# PEBBLING GRAPH PRODUCTS

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Abstract. A pebbling step on a graph consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. We consider all weight functions defined on the vertices of a graph that satisfy some property $P$. The $P$-pebbling number of a graph is the minimum number of pebbles needed in an arbitrary initial configuration so that, for any such weight function, there is a sequence of pebbling moves at the end of which each vertex has at least as many pebbles as required by the weight function. Some natural properties on graph products are induced by properties defined on the factor graphs. In this paper we give a bound for the $P'$-pebbling number associated with a particular kind of product property $P'$ in terms of the $P_i$-pebbling numbers associated with the factor properties $P_1$ and $P_2$. We do this by introducing color pebbling, which may be of interest in its own right.

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## 1. Introduction and definitions

The game of pebbling was first suggested by Lagarias and Saks as a tool for solving a number-theoretical conjecture of Erdös. Chung successfully used this tool to prove the conjecture and established other results concerning pebbling numbers. In doing so she introduced pebbling to the literature in 1989 [2]. Since this time, there have been over 50 papers published on the topic; an initial survey [5] and a recent update [4] provide an overview. We use the language and notation of [4].

Graph pebbling consists of moving pebbles around the vertices of a connected graph according to certain rules, and asking questions such as under what conditions a pebble may be moved to any vertex, to all vertices, etc. A configuration $C$ on the graph $G$ may be thought of as an initial placement of pebbles on the vertices of $G$, or equivalently as a function $C : V \to \mathbb{N}$, where for $v \in V$, $C(v)$ indicates the number of pebbles placed at $v$. The size of $C$, $|C|$, is $\sum_{v \in V} C(v)$. A pebbling step consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. If it is possible to begin with the configuration $C$ and, through a sequence of pebbling moves, place a pebble on vertex $v$, we say $v$ may be reached by $C$. In (regular) pebbling, a target vertex is selected, and the goal is to reach that target vertex. The pebbling number of $G$, denoted $\pi(G)$, is the minimum number $t$ such that it is possible to reach any target vertex when beginning with any configuration of size at least $t$.

Weighted cover pebbling number is a natural generalization of pebbling. We define a weight function $w : V(G) \to \mathbb{N}$. The weighted cover pebbling number, $\gamma_w(G)$, is the minimum number $k$ such that, beginning with any initial configuration of $k$ pebbles, there is a sequence of pebbling moves at the end of which each vertex $v$ of $G$ has at least $w(v)$ pebbles on it[1]. In this paper we will focus on weight functions that satisfy a specific goal property $P$, and we will denote the set of all weight functions (on a particular graph) having goal property $P$ as $\mathcal{W}_P$. For instance, if $\mathcal{W}_P$ consists of all weight functions $w$ satisfying the property that $w(v)$ is one for one vertex $v$ and zero for all other vertices, then the context is that of (regular) pebbling. Similarly, $t$-cover pebbling results from requiring a weight function that assigns $t$ to every vertex, while dominance pebbling results from weight functions that are one for a dominating set of vertices and zero for all others.

Given a goal property $P$, we say that the configuration $C$ is solvable for $P$ if, for every weight function $w \in \mathcal{W}_P$, it is possible, through a sequence of pebbling moves, to move from $C$ to a configuration $C'$ satisfying $C'(v) \geq w(v)$ for all $v \in V$. We define the $P$-pebbling number of $G$, $\gamma_P(G)$, to be the minimum number $k$ such that every configuration of size $k$ is solvable for $P$. In the case in which $P$ is the property that a weight function in $\mathcal{W}_P$ is one for exactly one vertex of $V$ and zero elsehwere, $\gamma_P(G) = \pi(G)$. Similarly, should property $P$ correspond to the (unique) weight function assigning one to each vertex of $G$, $\gamma_P(G)$ is the cover pebbling number, $\gamma(G)$.

The behavior of pebbling numbers under graph products has been the focus of much study. Throughout this paper we will assume that graphs $G$ and $H$ have vertex sets $V(G) = \{v_1, \ldots, v_g\}$ and $V(H) = \{x_1, \ldots, x_h\}$, respectively.

[1]This language and notation corresponds, for example, to [3], [4], and [1]; other authors refer to "demand" configurations, e.g. [7]

Definition 1.1. The product of $G$ and $H$, $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ (Cartesian product) and with edge set

$$\begin{aligned} E(G\Box H) =& \{((v_1,x_1),(v_2,x_2))\,|\, v_1 = v_2 \text{ and } (x_1,x_2) \in V(H)\} \\ & \cup \{((v_1,x_1),(v_2,x_2))\,|\, x_1 = x_2 \text{ and } (v_1,v_2) \in V(G)\}. \end{aligned}$$

The key open problem concerning pebbling numbers under graph products is due to Graham [2].

Graham's Conjecture. If $G\Box H$ is the product of $G$ and $H$, then $\pi(G\Box H) \leq \pi(G)\pi(H)$.

In this paper we discuss weight functions satisfying certain properties on graph products and determine their pebbling numbers. We introduce colors to "translate" a configuration on a graph product to a configuration on one of the factors.

Suppose $P_G$ and $P_H$ are goal properties on graphs $G$ and $H$ respectively. Then there is an induced product goal property $P_{G\Box H}$ on $G\Box H$ where the weight function $w^\star$ is said to satisfy $P_{G\Box H}$ if and only if there are weight functions $w_G$ and $w_H$ satisfying $P_G$ and $P_H$ respectively such that $w^\star(v_i, x_j) = w_G(v_i)w_H(x_j)$. Most natural goal properties on a product graph are in fact product goal properties. For example, regular pebbling on a product graph may be thought of as a product of goal properties requiring a weight of one for exactly one vertex of each of the factor graphs (and zero on all other vertices), and cover pebbling on product graphs results from factor goal properties requiring a weight of one for all vertices of the factor graphs. The product property on $G\Box H$ that we will explore in this paper is characterized by specifying that each restriction of the property to each copy of $G$ is identical.

We will prove the following:

Main Theorem. Let $P_G$ be any goal property on $G$ and let $P_H$ be the property on $H$ requiring $w(x) = 1$ for all vertices $x$ of $H$. The induced product goal property on $G\Box H$ is $P_{G\Box H}$. Then

$$\gamma_{P_{G\Box H}}(G\Box H) \leq \gamma_{P_G}(G) \cdot max_j\{\sum_{i=1}^{h} 2^{dist(x_j,x_i)}\} = \gamma_{P_G}(G)\gamma(H).$$

In other words, to find an upper bound for $\gamma_{P_{G\Box H}}(G\Box H)$ it is enough to just consider initial configurations that place all pebbles on a single copy of $G$.

Recall that $V(G\Box H) = \{(v_i, x_j) | i = 1, \ldots, g; j = 1, \ldots, h\}$. Let $G_j = G \times \{x_j\}$ be the subgraph with vertex set $\{(v_i, x_j) | i = 1, 2, \ldots g\}$ and edges induced from $G\Box H$; note that $G_j$ is isomorphic to $G$. We will associate to each configuration on $G\Box H$ a certain configuration of colored pebbles on $H$. We will call a configuration $t$-colored if each pebble in the configuration has been assigned one of $t$ possible colors. A color-respecting pebbling move for a colored configuration consists of taking two pebbles of the same color from some vertex and placing one of these pebbles on an adjacent vertex. When considering colored configurations we allow only color-respecting steps.

To each configuration $C$ on $G\Box H$ we associate a color configuration $\tilde{C}$ on $H$ in the following way: use colors $c_1, c_2, \ldots, c_g$ and assign color $c_i$ to each pebble that $C$ places on vertices $(v_i, x_j)$ (for any $j$). Collapse $G\Box H$ to a single copy of $H$, which we call $\tilde{H}$ for clarity, by identifying $G_i$ in $G\Box H$ with vertex $X_i$ in $\tilde{H}$. We place all pebbles from $G_i$ on $X_i$.

Lemma 2.1. Let

- $P_G$ be any goal property on $G$,
- $P_{\tilde{H}}$ be the goal property on $\tilde{H}$ requiring the unique weight function on $V(\tilde{H})$ satisfying $w(X) = \gamma_{P_G}(G)$ for all $X \in V(\tilde{H})$, and
- $P_{G\Box H}$ be the product goal property induced by $P_G$ and $P_H$.

Let $C$ be a configuration on $G\Box H$ and $\tilde{C}$ be its associated $g$-colored configuration on $\tilde{H}$. If $\tilde{C}$ is solvable for $P_{\tilde{H}}$ on $\tilde{H}$, then $C$ is solvable for $P_{G\Box H}$ on $G\Box H$.

Proof. By hypothesis, there is a sequence of color-respecting pebbling steps beginning with $\tilde{C}$ at the end of which there are $\gamma_{P_G}(G)$ pebbles on each vertex of $\tilde{H}$. Because the steps respect color we could have performed them in $G\Box H$: taking two pebbles of color $c_i$ from $X_j$, discarding one and placing the other one on $X_k$ in $\tilde{H}$ corresponds to taking two pebbles from vertex $(v_i, x_j)$ and placing one of them on $(v_i, x_k)$ in $G\Box H$. So there is a sequence of steps on $G\Box H$, consisting only of moving pebbles from one copy of $G$ to another, at the end of which each $G_i$ has $\gamma_{P_G}(G)$ pebbles. These $\gamma_{P_G}(G)$ pebbles suffice to satisfy property $P_G$ on each copy of $G$ in $G\Box H$, so $C$ is solvable for $P_{G\Box H}$. □

Within the usual concept of pebbling, we may move approximately half of the pebbles on one vertex to an adjacent vertex. An analogous statement holds for colored configurations.

Lemma 2.2. Suppose a $t$-colored configuration places $m \geq t$ pebbles on a vertex. Given any integer $n \leq m-t$, color-respecting steps may be used to move at least $\lceil n/2 \rceil$ pebbles to an adjacent vertex.

Proof. Consider the set $T$ of all pebbles on the given vertex $v$. If a color has an odd number of representatives in $T$, remove from consideration one pebble of that color. As there are only $t$ colors, at most $t$ pebbles are removed. Let $S$ be the subset of all remaining pebbles; note $|S| \geq m-t$ and $|S|$ is even. As $n \leq |S|$, we may then select pairs of same-colored pebbles to create a subset $N$ of $S$, with $|N| = n$ when $n$ is even and $|N| = n+1$ when $n$ is odd. Half of the pebbles in $N$ may be moved to a vertex adjacent to $v$ – using color-respecting steps – thus achieving our goal of moving $\lceil n/2 \rceil$ pebbles.

□

Main Theorem. Let $P_G$ be any goal property on $G$ and let $P_H$ be the property on $H$ requiring $w(x) = 1$ for all vertices $x$ of $H$. Let $P_{G\square H}$ be the induced product goal property on $G\square H$. Then

$$\gamma_{P_{G\square H}}(G\square H) \leq \gamma_{P_G}(G) \cdot max_{x_j}\{\sum_{i=1}^{h} 2^{dist(x_j,x_i)}\} = \gamma_{P_G}(G)\gamma(H).$$

Proof. Define $P'_H$ as the goal property on $H$ requiring the unique weight function on $V(H)$ satisfying $w(x) = \gamma_{P_G}(G)$ for all $x \in V(H)$. Let $P_G$ be any goal property on $G$. Recall $|V(G)| = g$. By Lemma 2.1, establishing $\gamma_{P_{G\square H}}(G\square H) \leq \gamma_{P_G}(G)\gamma(H)$ may be achieved by showing that every $g$-colored configuration on $H$ of size at least $\gamma_{P_G}(G)\gamma(H)$ is solvable for $P'_H$. In other words, it suffices to show that from any $g$-colored configuration of size $\gamma_{P_G}(G)\gamma(H)$ on $H$, there is a sequence of pebbling steps at the end of which every vertex of $H$ has at least $\gamma_{P_G}(G)$ pebbles. Note that $g \leq \gamma_{P_G}(G)$ for all goal properties of positive size. Let $C_H$ be a $g$-colored configuration on $H$ of size $\gamma_{P_G}(G)\gamma(H)$, and assume that $C_H$ is not solvable for $P'_H$. We model the proof of this result on the proof of the main theorem in [6] but we need to ascertain that the proof indeed carries over when colored configurations are considered.

Sjöstrand defines the value of a pebble so as to make the total value of a configuration invariant under pebbling moves [6]: "The value of a pebble... is the number of pebbles that have gone into it." In other words, instead of thinking of a pebbling step as moving one pebble and discarding one, we will think of a pebbling step as picking up two pebbles, joining them together by adding their values, and then placing the new, higher-valued pebble on an adjacent vertex. Thus the total value of the pebbles on the graph remains the same throughout any sequence of pebbling moves. As we only allow color-preserving steps, the total value of pebbles of any given color is also preserved. Initially, i.e., prior to any pebbles being moved, all pebbles have a value of one.

Call a vertex in $H$ fat if there are more than $\gamma_{P_G}(G)$ pebbles on it (regardless of their color), thin if there are fewer than $\gamma_{P_G}(G)$ pebbles on it, and perfect if there are exactly $\gamma_{P_G}(G)$ pebbles on it. Let $(f, t)$ be a pair of a fat and a thin vertex that are a minimum distance from each other amongst all such pairs, and let $p_1, p_2, ..., p_n$ be the shortest path from $f$ to $t$. Note that each vertex $p_i$ has exactly $\gamma_{P_G}(G)$ pebbles on it. By Lemma 2.2, as $g \leq \gamma_{P_G}(G)$, we can move one pebble from $f$ to $p_1$. Now $p_1$ is fat, so we can move one pebble from $p_1$ to $p_2$ etc., until finally $t$ has an additional pebble and each of $f, p_1, .., p_n$ has one fewer pebble. Then pick a new pair of a fat and a thin vertex at minimum distance, and repeat the precedure until – as is guaranteed by the unsolvability of the initial configuration – no fat vertices are left. This procedure does not result in any new fat vertices, so each time a pair of one thin and one fat vertex are chosen, all pebbles on the fat vertex have initial value of 1. So if a pebble is moved in a sequence of steps from a fat vertex $f$ to a thin vertex $t$, the final value of the pebble is $2^{dist(f,t)}$.

Let $x_j$ be the fat vertex that survived the above procedure the longest. Then, if at the end of the algorithm a pebble $p$ is on vertex $x_i$, this pebble must have come from a fat vertex whose distance from $x_i$ is less than or equal to $dist(x_i, x_j)$, so $value(p) \leq 2^{dist(x_j,x_i)}$. However, once done making moves, there are at most $\gamma_{P_G}(G)$ pebbles on each vertex of $H$, and at least one vertex has strictly fewer than $\gamma_{P_G}(G)$ pebbles (as $C_H$ was assumed to be unsolvable). Therefore the initial number of pebbles in $C_{\bar{H}}$ must have been fewer than $\gamma_{P_G}(G) \cdot max_j\{\sum_{i=1}^{h} 2^{dist(x_j,x_i)}\}$. As $max_j\{\sum_{i=1}^{h} 2^{dist(x_j,x_i)}\} = \gamma(H)$, this contradicts our hypothesis.

□

Finally we note that the result above is the best possible. If $P_G$ and $P_H$ are both the cover pebbling property (i.e., the goal is to place one pebble on each of the vertices) then it follows from [6] that $\gamma_{P_{G\Box H}}(G\Box H) = \gamma(G)\gamma(H)$.

On the other hand, consider $G = C_3$, the cycle on 3 vertices, and let $P_G$ be the property that we can place a pebble on any one vertex (i.e., $\gamma_P(C_3) = \pi(C_3) = 3$). Let $H = P_2$, the path on two vertices, and $P_H$ be the cover pebbling property again, so $\gamma_{P_H}(H) = 3$. We have $\gamma_{P_G}(G)\gamma_{P_H}(H) = 9$, but it is easy to check that every configuration of 8 pebbles on $\gamma_{P_{G\Box H}}$ is solvable for $P_{G\Box H}$, i.e., $\gamma_{P_{G\Box H}}(G\Box H) < \gamma_{P_G}(G)\gamma_{P_H}(H)$.

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