_5$ with $f(e) = \frac{\Delta-5}{\Delta-2}$; if v_s is adjacent to one ($\Delta-2$)-vertex or is only adjacent to ($\geq \Delta-1$)-vertices, then by Lemma 2.5 , Lemma 2.6 and (iv) we have that $\sum_{v_t \in T} f(v_s v_t) \geq \frac{\Delta-4}{2(\Delta-2)} + 4 \cdot \frac{\Delta-5}{\Delta-2}$ or $\geq \frac{\Delta-5}{\Delta-2} + \frac{1}{4} + \frac{3(\Delta-4)}{2(\Delta-2)}$ or $\geq \frac{1}{4} + 4 \cdot \frac{\Delta-5}{2(\Delta-3)}$ or $\geq 5 \cdot \frac{\Delta-4}{3(\Delta-3)}$.

when $\Delta \geq 7$, it is easy to check that

$\min\{5\cdot\frac{\Delta-5}{\Delta-2}, \frac{\Delta-4}{2(\Delta-2)}+4\cdot\frac{\Delta-5}{\Delta-2}, \frac{\Delta-5}{\Delta-2}+\frac{1}{4}+\frac{3(\Delta-4)}{2(\Delta-2)}, \frac{1}{4}+4\cdot\frac{\Delta-5}{2(\Delta-3)}, 5\cdot\frac{\Delta-4}{3(\Delta-3)}\} = 5\cdot\frac{\Delta-4}{3(\Delta-3)}$.

So we have that $\sum\limits_{e\in A_5} f(e) \geq 5\cdot\frac{\Delta-4}{3(\Delta-3)}s_5$.

$$|T| \geq \sum_{v\in T, vv_s\in A} f(vv_s) = \sum_{e\in A} f(e) = \sum_{i=2}^{\Delta-1}\sum_{e\in A_i} f(e) \geq 2s_2 + \frac{s_3}{2} + \frac{2(\Delta-3)s_3}{\Delta-2}$$

$$+\frac{4(\Delta-3)s_4}{2(\Delta-2)} + \frac{5(\Delta-4)s_5}{3(\Delta-3)} + \frac{6s_6}{5} + \cdots + \frac{(\Delta-1)s_{\Delta-1}}{\Delta-2}. \quad (2)$$

Since G is critical, so $|T|\Delta > \sum\limits_{i=2}^{\Delta} is_i$. Thus

$$|T| > \sum_{i=2}^{\Delta}\frac{i}{\Delta}s_i = |S| - \sum_{i=2}^{\Delta}\frac{\Delta-i}{\Delta}s_i. \quad (3)$$

Combining (2) with (3) as (2) + $\frac{2\Delta}{\Delta-2}$(3), we have that $\frac{3\Delta-2}{\Delta-2}|T| > \frac{2\Delta}{\Delta-2}|S| + \frac{s_3}{2} + \frac{2s_4}{\Delta-2} + [\frac{5(\Delta-4)}{3(\Delta-3)} - \frac{2(\Delta-5)}{\Delta-2}]s_5 + \sum\limits_{i=6}^{\Delta-1}\frac{2(i-2)(i-\frac{\Delta}{2})s_i}{(i-1)(\Delta-2)}$.

For $\Delta = 11, 12$, $\frac{5(\Delta-4)}{3(\Delta-3)} - \frac{2(\Delta-5)}{\Delta-2} > 0$, and $\frac{2(i-2)(i-\frac{\Delta}{2})}{(i-1)(\Delta-2)} \geq 0$ when $i \geq 6$.

So we have that $|T| > \frac{2\Delta}{3\Delta-2}|S|$. Since $n = |S| + |T| > \frac{5\Delta-2}{3\Delta-2}|S|$, so $|S| < \frac{3\Delta-2}{5\Delta-2}n$. Hence we have that $\alpha(G) \leq \frac{3\Delta-2}{5\Delta-2}n$ when $\Delta = 11, 12$.

Combining (2) with (3) as (2) + $\frac{6\Delta}{5(\Delta-6)}$(3), we have that $\frac{11\Delta-30}{5(\Delta-6)}|T| \geq \frac{6\Delta}{5(\Delta-6)}|S| + a_2s_2 + a_3s_3 + a_4s_4 + a_5s_5 + \sum\limits_{i=7}^{\Delta-1}(\frac{i}{i-1} - \frac{6(\Delta-i)}{5(\Delta-6)})s_i$, where $a_2 = 2-\frac{6\Delta}{5(\Delta-6)}\cdot\frac{\Delta-2}{\Delta}$, $a_3 = \frac{1}{2}+\frac{2(\Delta-3)}{\Delta-2}-\frac{6\Delta}{5(\Delta-6)}\cdot\frac{\Delta-3}{\Delta}$, $a_4 = \frac{4(\Delta-3)}{2(\Delta-2)}-\frac{6\Delta}{5(\Delta-6)}\cdot\frac{\Delta-4}{\Delta}$, $a_5 = \frac{5(\Delta-4)}{3(\Delta-3)} - \frac{6\Delta}{5(\Delta-6)}\cdot\frac{\Delta-5}{\Delta}$. When $\Delta > 12$, $a_i > 0$ for $i = 2, 3, 4, 5$. And $\frac{i}{i-1} - \frac{6(\Delta-i)}{5(\Delta-6)} \geq 0$ when $i \geq 7$ and $\Delta \leq 29$. So $\frac{11\Delta-30}{5(\Delta-6)}|T| \geq \frac{6\Delta}{5(\Delta-6)}|S|$, then we have that $|S| \leq \frac{11\Delta-30}{17\Delta-30}n$. That is $\alpha(G) \leq \frac{11\Delta-30}{17\Delta-30}n$ when $13 \leq \Delta \leq 29$. This completes the proof of Theorem 1.11.

4 Appendix

Let the edges of a graph be colored with colors from $C = \{1, \cdots, k\}$ and let $u \in V$. If an edge incident with u is colored i, we say u *sees* i. Otherwise, we say u *misses* i. Let $i, j \in \{1, \cdots, k\}$, an $i-j$ *edge chain* is a chain of edges colored alternatively i and j. Let $L_{i,j}(u)$ denote the maximal $i-j$ chain starting from u if u misses i or j.

Let G be a Δ-critical graph and xy be an edge of G. Consider $G-xy$ that is edge Δ-colorable. Let f be an edge coloring of $G-xy$ from $E(G)\setminus\{xy\}$ to $\{1,2,\cdots,\Delta\}$.

Then there are the following facts about f.

Fact 1[4] Let $u \in N(x)\setminus\{y\}$ and the edge xu be colored k. If y misses k, then u sees every color seen by only one of x, y.

Fact 2[4] Let $u \neq y$ be a neighbor of x and $v \neq x, y$ be a neighbor of u. Assume that ux is colored k and uv is colored l, and that $d(x) < \Delta$. If k is missing at y and l is missing at either x or y, then v sees every color seen by only one of x, y.

Proof of Lemma 2.6 Let $N(x) = \{y,z,u,v,w\}$ where $d_{\leq\Delta-3}(w) = 4$. We only prove the lemma for $d(u) = d(v) = d(w) = d(y) = d(z) = \Delta$. For other cases, a similar argument can be applied. Assume that $N(y) = \{x, y_2, y_3, \cdots, y_\Delta\}$ and $N(w) = \{x, w_2, w_3, \cdots, w_\Delta\}$. Consider $G - xw$. Since G is critical, $G - xw$ has an edge Δ-coloring. Without loss of generality, we assume that yy_i and ww_i are colored i, xy is colored 1, xz is colored 2, xu is colored 3 and xv is colored 4.

Claim 1. $d(w_i) \leq \Delta - 3$, for $i = 2,3,4$.
Proof of Claim 1. Otherwise, without loss of generality, suppose that $d(w_3) \geq \Delta - 2$. We consider the following two cases.

Case 1 $d(w_2) \geq \Delta - 2$.

Since $d_{\leq\Delta-3}(w) = 4$, there are two vertices $w_p, w_q \in N(w)$ with $p, q > 4$ such that $d(w_p) \leq \Delta-3$ and $d(w_q) \leq \Delta-3$. By Fact 1(taking w as x, and x as y), w_p and w_q see all the colors in $\{1, 5, \cdots, \Delta\}$. Therefore, w_p, w_q both miss colors $2, 3, 4$. Let $k \geq 5$ and $k \neq p, q$ (such k exists because $\Delta \geq 9$). If $L_{k,2}(w_p)$ doesn't end at x, swap colors along $L_{k,2}(w_p)$. Then w_p misses the color k which is seen by w but not by x. It contradicts Fact 1. Therefore, $L_{k,2}(w_p)$ ends at x. Similarly, $L_{k,2}(w_q)$ also ends at x. It contradicts that $L_{k,2}(w_p)$ and $L_{k,2}(w_q)$ are either identical or disjoint.

Case 2 $d(w_2) \leq \Delta - 3$.

Then $d(w_p) \leq \Delta - 3$ for some $p \geq 5$. Using an argument similar to the one in Case 1, one can conclude that $d(w_p) = \Delta-3$ and w_p misses colors 2, 3, 4. If w_2 misses a color $k \geq 5$ and $k \neq p$, then by our observation at least one of the paths $L_{2,k}(w_p), L_{2,k}(w_2)$ doesn't end at x. If $L_{2,k}(w_2)$ doesn't end at x, swap colors along $L_{2,k}(w_2)$. Then ww_2 is colored k and ww_p is still colored p. Note that both k and p are not seen by x. We are back to Case 1. If $L_{2,k}(w_p)$ doesn't end at x, swap colors along $L_{2,k}(w_p)$. Then w_p misses k which is seen by w but not by x. It contradicts Fact 1. Thus w_2 sees all the colors in $\{5, 6, \cdots, \Delta\}\setminus\{p\}$. Since $d(w_2) \leq \Delta-3$, w_2 must miss one of $1, p$. Let $k \in \{5, 6, \cdots, \Delta\}\setminus\{p\}$. If w_2 misses 1, then $L_{1,k}(w_2)$ doesn't pass x. Swap colors along $L_{1,k}(w_2)$. Then w_2 misses $k \in \{5, 6, \cdots, \Delta\}\setminus\{p\}$, a contradiction. Thus w_2 misses p and sees 1. Since $L_{1,p}(w_2)$ passes neither x nor w, swap colors along $L_{1,p}(w_2)$. Then w_2 misses 1, a contradiction. Thus Claim 1 is true.

Claim 2. If w_i misses a color $k \in \{5, 6, \cdots, \Delta\}$ for $i = 2, 3, 4$, then $L_{i,k}(w_i)$ must end at x.
Proof or Claim 2. Otherwise, swap colors along $L_{i,k}(w_i)$. Then w_i misses i and ww_i is colored k which is not seen by x. By Claim 1, $d(w_i) = \Delta$, a contradiction.

Claim 3. $d(y_i) \geq \Delta - 2$ for each $i \geq 5$.
Proof of Claim 3. Otherwise, suppose that $d(y_i) \leq \Delta - 3$ for some $i \geq 5$. Then y_i sees all the colors except 2, 3, 4 by Fact 2, and therefore, $d(y_i) = \Delta - 3$. If w_2 misses a color $k \in \{5, 6, \cdots, \Delta\}$ and $k \neq i$, then by Claim 2, $L_{2,k}(w_2)$ ends at x and thus doesn't pass y_i. Swap colors along $L_{2,k}(w_2)$. Then x sees k but not 2. By Fact 2, y_i must see 2, a contradiction. Therefore, w_2 sees every color in $\{5, 6, \cdots, \Delta\} \setminus \{i\}$. Moreover w_2 also sees the color 1, otherwise, swap colors along the path $L_{1,j}(w_2)(j \in \{5, 6, \cdots, \Delta\} \setminus \{i\})$, which doesn't pass x. Then w_2 misses the color j, a contradiction. Similarly, we can prove that w_2 also sees i. Thus, w_2 only misses 3, 4. It contradicts Claim 1 which claims that $d(w_2) \leq \Delta - 3$. Therefore, $d(y_i) \geq \Delta - 2$.

Claim 4. $d(y_i) \geq \Delta - 2$ for $i = 2, 3, 4$.
Proof of Claim 4. We will prove $d(y_2) \geq \Delta - 2$. The cases $d(y_3) \geq \Delta - 2$ and $d(y_4) \geq \Delta - 2$ are similar. By contradiction, suppose $d(y_2) \leq \Delta - 3$.

Claim 4-1. Every color in $\{1, 2, \cdots, \Delta\}$ is either seen by w_2 or by y_2.

The proof of Claim 4-1 is divided into three steps.

First, we prove that each color in $\{5, 6, \cdots, \Delta\}$ is seen by either y_2 or w_2. Otherwise, suppose that both w_2 and y_2 miss a color k in $\{5, 6, \cdots, \Delta\}$. By Claim 2, $L_{2,k}(w_2)$ ends at x and thus doesn't pass y_2. Swap colors along $L_{2,k}(w_2)$. Then 2 is not seen by x. By Claim 3, $d(y_2) \geq \Delta - 2$, a contradiction.

Second, we prove that 1 is seen by either w_2 or y_2. Otherwise the paths $L_{1,5}(w_2)$ and $L_{1,5}(y_2)$ don't pass x. Swap colors along $L_{1,5}(w_2) \cup L_{1,5}(y_2)$. Then neither y_2 nor w_2 sees 5, a contradiction. Therefore, 1 must be seen by y_2 or w_2.

Third, we prove that each of 3 and 4 is seen by either w_2 or y_2. Without loss of generality suppose that neither w_2 nor y_2 sees 3. If there is a color $k \in \{5, 6, \cdots, \Delta\}$ not seen by w_3, then by Claim 2, $L_{3,k}(w_3)$ ends at x and passes neither y_2 nor w_2. Swap colors along $L_{3,k}(w_2) \cup L_{3,k}(y_2)$. Then k is not seen by both y_2 and w_2, a contradiction. Thus w_3 sees every color in $\{5, 6, \cdots, \Delta\}$. If w_3 miss 1, then $L_{1,5}(w_3)$ doesn't pass x. Swap colors along $L_{1,5}(w_3)$. Then w_3 doesn't see 5, a contradiction. Thus w_3 sees 1 and one can conclude that $d(w_3) \geq \Delta - 2$, a contradiction. This completes the proof of Claim 4-1.

Since $d(w_2) \leq \Delta - 3$ and w_2 sees 2, w_2 must miss a color in $\{1, 5, 6, \cdots, \Delta\}$. Without loss of generality, we assume that w_2 misses 1. In the following, we consider two cases:

Case 4-1 There are two colors, say p, q, in $\{5, 6, \cdots, \Delta\}$ such that w_2 misses q and y_2 misses p. Then w_2 sees p and y_2 sees q by Claim 4-1.

Note that in this case, w and w_2 both miss 1. We change the color of ww_2 into 1. Then 2 is seen by x but not by w in the new coloring. Then $L_{2,p}(x)$ ends at w and doesn't pass w_2, y_2. Swap colors along $L_{2,p}(x)$. Now x sees 1, p, 3,4 and w doesn't see p. Thus $L_{p,q}(w_2)$ doesn't pass x and w. Swap colors along $L_{p,q}(w_2)$. Then w and w_2 both miss p. Recolor the edge ww_2 with p. Note that $p, 3, 4$ are seen by both x and w, xy is colored 1, and yy_2 is colored 2. Hence $d(y_2) \geq \Delta - 2$ by Claim 3, a contradiction.

Case 4-2 There are no such two colors described in Case 4-1.

If w_2 misses a color $k \in \{5, 6, \cdots, \Delta\}$, then y_2 sees k by Claim 4-1 and in fact, y_2 sees every color in $\{5, 6, \cdots, \Delta\}$ since there are no such two colors described in Case 4-1. Since $d(y_2) \leq \Delta - 3$, we have $d(y_2) = \Delta - 3$ and y_2 misses colors 1, 3, 4. It contradicts Claim 4-1 since w_2 also misses 1. Therefore, w_2 sees every color in $\{5, 6, \cdots, \Delta\}$ and w_2 misses colors 1, 3,4. Hence, y_2 sees 1, 3, 4 and y_2 must miss at least three colors in $\{5, 6, \cdots, \Delta\}$ since $d(y_2) \leq \Delta - 3$. Without loss of generality, assume that y_2 misses 5, 6 and 7. Note that $L_{5,4}(w_2)$ and $L_{5,4}(y_2)$ are either identical or disjoint. If they are disjoint, then one of them doesn't end at x. If $L_{5,4}(w_2)$ doesn't end at x, swap colors along it. Then w_2 misses 5. Note that y_2 also misses 5. It contradicts Claim 4-1. If $L_{5,4}(y_2)$ doesn't end at x, swap colors along it. Then neither w_2 nor y_2 sees 4. It contradicts Claim 4-1. If $L_{5,4}(y_2)$ and $L_{5,4}(w_2)$ are identical, the two ends of the path are y_2 and w_2 and hence, it doesn't pass x. Swap colors along $L_{5,4}(w_2)$. Then w_2 sees 6 and misses 5 and y_2 sees 5 and misses 6. Thus, colors 5, 6 become the colors described in Case 4-1. So we are back to Case 4-1.

In both cases, we obtain a contradiction. This completes the proof of Claim 4.

Claim 5. Each of z, u, v is adjacent to only one $(\leq \Delta - 3)$-vertex.

Proof of Claim 5. Since $d(w_2) \leq \Delta - 3$, w_2 misses either 1 or a color in $\{5, 6, \cdots, \Delta\}$. If w_2 misses 1, recolor the edge ww_2 with 1. Using an argument similar to the one of Claim 4, one can prove that z is adjacent to only one $(\leq \Delta - 3)$-vertex. If w_2 sees 1, then it misses a color $k \in \{5, 6, \cdots, \Delta\}$. Then $L_{k,1}(w_2)$ doesn't pass x. Swap colors along it to obtain a new edge coloring. In this new coloring, w_2 misses 1 and we are back to the case that we just discussed. Since $d(w_3) \leq \Delta - 3$ and $d(w_4) \leq \Delta 3$, similarly, we can prove that u and v are adjacent to only one $(\leq \Delta - 3)$-

vertex. This completes the proof of Lemma 2.6.

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