Minimum Metric Dimension of Silicate Networks

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Abstract

The silicates are the largest, the most interesting and the most complicated class of minerals by far. The basic chemical unit of silicates is the (SiO_4) tetrahedron. A silicate sheet is a ring of tetrahedrons which are linked by shared oxygen nodes to other rings in a two dimensional plane that produces a sheet-like structure. We consider the silicate sheet as a fixed interconnection parallel architecture and call it a silicate network. We solve the Minimum Metric Dimension problem which is NP-complete for general graphs.

Keywords: silicate networks, topological and structural properties of interconnection networks, mesh-like architectures, NP-complete, minimum metric dimension.

1 Introduction

Multiprocessor interconnection networks are often required to connect thousands of homogeneously replicated processor-memory pairs, each of which is called a *processing node.* Instead of using a shared memory, all synchronization and communication between processing nodes for program execution is often done via message passing. Design and use of multiprocessor interconnection networks have recently drawn considerable attention due to the availability of inexpensive, powerful microprocessors and memory chips [17]. The homogeneity of processing nodes and the interconnection network is very important because it allows for cost/performance benefits from the inexpensive replication of multiprocessor components.

Silicates are obtained by fusing metal oxides or metal carbonates with sand. Essentially all the silicates contain SiO_4 tetrahedra. In chemistry,

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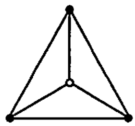


Figure 1: SiO_4 tetrahedra where the corner vertices represent oxygen ions and the center vertex the silicon ion

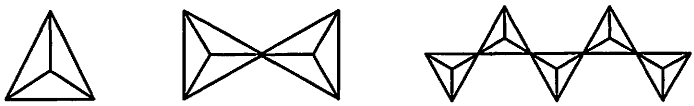


Figure 2: Different kinds of silicates (a) Orthosilicates (b) Pyrosilicates (c) Chain Silicates

the corner vertices of SiO_4 tetrahedran represent oxygen ions and the center vertex represents the silicon ion. In graph theory, we call the corner vertices as *oxygen nodes* and the center vertex as *silicon node*. See Figure 1.

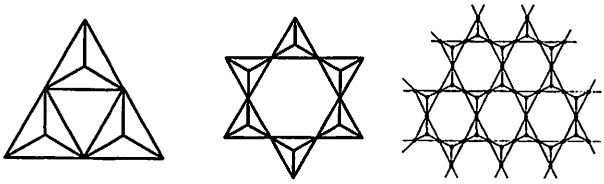
The minerals are obtained by successively fusing oxygen nodes of two tetrahedra of different silicates. The different types of silicate structure arise from the ways in which these tetrahedra are arranged: they may exist as separate unlinked entities, as linked finite arrays, as 1-dimensional chains, as 2-dimensional sheets or as 3-dimensional frameworks. Some of the structural units found in silicates are shown in Figures 2 and 3. They are termed orthosilicates, pyrosilicates, chain silicates, cyclic silicates and sheet silicates.

Simple orthosilicates contain discrete SiO_4 units. When two SiO_4 tetrahedra share an oxygen node, pyrosilicates are obtained. While tetrahedra are arranged linearly, chain silicates are obtained. See Figure 2. Cyclic silicates and sheet silicates are shown in Figure 3.

In this paper, we solve the minimum metric dimension problem for silicate networks. This problem is an NP-Complete problem for general graphs.

2 Properties of Silicate Networks

A silicate network can be constructed in different ways [13]. We describe the construction of a silicate network from a honeycomb network. A hon-



(a) (b) (c)

Figure 3: Cyclic and Sheet Silicates

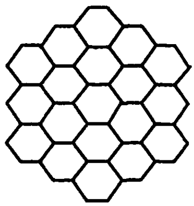


Figure 4: A honeycomb network $HC(3)$

eycomb network can be built from a hexagon in various ways [15, 16]. The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained by adding six hexagons to the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a layer of hexagons around the boundary of $HC(n-1)$. For instance, the graph in Figure 4 is $HC(3)$. The parameter n of $HC(n)$ is called the dimension of $HC(n)$.

Consider a honeycomb network $HC(n)$ of dimension n. Place silicon ions on all the vertices of $HC(n)$. Subdivide each edge of $HC(n)$ once. Place oxygen ions on the new vertices. Introduce $6n$ new pendant edges one each at the 2-degree silicon ions of $HC(n)$ and place oxygen ions at the pendent vertices. See Figure $5(a)$. With every silicon ion associate the three adjacent oxygen ions and form a tetrahedron as in Figure $5(b)$. The resulting network is a silicate network of dimension n, denoted $SL(n)$. The diameter of $SL(n)$ is $4n$. The graph in Figure $5(b)$ is a silicate network of dimension two.

The 3-degree oxygen nodes of silicates are called *boundary nodes.* In Figure $5(b)$, $c_1, c_2, ..., c_{12}$ are boundary nodes $SL(2)$.

Theorem 1 *[13] The number of nodes in $SL(n)$ is $15n^2 + 3n$ and the number of edges of $SL(n)$ is $36n^2$.* □

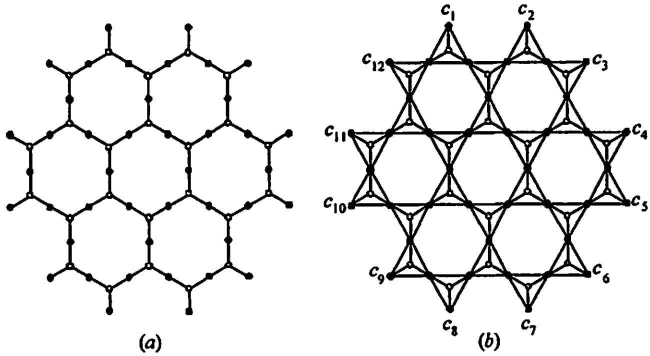


Figure 5: Silicate network construction and boundary nodes

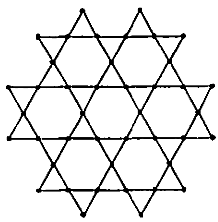


Figure 6: An Oxide Network $OX(2)$

3 Addressing Scheme in Silicate Networks

When we delete all the silicon nodes from a silicate network we obtain a new network which we shall call as an *Oxide Network*. See Figure 6. An n-dimensional oxide network is denoted by $OX(n)$. Even though $HC(n)$ and $OX(n)$ are sub graphs of $SL(n)$, $OX(n)$ plays more important role in studying the properties of $SL(n)$. We note that the diameter of silicate network $SL(n)$ is equal to the diameter of the oxide network $OX(n)$ [13].

A coordinate system is proposed that assigns an *id* to each node of oxide network. This coordinate system is then extended [13] to silicate network. The basic idea is due to Stojmenovic [16] and to Nocetti et al. [9] who proposed a system for a honeycomb network and a hexagonal network respectively. Three axes, α, β and γ parallel to three edge directions and at mutual angle of 120 degrees between any two of them are introduced. The three coordinate axes are $\alpha = 0$, $\beta = 0$, and $\gamma = 0$ respectively. We call lines parallel to the coordinate axes as α-lines, β-lines and γ-lines. Here $\alpha = h$ and $\alpha = -k$ are α-lines on either side of α-axis. A node of $OX(n)$ is

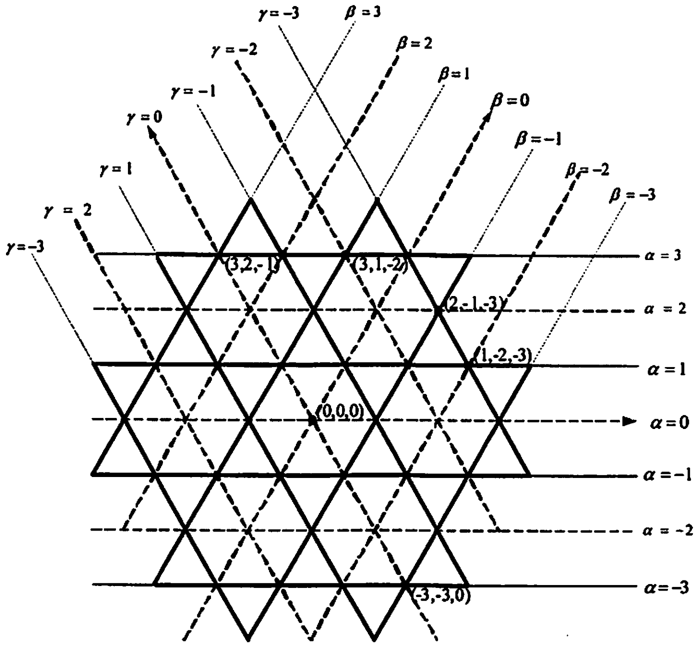


Figure 7: Coordinate System in Oxide Networks

assigned a triple (a, b, c) when the node is the intersection of lines $\alpha = a$, $\beta = b$, and $\gamma = c$. Each silicon node is at the centroid of three oxygen nodes of a tetrahedral SiO_4. One can assign *ids* to silicon nodes by applying the formula of centroid of an equilateral triangle. See Figure 7.

4 Equilateral Triangle Property of Silicate Sheets and Networks

Three vertices u, v, w of a graph $G(V, E)$ are said to form an *equilateral triangle* if $d(u, v) = d(v, w) = d(w, u)$ where $d(x, y)$ denotes the distance between x and y. There is an interesting equilateral triangular property of silicate networks.

Theorem 2 *Three vertices $A(x_1, x_2, x_3)$, $B(y_1, y_2, y_3)$ and $C(z_1, z_2, z_3)$ of an infinite silicate sheet form an equilateral triangle if $x_1 = y_1$, $y_2 = z_2$ and $z_3 = x_3$.*

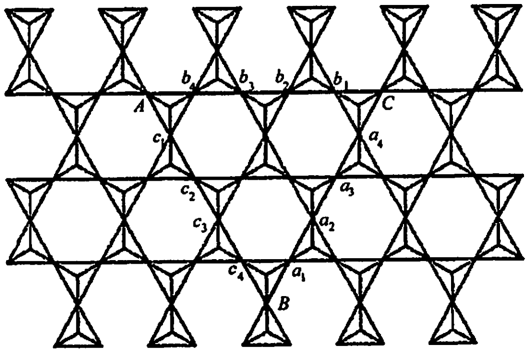


Figure 8: Equilateral triangle property

The proof is similar to the corresponding result for silicate networks [13].

Consider a triangle ABC formed by some α-line, β-line and γ-line. By the above theorem, ΔABC is equilateral. Let $a_1, a_2, ..., a_r$ be the nodes on the β-line between B and C. Let $b_1, b_2, ..., b_r$ be the nodes on the α-line between C and A. Let $c_1, c_2, ..., c_r$ be the nodes on the γ-line between A and B. See Figure 8. We know that $d(A, B) = d(A, C)$. The interesting observation is that $d(A, B) = d(A, a_i) = d(A, C)$ for $i = 1, 2, ..., r$.

5 Minimum Metric Dimension of Silicate Networks

A *metric basis* for a graph $G(V, E)$ is a subset of vertices $W \subseteq V$ such that for each pair of vertices u and v of $V \backslash W$, there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. A *minimum metric basis* is a metric basis of minimum cardinality. The cardinality of a minimum metric basis of G is called *minimum metric dimension* and is denoted by $\beta(G)$; the members of a minimum metric basis are called *landmarks*. A *minimum metric dimension* (MMD) *problem* is to find a minimum metric basis.

The problem of finding the metric dimension of a graph was first studied by Harary and Melter [5]. They gave a characterization for the metric dimension of trees. Melter and Tomescu [8] studied the metric dimension problem for grid graphs. Khuller et al. [7] have generalized a result of Melter and Tomescu and proved that the metric dimension of d-dimensional grids is d.

The problem of finding minimum metric dimension is NP-complete for general graphs [4]. Manuel et al. [11] have proved that this problem remains NP-complete for bipartite graphs.

The concept of metric basis and minimum metric basis has appeared in the literature under a different name as early as 1975. Slater in [14] had called metric basis and minimum metric basis as *locating sets* and *reference sets* respectively. Slater called the cardinality of a reference set as the *location number* of G. He described the usefulness of these ideas when working with sonar and loran stations. Chartrand et al. [3] have called a metric basis and a minimum metric basis as a *resolving set* and minimum resolving set. We adopt the terminology of Harary and Melter.

The minimum metric dimension problem has been studied for trees, multi-dimensional grids [7], Petersen graphs [1], Torus Networks [10], Benes Networks [11], Honeycomb Networks [12] and Enhanced Hypercubes [2]. Although the minimum metric dimension problem is a distance based problem the strategy adopted is different in each case. This paper uses yet another strategy to solve the problem for silicate networks. The following result provides a lower bound for the minimum metric dimension of silicate networks.

Theorem 3 $\beta(SL(n)) \geq 6n$.

Proof. Let u be a boundary node of $SL(n)$. Let v be the silicon node adjacent to u. Then for any node w in $SL(n)$, we have $d(u,w) = d(v,w)$.

Thus any metric basis will contain either u or v. There are $6n$ boundary nodes in $SL(n)$. Hence any metric basis of $SL(n)$ should contain at least $6n$ nodes of $SL(n)$. □

Theorem 4 $\beta(SL(n)) = 6n$.

Proof. We claim that the set of boundary nodes is a metric basis. In view of symmetry of the network we begin our discussion with respect to α-lines. A line $\alpha = k$ is said to be odd or even according as k is odd or even.

The dotted lines in Figure 9 are even axes and black lines are odd axes. For any oxygen node (a,b,c), $\alpha = a$, $\beta = b$, and $\gamma = c$ are the axis lines passing through (a,b,c). Among these three axis lines, two are odd and one is even. For the oxygen node X in Figure 9, α and β lines are odd axis lines and γ line is an even axis line. Similarly for node N, β and γ are odd axis lines and α is an even axis line. Further if we call an edge joining two oxygen nodes as an oxide edge then each such oxide edge is on some odd axis line.

Now consider any two oxygen nodes A and B. Let $AXBY$ be the parallelogram with nodes A and B as corner vertices. Let $P(A,B)$ be a shortest path between A and B. Since every shortest path between A and B lies inside the parallelogram $AXBY$, the path $P(A,B)$ also lies

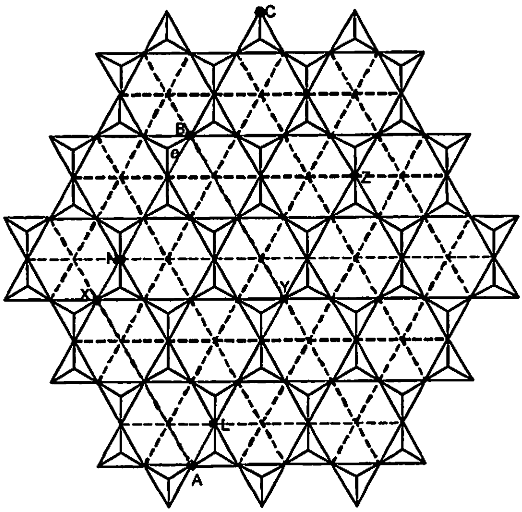


Figure 9: Proof cases in Theorem 4

inside parallelogram $AXBY$. Let $P(A,B)$ be an (A,L)-path followed by an (L,N)-path followed by an (N,B)-path. Let e be the last edge of $P(A,B)$ which is incident on node B. This edge e lies on one of the three odd axis lines. In Figure 9, e lies on an odd β-line. The node B divides this β-line into two segments one of which does not contain e. Let C denote the boundary node on this segment. Now consider the parallelogram $AXCZ$. Define a path $P(A,C)$ as (A,L)-path followed by an (L,N)-path followed by an (N,C)-path which is a shortest path between A and C passing through B. Thus $d(A,C) \neq d(B,C)$. Using the same argument, other cases can be proved. □

6 Conclusion

In this paper we have considered a new interconnection network motivated by the molecular structure of certain chemical compounds. The different forms of silicates available in nature led to the introduction of the silicate networks. The minimum metric dimension problem is investigated. This paper is an eye opener for researchers in the sense that different networks can be derived using the ores and compounds available in nature.

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