GENUS DISTRIBUTIONS FOR DOUBLE PEARL-LADDER GRAPHS

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ABSTRACT. On the basis of the joint tree model initiated and comprehensively described by Liu, we obtain the genus distributions of double pearl ladder graphs (a type of new 3-regular graphs) in orientable surfaces.

Key words: double pearl ladder graph, joint tree, genus distribution

1. Introduction

Determining the genus distribution of a graph is NP-complete [11], to compute the (non)orientable embedding genus distributions of some graphs has been an important research field in topological graph theory. About the embedding genus distributions of graphs and digraphs, there are many results in $[1-7, 10, 12, 13]$. In this paper, we extend the surface classification method [13] and obtain the genus distributions of double pearl ladder graphs in orientable surfaces. Throughout this paper, graphs considered are connected and their embeddings are orientable. Some concepts can be found in [4], [8], [12].

A surface considered is a compact 2-dimensional manifold without boundary. It can be represented as a polygon of even edges on the plane whose edges are labeled by letters, each letter occurring exactly twice, and each letter being given a indices $(\pm)$. The surface is obtained by identifying the pair of edges corresponding to a letter. Furthermore, an orientable surface S can be represented by a cyclic sequence of letters in parentheses, where each letter appears exactly twice and the two occurrences of each letter

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have distinct indices $+$ (always omitted) and $-$. This is called an algebraic representation of S (details to see [8, 9]).

Let $P = (abc \ldots st)$ be a cyclic sequence of letters, the reversion of P is denoted $\widehat{P} = (ts \ldots cba)$, the conversion of P is denoted $P' = (a^- b^- c^- \ldots s^- t^-)$. $A = abc$ is a section of successive letters in P and is a linear sequence.

Let A, B, C, D be sections of successive letters in an algebraic representation S and $\emptyset$ be the empty, the following each operation is called an elementary transformation:

Op.1 $S = (AB) \Leftrightarrow S = (Aaa^- B)$, *where* $AB \neq \emptyset$, $a \notin AB$;
Op.2 $S = (AabBb^- a^-) \Leftrightarrow S = (AcBc^-)$;
Op.3 $S = (AaBCa^- D) \Leftrightarrow S = (BaADa^- C)\}$.

If an algebraic representation S_1 can be obtained by some elementary transformations from another algebraic representation S_2, then they are called elementary equivalence, denoted by $S_1 \sim S_2$. The relation " $\sim$ " is indeed an equivalence relation and the genera of surfaces are invariant under the elementary transformations.

In what follows, the parentheses in each algebraic representation are always omitted for the sake of brevity.

On the basics of these operations, the following lemma can be done.

Lemma 1.1[8,9] *(1)* $AxByCx^- Dy^- E \sim ADCBExyx^- y^-$;
(2) $AxBCx^- D \sim AxCBx^- D \sim ADxBCx^-$;
(3) $AxBx^- yCy^- zDz^- \sim xBx^- AyCy^- zDz^- \sim xBx^- yCy^- AzDz^- \sim xAx^- ByCy^- zDz^- \sim CxBx^- yAy^- zDz^- \sim DxBx^- yCy^- zAz^-$
where A, B, C, D *and* E *be linear sequences,* $x, y, x^-, y^- \notin ABCDE$.

By using Lemma 1.1, the algebraic representation of each orientable surface is equivalent to one and only one of the following canonical forms:

$$O_i = \begin{cases} (a_0 a_0^-), & \text{if the surface is sphere;} \\ (\Pi_{k=1}^{i} a_k b_k a_k^- b_k^-), & \text{if the genus of a surface is } i. \end{cases}$$

i.e., O_i is the canonical representation of an orientable surface with genus i. The genus of a surface is called the genus of its algebraic representation which is denoted by $o(O_i) = i$.

Lemma 1.2 [8,9] *Let* S_1 *and* S_2 *be the algebraic representations of two orientable surfaces,* $a, b, a^-, b^- \notin S_2$, *if* $S_1 \sim S_2 aba^- b^-$, *then*

$$o(S_1) = o(S_2) + 1,$$

where $o(S)$ represents the genus of S.

A rotation σ_v at a vertex v is a cyclic permutation of edges incident with v, $\sigma_G = \prod_{v\in V(G)} \sigma_v$ is called a rotation system of G. An embedding (cellular embedding) of a graph G into a surface S is a homeomorphism i: $G \to S$, such that each component of $S - i(G)$ is homeomorphic to an open disc. Two embeddings f: $G \to S$ and g: $G \to S$ are the same if there is a homeomorphism h: $S \to S$ such that $f = gh$. The embedding is called orientable if S is orientable. An orientable embedding is determined by a rotation system.

Let G be a graph and T be a tree of G. For each nontree edge a, a is split into two semiedges a^+ (+ always omitted) and a^-. The graph obtained by splitting each of nontree edges with a rotation system σ_G is a joint tree J. A joint tree of G is determined by a spanning tree of G and a rotation system σ_G . For a joint tree J of G, a cyclic squence of all letters of semiedges along the clockwise or anticlockwise rotation is an associate surface. An associate surface is determined only by a joint tree.

The number of distinct embeddings of G on a surface with genus p is independent of the choice of a tree T on G.

Therefore, there is a 1-to-1 correspondence between the set of associate surfaces and the set of embeddings of a graph[8]. Let ρ_v denote the valence of vertex v which is the number of edges incident with v. The number of rotations systems of G, as well as the number of embeddings of G, is $\prod_{v\in V(G)} (\rho_v - 1)!$.

For a graph G, let $g_i(G)$ be the number of the distinct embeddings of G into the orientable surface with genus i ($i \geq 0$). The embedding genus distribution of G is: $g_0(G), g_1(G), g_2(G), \cdots$, then the genus polynomial of G is: $f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i$.

2 Main results

Let k_0h_0 and k_1h_1 be two parallel edges, adding n ($n \geq 1$) vertices $u_1, u_2, \ldots, u_n$ on k_0h_0 in sequence, n ($n \geq 1$) vertices $v_1, v_2, \ldots, v_n$ on k_1h_1 in sequence and adding edges h_0h_1, k_0k_1 and $u_tv_t (1 \leq t \leq n)$ such that u_tv_t and $u_{t+1}v_{t+1}$ ($n + 1 \equiv 1(mod n)$) are parallel edges, splitting $u_tv_t (1 \leq t \leq n)$ into $u_t\overline{u}_t$, $\overline{u}_t\overline{v}_t$ and $\overline{v}_tv_t$, deleting $\overline{u}_t\overline{v}_t$, then adding two

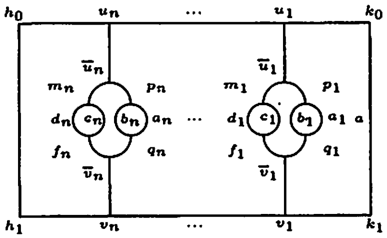

Fig.1: G_n

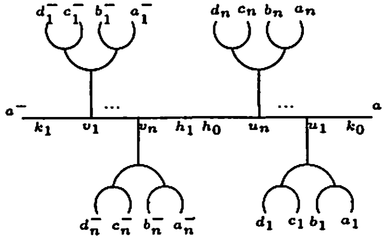

Fig.2: a joint tree J_n

parallel edges $\overline{u}_t\overline{v}_t$; thirdly, splitting one edge of the 2 mulit-edges $\overline{u}_t\overline{v}_t$ into three parts $\overline{u}_tm_t$, m_tf_t, $f_t\overline{v}_t$, splitting the other edge of the 2 mulit-edges $\overline{u}_t\overline{v}_t$ into three parts $\overline{u}_tp_t$, p_tq_t, $q_t\overline{v}_t$, then deleting m_tf_t and p_tq_t, adding two circles between two vertices m_t and f_t, two vertices p_t and q_t respectively $(1 \leq t \leq n)$. The graph attained is called a doule pearl-ladder graph, denoted G_n (see Fig.1).

In G_n, we select edges a, a_1, b_1,c_1, d_1 $\ldots$, a_n, b_n, c_n, d_n as nontree edges of G_n, then $G_n - \{a, a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n\}$ is a tree of the graph G_n, denoted T_n, its a joint tree is denoted J_n , at each vertex, the joint tree J_n has a anticlockwise rotation (see Fig.2).

Let a, b, c be distinct letters, by lemma1.1, the double pearl ladder surface sets can be divided into 24 types, they are S_k^n $(1 \leq k \leq 24)$:

$S_1^n = \{X_1^nX_2^nX_3^nX_4^n\}$, $S_2^n = \{X_1^nX_2^nX_4^nX_3^n\}$, $S_3^n = \{X_1^nX_3^nX_2^nX_4^n\}$,
$S_4^n = \{X_1^nX_3^nX_4^nX_2^n\}$, $S_5^n = \{X_1^nX_4^nX_2^nX_3^n\}$, $S_6^n = \{X_1^nX_4^nX_3^nX_2^n\}$,
$S_7^n = \{aX_1^nX_2^na^-X_3^nX_4^n\}$, $S_8^n = \{aX_1^nX_3^na^-X_2^nX_4^n\}$,
$S_9^n = \{aX_1^nX_4^na^-X_2^nX_3^n\}$, $S_{10}^n = \{aX_1^na^-X_2^nX_3^nX_4^n\}$,
$S_{11}^n = \{aX_1^na^-X_2^nX_4^nX_3^n\}$, $S_{12}^n = \{aX_2^na^-X_1^nX_3^nX_4^n\}$,
$S_{13}^n = \{aX_2^na^-X_1^nX_4^nX_3^n\}$, $S_{14}^n = \{aX_3^na^-X_1^nX_2^nX_4^n\}$,
$S_{15}^n = \{aX_3^na^-X_1^nX_4^nX_2^n\}$, $S_{16}^n = \{aX_4^na^-X_1^nX_2^nX_3^n\}$,
$S_{17}^n = \{aX_4^na^-X_1^nX_3^nX_2^n\}$, $S_{18}^n = \{aX_1^na^-bX_2^nb^-X_3^nX_4^n\}$,
$S_{19}^n = \{aX_1^na^-bX_3^nb^-X_2^nX_4^n\}$, $S_{20}^n = \{aX_1^na^-bX_4^nb^-X_2^nX_3^n\}$,
$S_{21}^n = \{aX_2^na^-bX_3^nb^-X_1^nX_4^n\}$, $S_{22}^n = \{aX_2^na^-bX_4^nb^-X_1^nX_3^n\}$,
$S_{23}^n = \{aX_3^na^-bX_4^nb^-X_1^nX_2^n\}$, $S_{24}^n = \{X_1^naX_2^na^-bX_3^nb^-cX_4^nc^-\}$,

where,

$X_1^n = w_{k_1}w_{k_2}w_{k_3}\cdots w_{k_r}$, $X_2^n = w_{k_{r+1}}w_{k_{r+2}}w_{k_{r+3}}\cdots w_{k_n}$,
$X_3^n = w_{t_1}^-w_{t_2}^-w_{t_3}^-\cdots w_{t_s}^-$, $X_4^n = w_{t_{s+1}}^-w_{t_{s+2}}^-w_{t_{s+3}}^-\cdots w_{t_n}^-$,
$n \geq k_1 > k_2 > k_3 > \ldots > k_r \geq 1, 1 \leq k_{r+1} < k_{r+2} < k_{r+3} < \ldots < k_n \leq n, n \geq t_1 > t_2 > t_3 > \ldots > t_s \geq 1, 1 \leq t_{s+1} < t_{s+2} < t_{s+3} < \ldots < t_n \leq n$ and $1 \leq r, s, p, q \leq n, k_p \neq k_q, t_p \neq t_q$, for $p \neq q$. For the graph G_n (Fig.1) and the joint tree J_n (Fig.2), each w_i has eight cases: $a_ib_ic_id_i$,

$a_ib_id_ic_i$, $b_ia_ic_id_i$, $b_ia_id_ic_i$, $c_id_ia_ib_i$, $c_id_ib_ia_i$, $d_ic_ia_ib_i$, $d_ic_ib_ia_i$, each w_i^- has eight cases: $a_i^-b_i^-c_i^-d_i^-$, $a_i^-b_i^-d_i^-c_i^-$, $b_i^-a_i^-c_i^-d_i^-$, $b_i^-a_i^-d_i^-c_i^-$, $c_i^-d_i^-a_i^-b_i^-$, $c_i^-d_i^-b_i^-a_i^-$, $d_i^-c_i^-a_i^-b_i^-$, $d_i^-c_i^-b_i^-a_i^-$ $(i = 1, 2, \ldots, n)$.

By the definitions of X_j^n $(j = 1, 2, 3, 4)$, we can obtain relations between X_j^n and X_j^{n-1} $(j = 1, 2, 3, 4)$ as follows:

Table 1

I	II
$X_1^n = a_nb_nc_nd_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = a_n^-b_n^-c_n^-d_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = a_nb_nd_nc_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = a_n^-b_n^-d_n^-c_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = b_na_nc_nd_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = b_n^-a_n^-c_n^-d_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = b_na_nd_nc_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = b_n^-a_n^-d_n^-c_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = c_nd_na_nb_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = c_n^-d_n^-a_n^-b_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = c_nd_nb_na_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = c_n^-d_n^-b_n^-a_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = d_nc_na_nb_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = d_n^-c_n^-a_n^-b_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = d_nc_nb_na_nX_1^{n-1}$, $X_2^n = X_2^{n-1}$	$X_3^n = d_n^-c_n^-b_n^-a_n^-X_3^{n-1}$, $X_4^n = X_4^{n-1}$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}a_nb_nc_nd_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}a_n^-b_n^-c_n^-d_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}a_nb_nd_nc_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}a_n^-b_n^-d_n^-c_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}b_na_nc_nd_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}b_n^-a_n^-c_n^-d_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}b_na_nd_nc_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}b_n^-a_n^-d_n^-c_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}c_nd_na_nb_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}c_n^-d_n^-a_n^-b_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}c_nd_nb_na_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}c_n^-d_n^-b_n^-a_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}d_nc_na_nb_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}d_n^-c_n^-a_n^-b_n^-$
$X_1^n = X_1^{n-1}$, $X_2^n = X_2^{n-1}d_nc_nb_na_n$	$X_3^n = X_3^{n-1}$, $X_4^n = X_4^{n-1}d_n^-c_n^-b_n^-a_n^-$

Theorem 2.1 *Let $g_{i_j}(n)$ be the number of surfaces in S_j^n with genus i for $j = 1, 2, 3, \ldots, 24$ and $n \geq 1$. Suppose $f_{S_j^0}(x) = 1$. Denote $g_{i_j}(n-1) = r_{i_j}$, then for $n \geq 1$,*

$$g_{i_j}(n) = \begin{cases} 16[r_{i_1} + r_{i_7} + 6r_{(i-1)_1} \\ +4r_{(i-1)_7} + 4r_{(i-2)_1}], 0 \leq i \leq 2n, j = 1; \\ 16[r_{i_{14}} + r_{i_{16}} + 2r_{(i-1)_2} + 4r_{(i-1)_{14}} \\ +4r_{(i-1)_{16}} + 4r_{(i-2)_2}], 0 \leq i \leq 2n, j = 2; \\ 8[r_{i_3} + r_{i_9} + r_{i_{11}} + r_{i_{17}} + 8r_{(i-1)_3} + 4r_{(i-1)_9} + 4r_{(i-1)_{11}} \\ +4r_{(i-1)_{17}} + 8r_{(i-2)_3}], 0 \leq i \leq 2n, j = 3; \\ 16[r_{i_{10}} + r_{i_{12}} + 2r_{(i-1)_4} + 4r_{(i-1)_{10}} \\ +4r_{(i-1)_{12}} + 4r_{(i-2)_4}], 0 \leq i \leq 2n, j = 4; \\ 8[r_{i_5} + r_{i_9} + r_{i_{13}} + r_{i_{15}} + 8r_{(i-1)_5} + 4r_{(i-1)_9} \\ +4r_{(i-1)_{13}} + 4r_{(i-1)_{15}} + 8r_{(i-2)_5}], 0 \leq i \leq 2n, j = 5; \end{cases}$$

$$
g_{i_j}(n) = \begin{cases}
32[r_{i_9} + r_{(i-1)_6} + 4r_{(i-1)_9} + 2r_{(i-2)_6}], 0 \le i \le 2n, j = 6; \\
32[r_{(i-1)_1} + r_{(i-1)_7} + 4r_{(i-2)_1} + 2r_{(i-2)_7}], 0 \le i \le 2n, j = 7; \\
8[r_{i_{19}} + r_{i_{22}} + r_{(i-1)_3} + r_{(i-1)_5} + 4r_{(i-1)_8} \\
+4r_{(i-1)_{19}} + 4r_{(i-1)_{22}} + 4r_{(i-2)_3} + 4r_{(i-2)_5} + 8r_{(i-2)_8}], \\
0 \le i \le 2n, j = 8; \\
16[r_{i_9} + r_{(i-1)_6} + 6r_{(i-1)_9} + 4r_{(i-2)_6} + 4r_{(i-2)_9}], \\
0 \le i \le 2n, j = 9; \\
8[r_{i_{10}} + r_{i_{18}} + r_{(i-1)_1} + r_{(i-1)_4} + 8r_{(i-1)_{10}} + 4r_{(i-1)_{18}} \\
+4r_{(i-2)_1} + 4r_{(i-2)_4} + 8r_{(i-2)_{10}}], 0 \le i \le 2n+1, j = 10; \\
16[r_{i_{20}} + r_{(i-1)_3} + 2r_{(i-1)_{11}} + 4r_{(i-1)_{20}} \\
+4r_{(i-2)_3} + 4r_{(i-2)_{11}}], 0 \le i \le 2n, j = 11; \\
8[r_{i_{12}} + r_{i_{18}} + r_{(i-1)_1} + r_{(i-1)_4} + 8r_{(i-1)_{12}} + 4r_{(i-1)_{18}} \\
+4r_{(i-2)_1} + 4r_{(i-2)_4} + 8r_{(i-2)_{12}}], 0 \le i \le 2n, j = 12; \\
16[r_{i_{21}} + r_{(i-1)_5} + 2r_{(i-1)_{13}} + 4r_{(i-1)_{21}} \\
+4r_{(i-2)_5} + 4r_{(i-2)_{13}}], 0 \le i \le 2n, j = 13; \\
8[r_{i_{14}} + r_{i_{23}} + r_{(i-1)_1} + r_{(i-1)_2} + 8r_{(i-1)_{14}} + 4r_{(i-1)_{23}} \\
+4r_{(i-2)_1} + 4r_{(i-2)_2} + 8r_{(i-2)_{14}}], 0 \le i \le 2n, j = 14; \\
16[r_{i_{21}} + r_{(i-1)_5} + 2r_{(i-1)_{15}} + 4r_{(i-1)_{21}} \\
+4r_{(i-2)_5} + 4r_{(i-2)_{15}}], 0 \le i \le 2n, j = 15; \\
8[r_{i_{16}} + r_{i_{23}} + r_{(i-1)_1} + r_{(i-1)_2} + 8r_{(i-1)_{16}} + 4r_{(i-1)_{23}} \\
+4r_{(i-2)_1} + 4r_{(i-2)_2} + 8r_{(i-2)_{16}}], 0 \le i \le 2n, j = 16; \\
16[r_{i_{20}} + r_{(i-1)_3} + 2r_{(i-1)_{17}} + 4r_{(i-1)_{20}} \\
+4r_{(i-2)_3} + 4r_{(i-2)_{17}}], 0 \le i \le 2n+1, j = 17; \\
16[r_{(i-1)_{10}} + r_{(i-1)_{12}} + 2r_{(i-1)_{18}} + 4r_{(i-2)_{10}} \\
+4r_{(i-2)_{12}} + 4r_{(i-2)_{18}}], 0 \le i \le 2n+1, j = 18; \\
8[r_{i_{24}} + r_{(i-1)_8} + r_{(i-1)_{10}} + r_{(i-1)_{14}} \\
+4r_{(i-1)_{19}} + 4r_{(i-1)_{24}} + 4r_{(i-2)_8} + 4r_{(i-2)_{10}} \\
+4r_{(i-2)_{14}} + 8r_{(i-2)_{19}}], 0 \le i \le 2n+1, j = 19; \\
8[r_{i_{20}} + r_{(i-1)_9} + r_{(i-1)_{11}} + r_{(i-1)_{17}} + 8r_{(i-1)_{20}} + 4r_{(i-2)_9} \\
+4r_{(i-2)_{11}} + 4r_{(i-2)_{17}} + 8r_{(i-2)_{20}}], 0 \le i \le 2n+1, j = 20; \\
8[r_{i_{21}} + r_{(i-1)_9} + r_{(i-1)_{13}} + r_{(i-1)_{15}} + 8r_{(i-1)_{21}} + 4r_{(i-2)_9} \\
+4r_{(i-2)_{13}} + 4r_{(i-2)_{15}} + 8r_{(i-2)_{21}}], 0 \le i \le 2n+1, j = 21; \\
8[r_{i_{24}} + r_{(i-1)_8} + r_{(i-1)_{12}} + r_{(i-1)_{16}} \\
+4r_{(i-1)_{22}} + 4r_{(i-1)_{24}} + 4r_{(i-2)_8} + 4r_{(i-2)_{12}} \\
+4r_{(i-2)_{16}} + 8r_{(i-2)_{22}}], 0 \le i \le 2n+1, j = 22;
\end{cases}
$$

$$g_{i_j}(n) = \begin{cases} 16[r_{(i-1)_{14}} + r_{(i-1)_{16}} + 2r_{(i-1)_{23}} + 4r_{(i-2)_{14}} \\ +4r_{(i-2)_{16}} + 4r_{(i-2)_{23}}], 0 \le i \le 2n+1, j = 23; \\ 8[r_{(i-1)_{19}} + r_{(i-1)_{20}} + r_{(i-1)_{21}} + r_{(i-1)_{22}} + 4r_{(i-1)_{24}} \\ +4r_{(i-2)_{19}} + 4r_{(i-2)_{20}} + 4r_{(i-2)_{21}} \\ +4r_{(i-2)_{22}} + 8r_{(i-2)_{24}}], 0 \le i \le 2n+1, j = 24; \\ 0, otherwise. \end{cases}$$

and the initial values are

Table 2

j	1	2	3	4	5	6	7	8	9	10	11	12
$g_{0_j}(1)$	32	32	32	32	32	32	0	16	16	16	16	16
$g_{1_j}(1)$	160	160	160	160	160	160	64	112	112	112	112	112
$g_{2_j}(1)$	64	64	64	64	64	64	192	128	128	128	128	128
j	13	14	15	16	17	18	19	20	21	22	23	24
$g_{0_j}(1)$	16	16	16	16	16	0	8	8	8	8	0	0
$g_{1_j}(1)$	112	112	112	112	112	64	88	88	88	88	64	64
$g_{2_j}(1)$	128	128	128	128	128	192	160	160	160	160	192	192

Proof In what follows, we consider $g_{i_j}(n)(j = 1, 2, \ldots, 24)$. Since the proof is similar for each j, we just prove $j = 1$. As a matter of convenience, we firstly find the relation between S_1^n and O_0, O_1, O_2, $S_j^{n-1}(j = 1, 2, \ldots, 24)$, then obtain the formula.

In Table 1, each group of X_1^n and X_2^n in (I) can combine 16 groups of X_3^n and X_4^n in (II) and constitute 16 cases of S_1^n. There are 16 groups of X_1^n and X_2^n in (I), therefore there are 256 cases in S_1^n. By using lemma 1.1, 1.2, OP.1, 2, 3, the 256 cases of S_1^n are:

$(1) S_1^n \sim a_n b_n c_n d_n X_1^{n-1} X_2^{n-1} a_n^- b_n^- c_n^- d_n^- X_3^{n-1} X_4^{n-1} \sim S_1^{n-1} O_2$

$(2) S_1^n \sim a_n b_n c_n d_n X_1^{n-1} X_2^{n-1} a_n^- b_n^- d_n^- c_n^- X_3^{n-1} X_4^{n-1} \sim S_7^{n-1} O_1$

...

$(255) S_1^n \sim X_1^{n-1} X_2^{n-1} d_n c_n b_n a_n X_3^{n-1} X_4^{n-1} d_n^- c_n^- a_n^- b_n^- \sim S_7^{n-1} O_1$

$(256) S_1^n \sim X_1^{n-1} X_2^{n-1} d_n c_n b_n a_n X_3^{n-1} X_4^{n-1} d_n^- c_n^- b_n^- a_n^- \sim S_1^{n-1} O_2$

As an example, by lemma 1.1 and op.1.2, $a_n^- b_n^- = (b_n a_n)^-$, $(255) S_1^n \sim X_1^{n-1} X_2^{n-1} d_n c_n b_n a_n X_3^{n-1} X_4^{n-1} d_n^- c_n^- a_n^- b_n^- \sim X_1^{n-1} X_2^{n-1} b_n a_n X_3^{n-1} X_4^{n-1} a_n^- b_n^- d_n c_n d_n^- c_n^- \sim (b_n a_n)^- X_1^{n-1} X_2^{n-1} (b_n a_n) X_3^{n-1} X_4^{n-1} d_n c_n d_n^- c_n^- \sim S_7^{n-1} O_1$.

$$S_1^n = \sum_{j=1}^{256} (j) S_1^n \sim 16(S_1^{n-1} + S_7^{n-1} + 6S_1^{n-1} O_1 + 4S_7^{n-1} O_1 + 4S_1^{n-1} O_2).$$

$S_1^1 \sim 32(O_0 + 5O_1 + 2O_2)$,

where, $\sum\limits_{j=m_1}^{m_2} (j)S_1^n$ represents the set of all surfaces in $(j)S_1^n, j = m_1, \ldots, m_2$, kS means the surface S appears k times, where m_1, m_2 and k are natural numbers and $m_1 < m_2$.

There are $8n$ letters in S_1^n, every 4 letters determine at most one genus, hence the genus range of S_1^n is $0 \le i \le 2n$.

Therefore,

$$g_{i_1}(n) = 16[r_{i_1} + r_{i_7} + 6r_{(i-1)_1} + 4r_{(i-1)_7} + 4r_{(i-2)_1}], 0 \le i \le 2n.$$

$g_{0_1}(1) = 32, g_{1_1}(1) = 160, g_{2_1}(1) = 64$.

Thus the theorem is proved.

By using theorem 2.1, we can obtain the genus distributions of the surfaces $S_j^2(j = 1, 2, \ldots, 24)$ are in Table 3 :

Table 3

j	1	2	3	4	5	6	7	8
$g_{0_j}(2)$	512	512	640	512	640	512	0	128
$g_{1_j}(2)$	6656	6656	7552	6656	7552	6656	1024	2944
$g_{2_j}(2)$	25600	25600	26624	25600	26624	25600	11264	17408
$g_{3_j}(2)$	28672	28672	26624	28672	26624	28672	32768	32768
$g_{4_j}(2)$	4096	4096	4096	4096	4096	4096	20480	12288
j	9	10	11	12	13	14	15	16
$g_{0_j}(2)$	256	128	128	128	128	128	128	128
$g_{1_j}(2)$	3840	2944	2944	2944	2944	2944	2944	2944
$g_{2_j}(2)$	18432	17408	17408	17408	17408	17408	17408	17408
$g_{3_j}(2)$	30720	32768	32768	32768	32768	32768	32768	32768
$g_{4_j}(2)$	12288	12288	12288	12288	12288	12288	12288	12288
j	17	18	19	20	21	22	23	24
$g_{0_j}(2)$	128	0	0	64	64	0	0	0
$g_{1_j}(2)$	2944	512	1152	1600	1600	1152	512	256
$g_{2_j}(2)$	17408	7680	11136	11648	11648	11136	7680	5888
$g_{3_j}(2)$	32768	28672	30720	29696	29696	30720	28672	26624
$g_{4_j}(2)$	12288	28672	22528	22528	22528	22528	28672	32768

By using theorem 2.1, we can obtain the genus distributions of the surfaces $S_j^3(j = 1, 2, \ldots, 24)$ are in Table 4 :

Table 4

j	1	2	3	4	5	6
$g_{0_j}(3)$	8192	4096	9216	4096	9216	8192
$g_{1_j}(3)$	172032	126976	195584	126976	195584	172032
$g_{2_j}(3)$	1327104	1179648	1474560	1179648	1474560	1327104
$g_{3_j}(3)$	4587520	4521984	4874240	4521984	4874240	4587520
$g_{4_j}(3)$	6881280	7143424	6815744	7143424	6815744	6881280
$g_{5_j}(3)$	3538944	3538944	3145728	3538944	3145728	3538944
$g_{6_j}(3)$	262144	262144	262144	262144	262144	262144

j	7	8	9	10	11	12
$g_{0_j}(3)$	0	0	4096	1024	1024	1024
$g_{1_j}(3)$	16384	32768	94208	44032	44032	44032
$g_{2_j}(3)$	311296	516096	819200	552960	552960	552960
$g_{3_j}(3)$	2097152	2859008	3342336	2875392	2875392	2875392
$g_{4_j}(3)$	5963776	6619136	6422528	6553600	6553600	6553600
$g_{5_j}(3)$	6553600	5701632	5046272	5701632	5701632	5701632
$g_{6_j}(3)$	1835008	1048576	1048576	1048576	1048576	1048576

j	13	14	15	16	17	18
$g_{0_j}(3)$	1024	1024	1024	1024	1024	0
$g_{1_j}(3)$	44032	44032	44032	44032	44032	4096
$g_{2_j}(3)$	552960	552960	552960	552960	552960	126976
$g_{3_j}(3)$	2875392	2875392	2875392	2875392	2875392	1212416
$g_{4_j}(3)$	6553600	6553600	6553600	6553600	6553600	4685824
$g_{5_j}(3)$	5701632	5701632	5701632	5701632	5701632	7340032
$g_{6_j}(3)$	1048576	1048576	1048576	1048576	1048576	3407872

j	19	20	21	22	23	24
$g_{0_j}(3)$	0	512	512	0	0	0
$g_{1_j}(3)$	5120	20992	20992	5120	4096	1024
$g_{2_j}(3)$	175104	293888	293888	175104	126976	56320
$g_{3_j}(3)$	1531904	1822720	1822720	1531904	1212416	745472
$g_{4_j}(3)$	5267456	5300224	5300224	5267456	4685824	3653632
$g_{5_j}(3)$	7176192	6717440	6717440	7176192	7340032	7340032
$g_{6_j}(3)$	2621440	2621440	2621440	2621440	3407872	4980736

By the joint tree J_n of G_n, we can obtain the set of the associate surfaces (embedding surfaces) of G_n:

$$U_n = \{X_1^n a X_2^n X_3^n a^- X_4^n\} \sim S_9^n.$$

By applying the theorem 2.1, the genus polynomial of G_n can be obtained for a given n. For $n = 1, 2, 3$, the genus polynomials of G_n are calculated as follows:

$$f_{G_1}(x) = 16 + 112x + 128x^2;$$

$$f_{G_2}(x) = 256 + 3840x + 18432x^2 + 30720x^3 + 12288x^4;$$
$$f_{G_3}(x) = 4096 + 94208x + 819200x^2 + 3342336x^3 + 6422528x^4 + 5046272x^5 + 1048576x^6.$$

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