Isoperimetric Edge Connectivity of Line Graphs and Path Graphs *

Zhao Zhang[†] Fengxia Liu
College of Mathematics and System Sciences, Xinjiang University
Urumqi, Xinjiang, 830046, People's Republic of China

Abstract

The k-th isoperimetric edge connectivity $\gamma_k(G) = \min\{|[U,\overline{U}]| : U \subset V(G), |U| \geq k, |\overline{U}| \geq k\}$. A graph G with $\gamma_k(G) = \beta_k(G)$ is said to be γ_k-optimal, where $\beta_k(G) = \min\{|[U,\overline{U}]| : U \subset V(G), |U| = k\}$. Let G be a connected d-regular graph. Write $L(G)$ and $P_2(G)$ the line graph and the 2-path graph of G, respectively. In this paper, we derive some sufficient conditions for $L(G)$ and $P_2(G)$ to be γ_k-optimal.

Keywords: Isoperimetric edge connectivity; Line graph; Path graph.

1 Introduction

Let G be a connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *line graph* of G is a graph $L(G)$ with vertex set $E(G)$, and two vertices u_1v_1, $u_2v_2 \in V(L(G))$ are adjacent in $L(G)$ if and only if they are adjacent as elements in $E(G)$.

It is well known that for any graph G, the edge connectivity $\lambda(G)$ is no

*This work is supported by NSFC and XJEDU.
[†]Corresponding author: Zhao Zhang, zhzhao@xju.edu.cn

greater than the minimum degree $\delta(G)$. A graph G with $\lambda(G) = \delta(G)$ is said to be *λ-optimal*. By Lemma 3.1 and Lemma 3.2 in [7], we see that the line graph of a 2-connected regular graph is λ-optimal. We will generalize this result to isoperimetric edge connectivity.

For any vertex set $U \subset V$, $[U, \overline{U}]$ denotes the set of edges with one end in U and the other end in $\overline{U} = V \setminus U$. For a positive integer k, the *k-th isoperimetric edge connectivity* of G, proposed by Hamidoune et al. in [6], is defined as

$$\gamma_k(G) = \min\{|[U, \overline{U}]| : U \subset V(G), |U| \geq k, |\overline{U}| \geq k\}.$$

It is obvious that $\gamma_k(G)$ exists for any positive integer $k \leq |V(G)|/2$. $\gamma_1(G)$ is exactly $\lambda(G)$. So, isoperimetric edge connectivity can be regarded as a generalization of the edge connectivity. It also has a close relation with restricted edge connectivity $\lambda_k(G)$ which plays an important role in measuring the reliability of a network (see for example [3, 8, 9], restricted edge connectivity is also called extra edge connectivity in [5]). In [10], the author showed that $\lambda_k(G)$ coincides with $\gamma_k(G)$ if G is a regular graph with girth $g \geq k/2$. We are interested in maximizing $\gamma_k(G)$.

Suppose $|V(G)| \geq 2k$. Let

$$\beta_k(G) = \min\{|[U, \overline{U}]| : U \subset V(G), |U| = k\}.$$

Clearly, $\gamma_k(G) \leq \beta_k(G)$. A graph G with $\gamma_k(G) = \beta_k(G)$ is said to be *γ_k-optimal*. In this paper, we show that the line graph of a connected d-regular graph G with either $\kappa(G) \geq 2k$ or $\kappa(G) = d$ is γ_k-optimal for $k \leq d$, where $\kappa(G)$ is the connectivity of G.

The concept of path graphs were proposed by Broersma and Hoede as a generalization of line graphs [4]. The *r-path graph* $P_r(G)$ is a graph with vertex set $\mathcal{V}_r(G) = \{u_1u_2...u_r \mid u_1u_2...u_r$ is an r-path in $G\}$, and two vertices in $P_r(G)$ are adjacent if and only if the union of the corresponding paths in G forms a path or a cycle of length $r+1$, in another word, if and only if one can be obtained from the other by 'shifting' the corresponding paths in G.

It is shown by Balbuena and Ferrero [1] that a regular graph G with $\lambda(G) \geq 4$ is γ_2-optimal. We will generalize this result in this paper, showing that a regular graph G with $\lambda(G) \geq 2k$ is γ_k-optimal.

A vertex set $U \subset V$ is called a *γ_k-fragment*, if $|U| \geq k, |\overline{U}| \geq k$ and $|[U, \overline{U}]| = \gamma_k(G)$. The cardinality of a minimum γ_k-fragment is denoted by $\alpha_k(G)$. It is easy to see that a graph G is γ_k-optimal if and only if $\alpha_k(G) = k$. We refer [2] for notation and terminologies not defined here.

2 Isoperimetric edge connectivity in line graph

An edge in $E(L(G))$ with the form $\{uv, uw\}$ is called an u-edge. By the definition of line graphs, the following lemma is obvious.

Lemma 1. *Let $S = [U, \overline{U}]$ be an edge cut of $L(G)$, and let u be a vertex in $V(G)$. If U contains s vertices of the form uv, then S contains $s(d(u)-s)$ u-edges. In particular, if G is a d-regular graph and S contains u-edges, then S contains at least $d-1$ u-edges.*

Theorem 1. *Let G be a d-regular graph of connectivity $\kappa = \kappa(G)$, and k a positive integer with $k \leq d$. If $L(G)$ is not γ_k-optimal, then $\gamma_k(L(G)) \geq \kappa(d-1)$.*

Proof. Clearly, $L(G)$ is $(2d-2)$-regular. Since $d \geq k$, $L(G)$ has k-cliques. So $\beta_k(L(G)) = k(2d-2) - k(k-1)$.

Let U be a γ_k-fragment of $L(G)$. Since $L(G)$ is not γ_k-optimal, we have $|U| \geq k+1$. Set $S = [U, \overline{U}]$. The theorem follows from the following assertion and Lemma 1.

Assertion. There exist κ distinct vertices $u_1, u_2, ..., u_\kappa \in V(G)$ such that S contains u_i-edges ($i = 1, 2, ..., \kappa$).

By contradiction. Suppose S contains $u_1, u_2, ..., u_t$-edges with $t \leq \kappa - 1$. Then $t \leq d - 1$.

We first show that any vertex in one of U or $\overline{U}$, say U, has the form $u_i u_j$ ($i \neq j; i, j \in \{1, 2, ..., t\}$). In fact, if this is not true, then there are $v_1, v_2 \in V(G) \setminus \{u_1, u_2, ..., u_t\}$ such that $v_1 w_1 \in U$ for some $w_1 \in V(G)$, and $v_2 w_2 \in \overline{U}$ for some $w_2 \in V(G)$. Let $P = x_0 x_1 ... x_l$ be a (v_1, v_2)-path in $G - \{u_1, u_2, ..., u_t\}$, where $x_0 = v_1$ and $x_l = v_2$ (note that such a path exists since $t < \kappa$). As $x_0 \notin \{u_1, u_2, ..., u_t\}$, S contains no x_0-edges. So, it follows from $x_0 w_1 \in U$ that $x_0 x_1 \in U$. Similarly, as $x_1 \notin \{u_1, u_2, ..., u_t\}$, S contains no x_1-edge, and thus $x_1 x_2 \in U$. Proceeding like this, we see that $x_{l-1} v_2 \in U$. But then $\{x_{l-1} v_2, v_2 w_2\}$ is a v_2-edge contained in S, a contradiction.

As a consequence, we have $t \geq 3$, since otherwise $|U| = 1$, contradicting that $|U| \geq k + 1 \geq 2$.

Suppose u_i occurs s_i times in U. Then $s_i \leq |\{u_i u_j \mid j \neq i, j \in \{1, 2, ..., t\}\}| = t - 1$ ($i = 1, 2, ..., t$). Write $f(s_1, s_2, ..., s_t) = \sum_{i=1}^{t} s_i(d - s_i)$. Then $\gamma_k(L(G)) = |S| = f(s_1, s_2, ..., s_t)$. In the following, we show that

$$f(s_1, ..., s_t) \geq \beta_k(L(G)), \tag{1}$$

and thus arrive at a contradiction to the assumption that $L(G)$ is not γ_k-optimal.

Since $|U| \geq k + 1$, we have

$$\sum_{i=1}^{t} s_i = 2|U| \geq 2(k + 1). \tag{2}$$

So, (1) is equivalent to

$$h(s_1, ..., s_t) \triangleq \frac{\sum_{i=1}^{t} s_i^2 - k(k+1)}{\sum_{i=1}^{t} s_i - 2k} \leq d. \tag{3}$$

For fixed t, $h(s_1, ..., s_t)$ is maximum when $s_1 = s_2 = ... = s_t = t - 1$. So $h(s_1, ..., s_t) \leq g(t)$, where

$$g(t) = \frac{t(t-1)^2 - k(k+1)}{t(t-1) - 2k}.$$

Note that the denominator of G is greater than zero. So, when t satisfies $t(t-1)^2 \leq k(k+1)$, it is obvious that $g(t) \leq d$. When $t(t-1)^2 > k(k+1)$, $g(t)$ is monotonously increasing, and thus $g(t) \leq g(d-1)$. If $d = 3$, then $t \leq d - 1 = 2$, contradicting that $t \geq 3$. So, $d \geq 4$. In this case, it is easy to see that $g(d-1) \leq d$ (note that $k \leq d$). □

As a consequence, we see that

Corollary 1. *Let G be a connected d-regular graph, k be an integer with $k \leq d$. If $\kappa(G) \geq (2kd - k - k^2)/(d-1)$, then $L(G)$ is γ_k-optimal.*

In particular, we have the following two sufficient conditions for a line graph to be γ_k-optimal.

Corollary 2. *Let G be a connected regular graph with connectivity $\kappa(G) \geq 2k$. Then $L(G)$ is γ_k-optimal.*

Corollary 3. *Let G be a connected d-regular graph with $\kappa(G) = d$. Then $L(G)$ is γ_k-optimal for any $k = 1, 2, ..., d$.*

3 Isoperimetric edge connectivity in 2-path graph

An edge in $P_2(G)$ with the form (xuv, uvy) is called an uv-edge.

Lemma 2. *Let G be a d-regular connected graph with $d \geq 2$, and $S = [U, \overline{U}]$ an edge cut of $P_2(G)$. Suppose there are s vertices in U with the form xuv and t vertices in U with the form uvy, then S has $s(d-1-t)+t(d-1-s)$ uv-edges. So, if S contains uv-edges, then S contains at least $d-1$ uv-edges.*

Proof. Note that $P_2(G)$ has $d-1$ vertices with the form xuv and $d-1$ vertices with the form uvy. So, $\overline{U}$ has $d-1-s$ vertices with the form xuv and $d-1-t$ vertices with the form uvy. Since every vertex with the form xuv is adjacent to every vertex with the form uvy, the result follows. □

For an edge set $S \subset E(P_2(G))$, write $S' = \{(u,v) : S \text{ contains } uv\text{-edges}\}$. The following lemma is shown in [1].

Lemma 3. *Let G be a connected graph with $\delta(G) \geq 2$, and $S = [U, \overline{U}]$ an edge cut of $P_2(G)$. If there exists a vertex $uvw \in U$ and a vertex $u'v'w' \in \overline{U}$ with $(u,v) \notin S'$ or $(v,w) \notin S'$ and $(u',v') \notin S'$ or $(v',w') \notin S'$, then S' is an edge cut of G.*

Theorem 2. *Let G be a d-regular connected graph with $\delta(G) \geq 2$, and k a positive integer with $k \leq d-2$. If $P_2(G)$ is not γ_k-optimal, then $\gamma_k(P_2(G)) \geq \lambda(G)(d-1)$.*

Proof. When $d = 2$, $G \cong P_2(G) \cong C_n$, which is obviously γ_k-optimal. So, suppose $d \geq 3$.

Let U be a γ_k-fragment of $P_2(G)$. Set $S = [U, \overline{U}]$, and suppose $S' = \{(u_i, v_i)\}_{i=1}^{t}$. Since $P_2(G)$ is not γ_k-optimal, we have $|U| \geq k+1$. Suppose the theorem is not true, then $|S'| = t < \lambda(G)$ by Lemma 2. So, S' is not an edge cut of G. Then it follows from Lemma 3 that at least one of U and $\overline{U}$, say U, has the following property: for any vertex $uvw \in U$, both $(u,v) \in S'$ and $(v,w) \in S'$.

For each $i \in \{1, ..., t\}$, suppose there are s_i vertices in U with the form xu_iv_i, and t_i vertices in U with the form u_iv_iy. Then, $s_i \leq |\{xu_iv_i \mid (x, u_i) \in S', (x, u_i) \neq (u_i, v_i)\}| \leq t-1$, and $t_i \leq |\{u_iv_iy \mid (v_i, y) \in S', (v_i, y) \neq (u_i, v_i)\}| \leq t-1$. Furthermore,

$$\sum_{i=1}^{t}(s_i + t_i) = 2|U| \geq 2(k+1), \tag{4}$$

and

$$|S| = \sum_{i=1}^{t}[s_i(d-1-t_i) + t_i(d-1-s_i)] = (d-1)\sum_{i=1}^{t}(s_i+t_i) - 2\sum_{i=1}^{t} s_it_i.$$

We are to show that

$$|S| \geq \beta_k(P_2(G)), \tag{5}$$

and thus arrive at a contradiction. Since $P_2(G)$ is $(2d-2)$-regular, we have

$$\beta_k(P_2(G)) \leq k(2d-2) - 2(k-1). \tag{6}$$

By (4) and (6), to show inequality (5), it suffices to show that

$$\frac{2(\sum_{i=1}^{t} s_it_i - k + 1)}{\sum_{i=1}^{t}(s_i+t_i) - 2k} \leq d-1.$$

The maximum of the left term is realized when $s_i = t_i = t-1$ $(i = 1, 2, ..., t)$. So, it suffices to show that

$$g(t) \triangleq \frac{t(t-1)^2 - k + 1}{t(t-1) - k} \leq d-1. \tag{7}$$

Note that $g(t)$ is monotonously increasing. So, $g(t) \leq g(\lambda(G)-1) \leq g(d-1)$. Since $k \leq d-2$, we have $g(d-1) \leq d-1$. So (7), and thus (5) is proved. □

Combining Theorem 2 with inequality (6), we see that

Corollary 4. *Let G be a d-regular connected graph ($d \geq 2$), and k a positive integer with $k \leq d-2$. If $\lambda(G) \geq 2k - \frac{2(k-1)}{d-1}$, then $P_2(G)$ is γ_k-optimal.*

In particular, we have

Corollary 5. *Let k be a positive integer, and G a d-regular connected graph with $\lambda(G) \geq 2k$ ($d \geq 4$). Then $P_2(G)$ is γ_k-optimal.*

Corollary 6. *Let G be a d-regular connected graph with $\lambda(G) = d \geq 2$. Then for any positive integer $k \leq \min\{(d+1)/2, d-2\}$, $P_2(G)$ is γ_k-optimal.*

References

[1] C. Balbuena and D. Ferrero, Edge-connectivity and super edge-connectivity of P_2-path graphs, *Discrete Math.*, **269** (2003) 13-20.

[2] J.A. Bondy and U.S.R. Murty, Graph theory with applications, *The Macmillan Press LTD, London and Basingstoke*, (1976) 47-49.

[3] P. Bonsma, N. Ueffing and L. Volkmanns, Edge-cuts leaving components of order at least three, *Discrete Math.*, **256** (2002) 431-439.

[4] H.J. Broersma and C. Hoede, Path graphs, *J. Graph Theory*, **13** (1989) 427-444.

[5] J. Fàbrega and M. A. Foil, On the extraconnectivity of graphs. *Discrete Math.*, **155** (1996) 49-57.

[6] Y.O. Hamidoune, A.S. Lladó, O. Serra and R. Tindell, On isoperimetric connectivity in vertex-transitive graphs, *SIAM J. Discrete Math.*, **13** (2000) 139-144.

[7] J.X. Meng, Superconnectivity and super edge connectivity of line graphs, *Graph Theory Notes of New York*, **XL** (2001) 12-14.

[8] M. Wang and Q. Li, Conditional edge connectivity properties, reliability comparisons and transitivity of graphs, *Discrete Math.*, **258** (2002) 205-214.

[9] Z. Zhang and J.X. Meng, On optimally-$\lambda^{(3)}$ transitive graphs, *Discrete Appl. Math.*, **154-6** (2006) 1011-1018.

[10] Z. Zhang, Extra edge connectivity and isoperimetric edge connectivity, submitted.