Weak Edge Detour Number of a Graph

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ABSTRACT

For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. A set $S \subseteq V$ is called a *weak edge detour set* if every edge in G has both its ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a *weak edge detour basis* of G. Certain general properties of these concepts are studied. The weak edge detour numbers of certain classes of graphs are determined. Its relationship with the detour diameter is discussed and it is proved that for each triple D, k, p of integers with $3 \leq k \leq p - D + 1$ and $D \geq 3$ there is a connected graph G of order p with detour diameter D and $dn_w(G) = k$. It is also proved that for any three positive integers a, b, k with $k \geq 3$ and $a < b \leq 2a$, there is a connected graph G with detour radius a, detour diameter b and $dn_w(G) = k$. Graphs G with detour diameter $D \leq 4$ are characterized for $dn_w(G) = p - 1$ and $dn_w(G) = p - 2$ and trees with these numbers are characterized. A weak edge detour set S, no proper subset of which is a weak edge detour set, is a *minimal weak edge detour set*. The *upper weak edge detour number* $dn_w^+(G)$ of a graph G is the maximum cardinality of a minimal weak edge detour set of G. It is shown that for every pair a, b of integers with $2 \leq a \leq b$, there is a connected graph G with $dn_w(G) = a$ and $dn_w^+(G) = b$.

Key Words: Detour, Detour set, Detour number, Weak edge detour set, Weak edge detour basis, Weak edge detour number.

AMS Subject Classification: 05C12

1 Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1, 4].

For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest u–v path in G. A u–v path of length $d(u, v)$ is called a *u–v geodesic.* For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, $rad\,G$ and the maximum eccentricity is its *diameter*, $diam\,G$ of G. For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour.* It is known that the distance and the detour distance are metrics on the vertex set V. The *detour eccentricity* $e_D(v)$ of a vertex in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D\,G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D\,G$ of G is the maximum detour eccentricity among the vertices of G.

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G.

Two vertices u and v of G are *antipodal* if $d(u, v) = diam\,G$. A *caterpillar* is a tree for which the removal of all end-vertices leaves a path. A *wounded spider* is the graph formed by subdividing at most $t - 1$ of the edges of a *star* $K_{1,t}$ for $t \geq 0$. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v to $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. Thus every end-block is a branch of G at the cut-vertex v of G. The following theorems are used in the sequel.

Theorem 1.1 ([3]) *Every end-vertex of a non-trivial connected graph G belongs to every detour set of G. Also if the set S of all end-vertices of G is a detour set, then S is the unique detour basis for G.*

Theorem 1.2 ([3]) *If T is a tree with k end-vertices, then $dn(T) = k$.*

In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 1.1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivates us to introduce the concept of *weak edge detour set* of a graph.

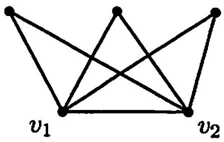


Figure 1.1: G

Throughout this paper G denotes a connected graph with at least two vertices.

2 Weak Edge Detour Number of a Graph

Definition 2.1 Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge in G has both its ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a *weak edge detour* basis of G.

Example 2.2 For the graph G given in Figure 2.1, it is clear that no two element subset of V is a weak edge detour set of G. The set $S = \{v_1, v_2, v_3\}$ is a weak edge detour basis of G so that $dn_w(G) = 3$. The set $S_1 = \{v_1, v_4, v_5\}$ is another weak edge detour basis of G. Thus there can be more than one weak edge detour basis for a graph G. Also $U = \{v_1, v_2\}$ is a detour basis of G so that $dn(G) = 2$ and hence the detour number and the weak edge detour number of a graph G are different.

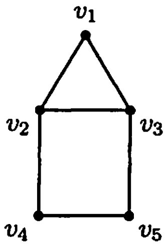


Figure 2.1: G

Example 2.3 For the graph G given in Figure 2.2, $S_1 = \{v_1, v_4\}$, and $S_2 = \{v_2, v_3\}$ are weak edge detour bases for G and $dn_w(G) = 2$.

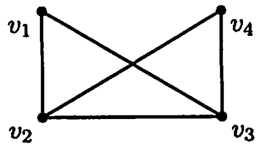


Figure 2.2: G

Theorem 2.4 *For any graph G of order $p \geq 2$, $2 \leq dn_w(G) \leq p$.*

Proof. A weak edge detour set needs at least two vertices so that $dn_w(G) \geq 2$ and the set of all vertices of G is a weak edge detour set of G so that $dn_w(G) \leq p$. Thus $2 \leq dn_w(G) \leq p$. ■

Remark 2.5 The bounds in Theorem 2.4 are sharp. For the complete graph K_2, $dn_w(K_2) = p$. The set of two end-vertices of a path P_n $(n \geq 2)$ is its unique weak edge detour set so that $dn_w(P_n) = 2$. Thus the complete graph K_2 has the largest possible weak edge detour number p and the non-trivial paths have the smallest weak edge detour number 2.

This suggests the following question.

Problem 2.6 *Is the upper bound in Theorem 2.4 sharp if $p > 2$?*

Definition 2.7 A vertex v in a graph G is a *weak edge detour vertex* if v belongs to every weak edge detour basis of G. If G has a unique weak edge detour basis S, then every vertex in S is a weak edge detour vertex of G.

Example 2.8 For the graph G given in Figure 2.3, $S = \{u, v, w\}$ is the unique weak edge detour basis so that every vertex of S is a weak edge detour vertex of G.

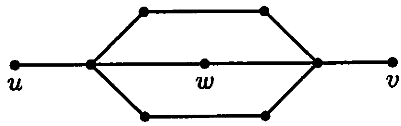


Figure 2.3: G

Example 2.9 For the graph G given Figure 2.4, $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, w\}$ are the only weak edge detour bases of G so that u and v are the weak edge detour vertices of G.

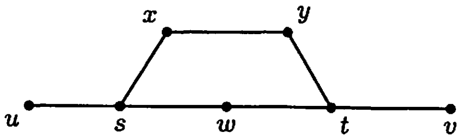


Figure 2.4: G

Remark 2.10 A cut-vertex may or may not belong to a weak edge detour basis of a graph G. For the graph G given in Figure 2.5(a), $S_1 = \{u, w, x\}$, $S_2 = \{u, w, y\}$, $S_3 = \{v, w, x\}$ and $S_4 = \{v, w, y\}$ are the four weak edge detour bases. The cut-vertex w belongs to every weak edge detour basis so that the cut-vertex w is the unique weak edge detour vertex of G. For the graph G in Figure 2.5(b), $S = \{u, v\}$ is a weak edge detour basis and the cut-vertex w is not a weak edge detour vertex.

Figure 2.5: G

In the following theorem we show that there are certain vertices in a non-trivial connected graph G that are weak edge detour vertices of G.

Theorem 2.11 *Every end-vertex of a non-trivial connected graph G belongs to every weak edge detour set of G. Also if the set S of all end-vertices of G is a weak edge detour set, then S is the unique weak edge detour basis for G.*

Proof. Let v be an end-vertex of G and uv an edge in G incident with v. Then uv is either an initial edge or the terminal edge of any detour containing the edge uv. Hence it follows that v belongs to every weak edge detour set of G. If S is the set of all end-vertices of G, then by the first part of this theorem $dn_w(G) \geq |S|$. If S is a weak edge detour set of G, then $dn_w(G) \leq |S|$. Hence $dn_w(G) = |S|$ and S is the unique weak edge detour basis for G. ■

Corollary 2.12 *If T is a tree with k end-vertices, then $dn(T) = dn_w(T) = k$.*

Proof. This follows from Theorems 1.2 and 2.11. ■

Corollary 2.13 *Every end-vertex of a connected graph G is a detour vertex as well as weak edge detour vertex.*

Proof. This follows from Theorems 1.1 and 2.11. ■

Corollary 2.14 *For any connected graph G with k end-vertices, $\max\{2, k\} \leq dn_w(G) \leq p$.*

Proof. This follows from the Theorems 2.4 and 2.11. ■

Theorem 2.15 *Let G be a connected graph with cut-vertices and S a weak edge detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Proof. Let v be a cut-vertex of G such that one of the components, say C of $G - v$ contains no vertex of S. Then by Theorem 2.11, C does not contain any end-vertex of G. Hence C contains at least one edge, say uw. Since S is a weak edge detour set, there exist vertices $x, y \in S$ such that uw lies on some x–y detour P: $x = u_0, u_1, \ldots, u, w, \ldots, u_t = y$ in G or both the ends u and w of the edge uw are in S. Suppose that uw lies on the detour P. Let P_1 be the x–u subpath of P and P_2 be the u–y subpath of P. Since v is a cut-vertex of G, both P_1 and P_2 contain v so that P is not a detour, which is a contradiction. Suppose that u and w are in S. Then C contains vertices of S, which is a contradiction. Thus every component of $G - v$ contains an element of S. ■

Corollary 2.16 *Let G be a connected graph with cut-vertices and S a weak edge detour set of G. Then every branch of G contains an element of S.*

Remark 2.17 By Corollary 2.16, if S is a weak edge detour set of a connected graph G, then every end-block of G must contain at least one element of S. However, it is possible that some blocks of G that are not end-blocks must contain an element of S as well. For example, consider the graph G of Figure 2.4, where the cycle $C_5 : x, y, t, w, s, x$ is a block of G that is not an end-block. By Theorem 2.11, every weak edge detour set of G must contain u and v. Since the u–v detour does not contain the edges sw and wt, it follows that $\{u, v\}$ is not a weak edge detour set. Thus every weak edge detour set of G must contain at least one vertex from the block C_5.

Corollary 2.18 *If G is a connected graph with $k \geq 2$ end-blocks, then $dn_w(G) \geq k$.*

Corollary 2.19 *If G is a connected graph with a cut-vertex v and the number of components of $G - v$ is r, then $dn_w(G) \geq r$.*

For the graph H and an integer k, we write kH for the union of the k disjoint copies of H.

Theorem 2.20 *Let $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 5$ such that $r \geq 2$, each $n_i \geq 2$ and $n_1 + n_2 + \cdots + n_r + k = p - 1$. Then $dn_w(G) = r + k + 1$.*

Proof. Let $u_1, u_2, \ldots, u_k$ be the end-vertices of G. Let S be any weak edge detour set of G. Then by Theorem 2.11, $u_i \in S\,(1 \leq i \leq k)$ and by Theorem 2.15, S contains a vertex from each component K_{n_i} $(1 \leq i \leq r)$. Choose exactly one vertex v_i from each K_{n_i} such that $v_i \in S$. Then $|S| \geq r + k$. Let $T = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_k\}$. Since any of the edges $vv_i\,(1 \leq i \leq r)$ neither lies on any detour joining a pair of vertices of T nor has both its ends in T, T is not a weak edge detour set of G. Hence it follows that $dn_w(G) \geq r + k + 1$. Now, $T \cup \{v\}$ is clearly is a weak edge detour set of G and so $dn_w(G) = r + k + 1$. ■

Remark 2.21 If the blocks of the graph G in Theorem 2.20 are not complete, then the theorem is not true. For the graph G given in Figure 2.6, there are two blocks and $\{v_1, v_2, v_5, v_7\}$ is a weak edge detour basis so that $dn_w(G) = 4$.

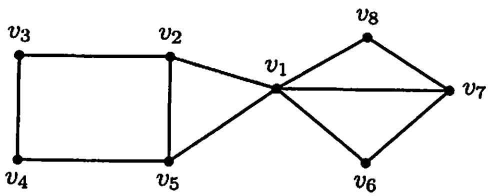


Figure 2.6: G

In the following theorem we give certain graphs G for which $dn_w(G) = 2$.

Theorem 2.22 *If G is the complete graph $K_p\,(p \geq 2)$ or $K_p - e\,(p \geq 3)$ or a cycle C_n or a non-trivial path P_n or a complete bipartite graph $K_{m,n}\,(m, n \geq 2)$, then $dn_w(G) = 2$.*

Proof. It is clear that any set of two vertices in $K_p\,(p \geq 2)$ or $K_{m,n}\,(m, n \geq 2)$ is a weak edge detour set. Also it is clear that any set of two adjacent vertices in C_n, those two vertices of degree $p-2$ in $K_p - e\,(p \geq 3)$ and the two end-vertices of the non-trivial path P_n are weak edge detour sets in C_n, $K_p - e$ and P_n respectively. Hence the result follows. ■

The following theorems give realization results.

Theorem 2.23 *For each pair of integers k and p with $2 \leq k < p$, there exists a connected graph G of order p with $dn_w(G) = k$.*

Proof. For $2 \leq k < p$, let P be a path of order $p - k + 2$. Then the graph G obtained from P by adding $k-2$ new vertices to P and joining them to any cut-vertex of P is a tree of order p and so by Corollary 2.12, $dn_w(G) = k$. ■

Theorem 2.24 *For each positive integer $k \geq 2$ there exists a connected graph G and a vertex v of degree k in G such that v belongs to a weak edge detour basis of G and $dn_w(G) = k$.*

Proof. For $k \geq 2$, let G be the graph obtained from the complete graph K_3, where $V(K_3) = \{v_1, v_2, v_3\}$, by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ and joining each $u_i\,(1 \leq i \leq k-2)$ to v_1. Then $deg_G v_1 = k$. Let $S = \{u_1, u_2, \ldots, u_{k-2}\}$. Then neither S nor $S \cup \{v_i\}\,(1 \leq i \leq 3)$ is a weak edge detour set of G. However, $S \cup \{v_1, v_2\}$ is a weak edge detour set of G and hence by Theorem 2.11, $S \cup \{v_1, v_2\}$ is a weak edge detour basis of G so that $dn_w(G) = k$. ■

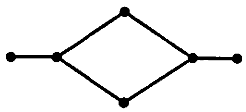
3 Weak Edge Detour Number and Detour Diameter of a Graph

In [3], an upper bound for the detour number of a graph is given in terms of its order and detour diameter D as follows:

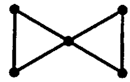
Proposition A([3]) If G is a non-trivial connected graph of order $p \geq 3$ and detour diameter D, then $dn(G) \leq p - D + 1$.

Remark 3.1 In the case of weak edge detour number $dn_w(G)$ of a graph

G, there are graphs for which $dn_w(G) = p - D + 1$, $dn_w(G) < p - D + 1$ and $dn_w(G) > p - D + 1$. For any cycle C of order $p \geq 3$, $D = p - 1$ and $dn_w(C) = 2$ so that $dn_w(C) = p - D + 1$. For the graph G in Figure 3.1(a), $p = 6$, $D = 4$ and $dn_w(G) = 2$ so that $dn_w(G) < p - D + 1$. For the graph G in Figure 3.1(b), $p = 5$, $D = 4$ and $dn_w(G) = 3$ so that $dn_w(G) > p - D + 1$.



(a)



(b)

Figure 3.1: G

In the following we give conditions for the graph G so that $dn_w(G) \geq p - D + 1$.

Theorem 3.2 *Let G be a graph of order $p \geq 2$. If $D = p - 1$, then $dn_w(G) \geq p - D + 1$.*

Proof. For any graph G, $dn_w(G) \geq 2$. Since $D = p - 1$, we have $p - D + 1 = 2$ and so $dn_w(G) \geq p - D + 1$. ∎

Remark 3.3 The converse of Theorem 3.2 is not true. For the graph G given in Figure 3.2, $p = 6$ and $D = 4$ so that $p - D + 1 = 3$ and $dn_w(G) = 4$. Thus $dn_w(G) > p - D + 1$, but $D \neq p - 1$.

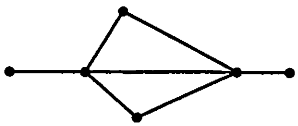


Figure 3.2: G

Theorem 3.4 *If G is a non-trivial tree of order p, then $dn_w(G) \leq p - D + 1$.*

Proof. Let u and v be the vertices of G for which $D(u, v) = D$ and let $P : u = v_0, v_1, \ldots, v_{D-1}, v_D = v$ be a u–v detour of length D. Let $S = V(G) - \{v_1, v_2, \ldots, v_{D-1}\}$. Clearly, S is a weak edge detour set of G and so $dn_w(G) \leq |S| = p - D + 1$. ∎

We give below a characterization theorem for trees.

Theorem 3.5 *For every non-trivial tree T of order p, $dn_w(T) = p-D+1$ if only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $D = D(u,v)$ and $P : u = v_0$, $v_1, \ldots, v_{D-1}, v_D = v$ be a detour diametral path. Let k be the number of end-vertices of T and l the number of internal vertices of T other than $v_1, v_2, \ldots, v_{D-1}$. Then $D-1+l+k = p$. By Corollary 2.12, $dn_w(T) = k = p - D - l + 1$. Hence $dn_w(T) = p - D + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the detour diameteral path P, if and only if T is a caterpillar. ■

Corollary 3.6 *For a wounded spider T of order p, $dn_w(T) = p - D + 1$ if and only if T is obtained from $K_{1,t}$ $(t \geq 1)$ by subdividing at most two of its edges.*

Proof. It is clear that a wounded spider T is a caterpillar if and only if T is obtained from $K_{1,t}$ $(t \geq 1)$ by subdividing at most two of its edges. Then the result follows from Theorem 3.5. ■

The following theorems give realization results.

Theorem 3.7 *For each triple D, k, p of integers with $3 \leq k \leq p-D+1$ and $D \geq 3$, there exists a connected graph G of order p with detour diameter D and $dn_w(G) = k$.*

Proof. Case 1. When D is even, let G be the graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by adding $k-1$ new vertices $v_1, v_2, \ldots, v_{k-1}$ and joining each vertex v_i $(1 \leq i \leq k-1)$ to u_1 and adding $p - D - k + 1$ new vertices $w_1, w_2, \ldots, w_{p-D-k+1}$ and joining each vertex w_i $(1 \leq i \leq p - D - k + 1)$ to both u_1 and u_3. The graph G is connected of order p and detour diameter D and is shown in Figure 3.3(a).

Now, we show that $dn_w(G) = k$. Let $S = \{v_1, v_2, \ldots, v_{k-1}\}$ be the set of all end-vertices of G. Since no edge of G other than the edges u_1v_i $(1 \leq i \leq k-1)$ lies on a detour joining a pair of vertices of S or has both its ends in S, S is not a weak edge detour set of G. Let $T = S \cup \{v\}$, where v is the antipodal vertex of u_1 in C_D. Then every edge of G lies on a detour joining a vertex v_i $(1 \leq i \leq k-1)$ and v so that T is a weak edge detour set of G. Now, it follows from Theorem 2.11 that T is a weak edge detour basis of G and so $dn_w(G) = k$.

Case 2. When D is odd, let G be the graph obtained from the cycle C_D : $u_1, u_2, \ldots, u_D, u_1$ of order D by adding $k-2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ and joining each vertex v_i $(1 \leq i \leq k-2)$ to u_1 and adding $p-D-k+2$ new vertices $w_1, w_2, \ldots, w_{p-D-k+2}$ and joining each vertex w_i $(1 \leq i \leq p-D-k+2)$ to both u_1 and u_3. The graph G is connected of order p and detour diameter D and is shown in Figure 3.3(b).

Now, we show that $dn_w(G) = k$. Let $S = \{v_1, v_2, \ldots, v_{k-2}\}$ be the set of all end-vertices of G. As in Case 1, S is not a weak edge detour set of G. Let $S_1 = S \cup \{v\}$, where v is any vertex of G such that $v \neq v_i$ $(1 \leq i \leq k-2)$. It is easy to see that S_1 is not a weak edge detour set of G. Now, the set $T = S \cup \{u_1, u_D\}$ is clearly a weak edge detour set of G. Hence it follows from Theorem 2.11 that T is a weak edge detour basis of G and so $dn_w(G) = k$. ■

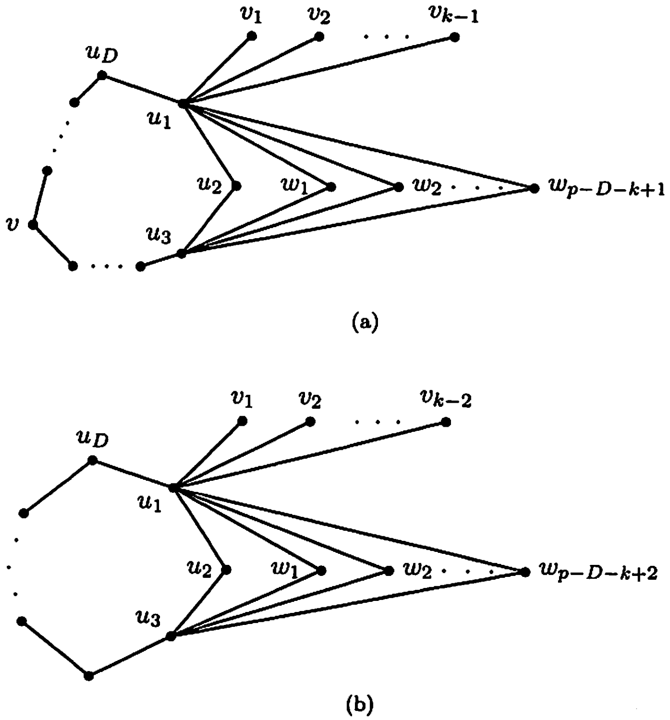


Figure 3.3: G

Chartrand, Escuadro and Zang [2] proved that the detour radius and detour diameter of a connected graph G satisfy $rad_D\, G \leq diam_D\, G \leq 2\, rad_D\, G$. Also they proved that every pair a, b of positive integers can be realized as the detour radius and detour diameter respectively of some connected graph provided $a \leq b \leq 2a$. We extend this theorem so that the weak edge detour number can be prescribed as well when $a < b \leq 2a$.

Theorem 3.8 *Let R, D, k be three positive integers such that $k \geq 3$ and $R < D \leq 2R$. Then there exists a connected graph G such that* $\mathrm{rad}_D\, G = R$, $\mathrm{diam}_D\, G = D$ *and* $dn_w(G) = k$.

Proof. Case 1. Let R be an odd integer. When $R = 1$, let $G = K_{1,k}$. Clearly, $rad_D\, G = 1$, $diam_D\, G = 2$ and by Corollary 2.12, $dn_w(G) = k$. When $R \geq 3$ and $R < D \leq 2R$, we construct a graph G with the desired properties as follows: Let $C_{R+1} : v_0, v_1, \ldots, v_R, v_0$ be a cycle of order $R+1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let H be the graph obtained from C_{R+1} and P_{D-R+1} by identifying v_0 of C_{R+1} with u_0 of P_{D-R+1}. The required graph G is obtained from H by adding $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ to H and joining each w_i $(1 \leq i \leq k-2)$ to the vertex u_{D-R-1} and is shown in Figure 3.4(a). Clearly, G is connected such that $rad_D\, G = R$ and $diam_D\, G = D$.

Now, we show that $dn_w(G) = k$. Let $S = \{w_1, w_2, \ldots, w_{k-2}, u_{D-R}\}$ be the set of all end-vertices of G. Since no edge of G other than the edges $w_i u_{D-R-1}$ $(1 \leq i \leq k-2)$ and the edge $u_{D-R} u_{D-R-1}$ lies on a detour joining a pair of vertices of S or has both its ends in S, S is not a weak edge detour set of G. Let $T = S \cup \{v\}$, where v is the antipodal vertex of v_0 in C_{R+1}. Then T is a weak edge detour set of G and hence it follows from Theorem 2.11 that T is a weak edge detour basis of G so that $dn_w(G) = k$.

Case 2. Let R be an even integer. Construct the graph H as in Case 1. Then G is obtained from H by adding $k-3$ new vertices $w_1, w_2, \ldots, w_{k-3}$ to H and joining each w_i $(1 \leq i \leq k-3)$ to the vertex u_{D-R-1} and is shown in Figure 3.4(b). Clearly G is connected such that $rad_D\, G = R$ and $diam_D\, G = D$.

Now, we show that $dn_w(G) = k$. Let $S = \{w_1, w_2, \ldots, w_{k-3}, u_{D-R}\}$ be the set of all end-vertices of G. As in Case 1, S is not a weak edge detour set of G. Let $S_1 = S \cup \{v\}$, where v is any vertex of G such that $v \notin S$. It is easy to see that S_1 is not a weak edge detour set of G. Now the set $T = S \cup \{v_1, v_R\}$ is clearly a weak edge detour set of G. Hence it follows from Theorem 2.11 that T is a weak edge detour basis of G and so $dn_w(G) = k$. ∎

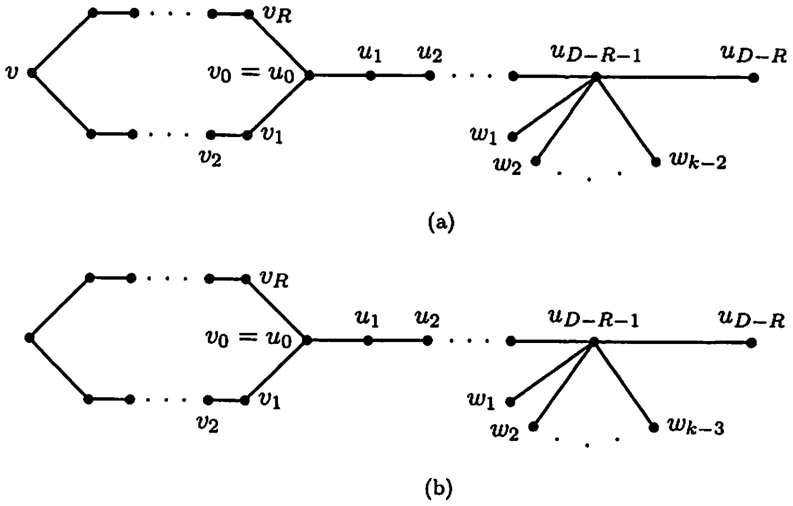


Figure 3.4: G

Now we proceed to study graphs G for which the weak edge detour number dn_w of G is either $p-2$ or $p-1$ when the detour diameter is known. In the following we characterize graphs G with detour diameter $D \leq 4$ for which $dn_w(G) = p-2$ and $dn_w(G) = p-1$. For this purpose we introduce the collection $\mathscr{G}$ of graphs given below.

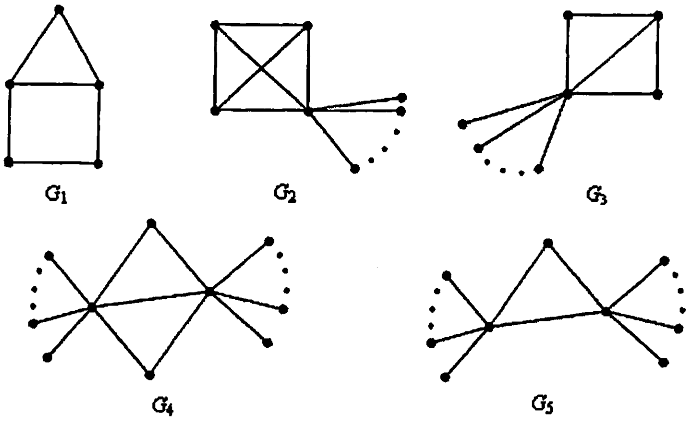


Figure 3.5: Graphs in family $\mathscr{G}$

Theorem 3.9 *Let G be a connected graph of order $p \geq 5$ with detour diameter $D \leq 4$. Then $dn_w(G) = p - 2$ if and only if G is a double star T or $T + e$ or $K_{1,p-1} + e + f$ or $G \in \mathcal{G}$.*

Proof. It is straightforward to verify that if G is a double star T or $T + e$ or $K_{1,p-1} + e + f$ or $G \in \mathcal{G}$, then $dn_w(G) = p - 2$. For the converse, let G be a connected graph of order $p \geq 5$, $D \leq 4$ and $dn_w(G) = p - 2$.

If $D \leq 2$, it is clear that there are no graphs G for which $dn_w(G) = p - 2$.

Suppose $D = 3$. If G is a tree, then G is a double star T and the result follows from Corollary 2.12. Assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D = 3$, it follows that $c(G) \leq 4$. We consider two cases.

Case 1: Let $c(G) = 4$. Let $C : v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G. Since $p \geq 5$ and G is connected, there exists a vertex x not on C such that it is adjacent to some vertex, say v_1 of C. Then x, v_1, v_2, v_3, v_4 is a path of length 4 in G so that $D \geq 4$, which is a contradiction.

Case 2: Let $c(G) = 3$. We claim that G is a double star. If G contains two or more triangles, then $c(G) = 4$ or $D \geq 4$, which is a contradiction. Hence G contains a unique triangle $C_3 : v_1, v_2, v_3, v_1$. Now, if there are two or more vertices of C_3 having degree 3 or more, then $D \geq 4$, which is contradiction. Thus exactly one vertex in C_3 has degree 3 or more. Since $D = 3$, it follows that $G = K_{1,p-1} + e$ and so $dn_w(K_{1,p-1} + e) = p - 1$, which is a contradiction. Thus it follows that G is a double star T.

Suppose $D = 4$. If G is a tree, then there exists a path of length 4 so that there are at least 3 internal vertices of G. Hence there are at most $(p-3)$ end-vertices of G so that by Corollary 2.12, $dn_w(G) \leq p-3$, which is contradiction. So, assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D = 4$, it follows that $c(G) \leq 5$. We consider three cases.

Case 1: Let $c(G) = 5$. Then, since $D = 4$, it is clear that G has exactly five vertices. Now, it is easily verified that the graph $G_1 \in \mathcal{G}$ given in Figure 3.5 is the only graph with $dn_w(G_1) = p-2$ among all graphs on five vertices having a largest cycle of length 5.

Case 2: Let $c(G) = 4$. Suppose that G contains K_4 as an induced subgraph. Since $p \geq 5$, $D = 4$ and $c(G) = 4$, every vertex not on K_4 is pendant and adjacent to exactly one vertex of K_4. Thus the graph reduces to the graph $G_2 \in \mathcal{G}$ given in Figure 3.5. Also since $dn_w(G_2) = p - 2$, G_2 is the only graph in this case satisfying the requirements of the theorem.

Now, suppose that G does not contain K_4 as an induced subgraph. We claim that G contains exactly one 4-cycle C_4. Suppose that G contains two or more 4-cycles. If two 4-cycles in G have no edges in common, then it is clear that $D \geq 5$, which is a contradiction. If two 4-cycles in G have exactly one edge in common, then G must contain the graphs given in Figure 3.6 as subgraphs or induced subgraphs. In any case, $D \geq 5$ or $c(G) \geq 5$, which is a contradiction.

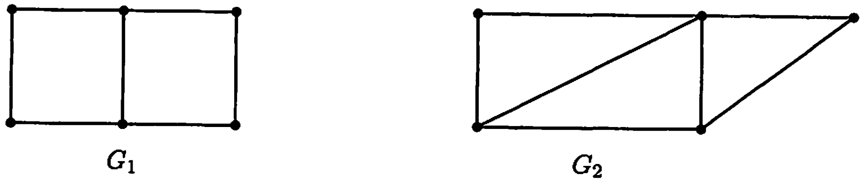


Figure 3.6:

If two 4-cycles in G have exactly two edges in common, then G must contain only the graphs given in Figure 3.7 as subgraphs. It is easily verified that all other subgraphs having two edges in common will have cycles of length ≥ 5 so that $D \geq 5$, which is a contradiction.

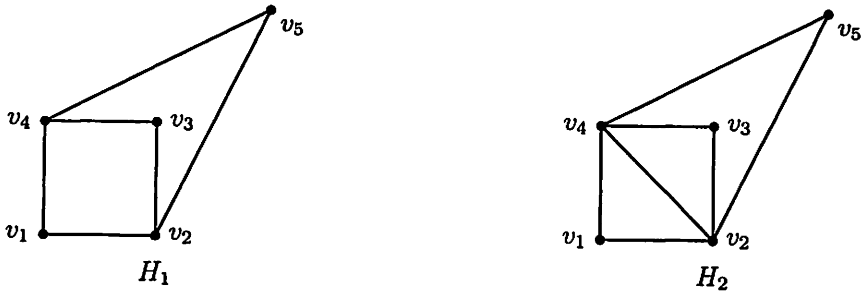


Figure 3.7:

Now, if G is one of these H_i $(i = 1, 2)$, then $dn_w(G) = p - 3$, which is a contradiction. Assume first that G contains H_1 as a proper subgraph. Then there is a vertex x such that $x \notin V(H_1)$ and x is adjacent to at least one vertex of H_1. If x is adjacent to v_1, we get a path $x, v_1, v_2, v_3, v_4, v_5$ of length 5 so that $D \geq 5$, which is a contradiction. Hence x cannot be adjacent to v_1. Similarly x cannot be adjacent to v_3 and v_5. Thus x is adjacent to v_2 or v_4 or both. If x is adjacent only to v_2, then x must be a pendant vertex of G, for otherwise, we get a path of length 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph G reduces to the one given in Figure 3.8.

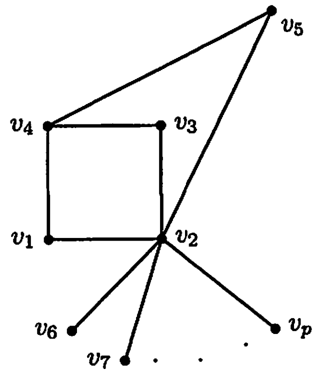


Figure 3.8: G

But for this graph G, $\{v_4, v_6, v_7, \ldots, v_p\}$ is a weak edge detour basis so that $dn_w(G) = p - 4$, which is a contradiction. So, in this case there are no graphs satisfying the requirements of the theorem. If x is adjacent only to v_4, then we get a graph G isomorphic to the one given in Figure 3.8 and hence in this case also there are no graphs satisfying the requirements of the theorem. If x is adjacent to both v_2 and v_4, then the graph reduces to the one given in Figure 3.9.

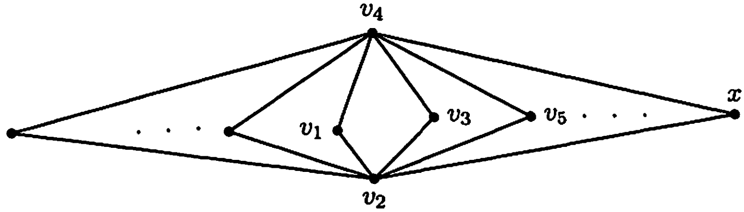


Figure 3.9: G

However for this graph, $\{x, v_3\}$ is a weak edge detour basis so that $dn_w(G) = 2$ and hence $dn_w(G) \leq p - 4$, which is a contradiction. Thus a vertex not in H_1 cannot be adjacent to both v_2 and v_4.

Next, if a vertex x not on H_1 is adjacent only to v_2 and a vertex y not on H_1 is adjacent only to v_4, then x and y must be pendant vertices of G, for otherwise, we get either a path or a cycle of length ≥ 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph reduces to the one given in Figure 3.10. But for this graph the set of all end-vertices is a weak edge detour basis so that $dn_w(G) = p - 5$, which is a contradiction.

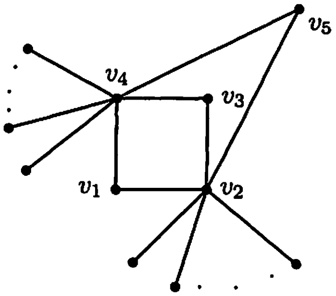


Figure 3.10: G

So, in this case also there are no graphs satisfying the requirements of the theorem. Thus we conclude that in this case there are no graphs G with H_1 as proper subgraph.

Next, if G contains H_2 as a proper subgraph, then as in the case of H_1, it is easily seen that there are no graphs G with H_2 as a proper subgraph satisfying $dn_w(G) = p - 2$. Thus, we conclude that, if G does not contain K_4 as an induced subgraph, then G has a unique 4-cycle. Now we consider two subcases.

Subcase 1: The unique cycle $C_4 : v_1, v_2, v_3, v_4, v_1$ contains exactly one chord v_2v_4. Since $p \geq 5$, $D = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to at least one vertex of C_4. The vertex x cannot be adjacent to both v_1 and v_3, for in this case we get $c(G) = 5$, which is a contradiction. Suppose that x is adjacent to v_1 or v_3, say v_1. Also if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to v_2 or v_3 or v_4, for in each case $D \geq 5$, which is a contradiction. Hence y is a pendant vertex and cannot be adjacent to x or v_2 or v_3 or v_4 so that in this case the graph G reduces to the one given in Figure 3.11. But for this graph G, $dn_w(G) = p - 3$, which is a contradiction.

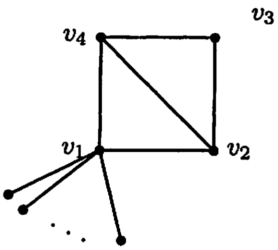


Figure 3.11: G

Similarly, if x is adjacent to v_3, we get a contradiction.

Now, if x is adjacent to both v_2 and v_4, we get the graph H given in Figure 3.12 as a subgraph which is isomorphic to the graph H_2 given in Figure 3.7. Then as in the first part of case 2, we see that there are no graphs which satisfy the requirements of the theorem.

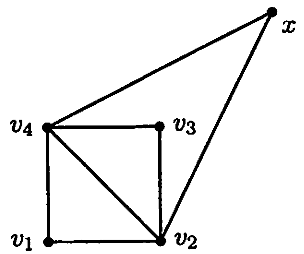


Figure 3.12: H

Thus x is adjacent to exactly one of v_2 or v_4, say v_2. Also if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to x or v_1 or v_3, for in each case $D \geq 5$, which is a contradiction. If y is adjacent to v_2 and v_4, then we get the graph H given in Figure 3.13 as a subgraph. Then exactly as in the first part of case 2 it can be seen that there are no graphs satisfying the requirements of the theorem.

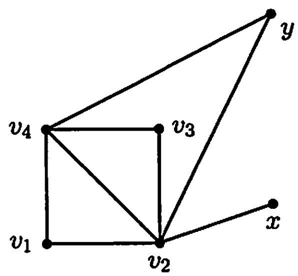


Figure 3.13: H

Thus y must be adjacent to v_2 or v_4 only. Hence we conclude that in either case the graph G must reduce to the graph G_3 or $G_4 \in \mathcal{G}$ as given in Figure 3.5. Similarly, if x is adjacent to v_4, then the graph G reduces to the graph G_3 or $G_4 \in \mathcal{G}$ as given in Figure 3.5. It is clear that $dn_w(G) = p - 2$ for these two classes of graphs G. Thus these two classes of graphs satisfy the requirements of the theorem. It is to be noted that the graph G_3 is nothing but $K_{1,p-1} + e + f$ where e and f are adjacent edges.

Subcase 2: The unique cycle $C_4 : v_1, v_2, v_3, v_4, v_1$ has no chord. In this case we claim that G contains no triangle. Suppose that G contains a triangle C_3. If C_3 has no vertex in common with C_4 or exactly one vertex in common with C_4, we get a path of length at least 5 so that $D \geq 5$. If C_3 has exactly two vertices in common with C_4, we get a cycle of length 5. Thus, in all cases, we have a contradiction and hence it follows that G contains a unique chordless cycle C_4 with no triangles. Since $p \geq 5$, $D = 4$, $c(G) = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to exactly one vertex of C_4, say v_1. Also if y is a vertex such that $y \neq x, v_1, v_2, v_3, v_4$, then y cannot be adjacent to v_2 or v_4, for in this case $D \geq 5$, which is a contradiction. Thus y must be adjacent to v_3 only. Hence we conclude that in either case G must reduce to the graphs H_1 or H_2 as given in Figure 3.14.

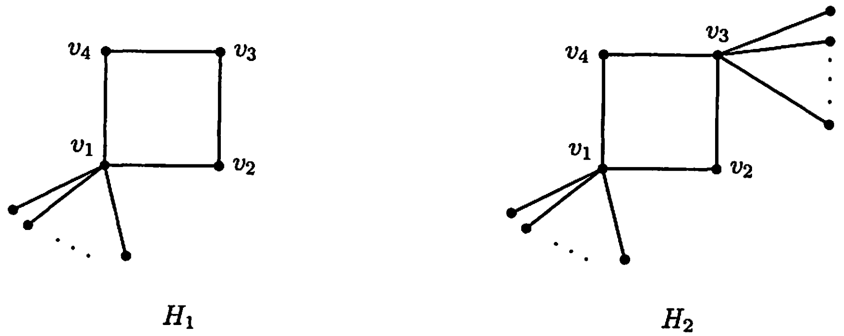
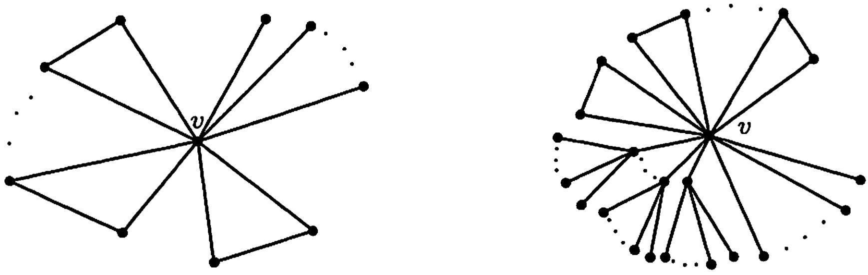


Figure 3.14:

But for these graphs H_1 and H_2 in Figure 3.14, $dn_w(H_1) = p - 3$ and $dn_w(H_2) = p - 4$. Hence there are no graphs satisfying the requirements of the theorem. Thus, when $D = 4$ and $c(G) = 4$, the graphs satisfying the requirements of the theorem are G_2, G_3 and $G_4 \in \mathscr{G}$ as in Figure 3.5.

Case 3: Let $c(G) = 3$. First we prove that the graph contains at most two triangles. If G contains more than two triangles, since $D = 4$, it is clear that all the triangles must have a vertex v in common. Now, if two triangles have two vertices in common, then it is clear that $c(G) \geq 4$. Hence all triangles must have exactly one vertex in common. Since $D = 4$, all the vertices of all the triangles are of degree 2 except v. Thus the graph reduces to the graphs given in Figure 3.15. By Theorem 2.11, every end-vertex of G belongs to every weak edge detour set of G. It is easy to see that the set S consisting of all end-vertices, the cut-vertex v and exactly one vertex other than v from each of the triangles is a weak edge detour set of G so that $dn_w(H_i) \leq |S|$ $(i = 1, 2)$. Since G contains more than two triangles, it follows that $|S| \leq p - 3$ and so $dn_w(H_i) \leq p - 3$ $(i = 1, 2)$, which is a contradiction to the assumption that $dn_w(G) = p - 2$.



H_1 H_2

Figure 3.15:

Thus the graph G contains at most two triangles. Now we consider two cases.

Case 3a: If G contains exactly one triangle $C_3 : v_1, v_2, v_3, v_1$. Since $p \geq 5$, there are vertices not on C_3. If all the vertices of C_3 have degree three or more, then since $D = 4$, the graph G must reduce to the one given in Figure 3.16. But in this case $dn_w(G) = p - 3$, which is a contradiction.

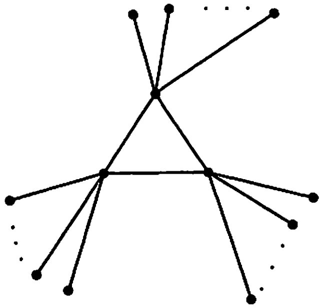


Figure 3.16: G

Hence we conclude that at most two vertices of C_3 have degree ≥ 3.

Subcase 1: Exactly two vertices of C_3 have degree 3 or more. Let $deg_G v_3 = 2$. Now, since $p \geq 5$, $D = 4$, $c(G) = 3$ and G is connected, we see that the graph reduces to the graph $G_5 \in \mathscr{G}$ as given in Figure 3.5, for which $dn_w(G) = p - 2$. Thus, in this case the graph $G_5 \in \mathscr{G}$ satisfies the requirements of the theorem.

Subcase 2: Exactly one vertex v_1 of C_3 has degree 3 or more. Since G is connected, $p \geq 5$, $D = 4$ and $c(G) = 3$, the graph reduces to the one given in Figure 3.17.

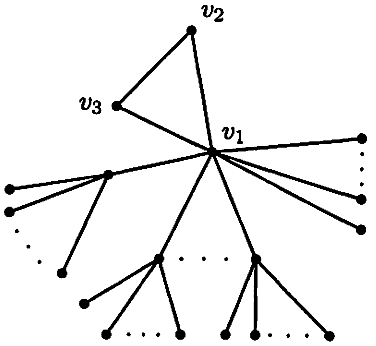


Figure 3.17: G

Now, we claim that exactly one neighbor of v_1 other than v_2 and v_3 has degree ≥ 2. If the claim is not true, then more than one neighbor of v_1 other than v_2 and v_3 has degree ≥ 2 and so the set of all end-vertices together with v_2 and v_3 forms a weak edge detour set of G. Hence $dn_w(G) \leq p-3$, which is a contradiction. Thus in this case the graph reduces to $T+e$ where T is a double star, which satisfies the requirements of the theorem.

Case 3b: Suppose that G contains exactly two triangles. Since G is connected, $p \geq 5$, $c(G) = 3$ and $D = 4$ the two triangles do not have two vertices in common and the graph reduces to the one given in Figure 3.18.

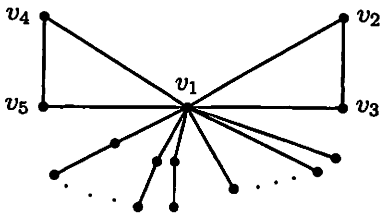


Figure 3.18: G

Now, we claim that all neighbors of v_1 which are not on the two triangles, are pendant vertices of G. Otherwise, the set of all end-vertices together with v_1, v_2 and v_4 forms a weak edge detour set of G and so $dn_w(G) \leq p-3$, which is a contradiction. Thus in this case the graph reduces to $K_{1,p-1} + e + f$ where e and f are non adjacent edges. This completes the proof of the theorem. ∎

Remark 3.10 For $p = 4$, the graphs are $G = P_4$, C_4, $K_4 - e$ and K_4

and $dn_w(G) = p-2$. For $p = 2$ and 3, there are no graphs G for which $dn_w(G) = p-2$.

In view of Theorem 3.9 we leave the following problem as an open question.

Problem 3.11 *Characterize connected graphs G of order $p \geq 5$ with detour diameter $D \geq 5$ for which $dn_w(G) = p-2$.*

Now we characterize the trees T for which $dn_w(T) = p-2$.

Theorem 3.12 *For any tree T of order $p \geq 5$, $dn_w(T) = p-2$ if and only if T is a double star.*

Proof. If T is a double star, then by Corollary 2.12, $dn_w(T) = p-2$. Conversely, assume that $dn_w(T) = p-2$. If $D \leq 2$, then it is proved in Theorem 3.9 that there are no graphs G for which $dn_w(G) = p-2$. If $D = 3$, then it is proved in Theorem 3.9 that T is a double star. If $D \geq 4$, then there exist at least three internal vertices of T so that there are at most $p-3$ end-vertices of T so that by Corollary 2.12, $dn_w(T) \leq p-3$, which is a contradiction. This completes the proof. ■

Theorem 3.13 *Let G be a connected graph of order $p \geq 3$ with detour diameter $D \leq 4$. Then $dn_w(G) = p-1$ if and only if $G = K_3$ or $K_{1,p-1}$ or $K_{1,p-1}+e$.*

Proof. It is straightforward to verify that if $G = K_3$ or $K_{1,p-1}$ or $K_{1,p-1}+e$, then $dn_w(G) = p-1$. For the converse, let G be a connected graph of order $p \geq 3$, $D \leq 4$ and $dn_w(G) = p-1$. If $p = 3$, it is easy to see that K_3 and $K_{1,2}$ are the only two graphs satisfying the requirements of the theorem. Now, let $p \geq 4$.

Suppose $D = 2$. If G is a tree, then G is the star $K_{1,p-1}$ and the result follows from Corollary 2.12. If G is not a tree, let $c(G)$ denote the length of a longest cycle in G. Since $D = 2$, it follows that $c(G) = 3$. Let $C_3 : v_1, v_2, v_3, v_1$ be a 3-cycle in G. Since $p \geq 4$ and G is connected, there exists a vertex x such that x is adjacent to some vertex of C_3, say v_1. Then there is a path of length ≥ 3 so that $D \geq 3$, which is a contradiction. Thus when $D = 2$, $G = K_{1,p-1}$ is the only graph satisfying the requirements of the theorem.

Suppose $D = 3$. If G is a tree, then G is a double star for which $dn_w(G) = p-2$, which is a contradiction. Hence, assume that G is not a tree. Let

$c(G)$ be the length of a longest cycle in G. Since $D = 3$, it follows that $c(G) \leq 4$. We consider two cases.

Case 1. Let $c(G) = 4$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G. For $p = 4$, it is easily seen that there are no graphs G in this case for which $dn_w(G) = p - 1$. If $p \geq 5$, since G is connected, there exists a vertex x such that x is adjacent to some vertex of C_4, say v_1. Then there is a path x, v_1, v_2, v_3, v_4 of length 4 in G so that $D \geq 4$, which is a contradiction. Thus, in this case there are no graphs satisfying the requirements of the theorem.

Case 2. Let $c(G) = 3$. First, we claim that G contains exactly one triangle. If G contains two or more triangles, then $c(G) \geq 4$ or $D \geq 4$, which is a contradiction. Thus G contains a unique triangle $C_3 : v_1, v_2, v_3, v_1$. Now, if there are two or more vertices of C_3 having degree ≥ 3, then $D \geq 4$, which is a contradiction. Thus, exactly one vertex in C_3 has degree 3 or more. Since $D = 3$ and $c(G) = 3$, it follows that $G = K_{1,p-1} + e$, for which $dn_w(G) = p - 1$. Thus, when $D = 3$, $G = K_{1,p-1} + e$ is the only graph with $dn_w(G) = p - 1$.

Suppose $D = 4$. Then we proceed as in the case of Theorem 3.9 and see that there are no graphs G satisfying $dn_w(G) = p - 1$. This completes the proof. ■

From the proof of Theorem 3.13, we anticipate that there are no graphs G with detour diameter $D \geq 4$ for which $dn_w(G) = p - 1$.

Conjecture 3.14 *For any connected graph G of order $p \geq 3$, $dn_w(G) = p - 1$ if and only if $G = K_3$ or $K_{1,p-1}$ or $K_{1,p-1} + e$.*

Now we characterize the trees T for which $dn_w(T) = p - 1$.

Theorem 3.15 *For any tree T of order $p \geq 3$, $dn_w(T) = p - 1$ if and only if T is the star $K_{1,p-1}$.*

Proof. If T is the star $K_{1,p-1}$, then by Corollary 2.12, $dn_w(T) = p - 1$. Conversely, assume that $dn_w(T) = p - 1$. If $D \leq 2$, then it is proved in Theorem 3.13 that T is a star. If $D \geq 3$, then there exist at least two internal vertices of T so that there are at most $(p - 2)$ end-vertices of T so that by Corollary 2.12, $dn_w(T) \leq p - 2$, which is a contradiction. This completes the proof. ■

4 Minimal Weak Edge Detour Sets in a Graph

Definition 4.1 A weak edge detour set S in a connected graph G is called a *minimal weak edge detour set* of G if no proper subset of S is a weak edge detour set of G.

Example 4.2 For the graph G given in Figure 4.1, $S_1 = \{u, w, x, z\}$, $S_2 = \{u, w, y, z\}$, $S_3 = \{v, w, x, z\}$, $S_4 = \{v, w, y, z\}$ and $S_5 = \{u, v, x, y, z\}$ are the minimal weak edge detour sets of G.

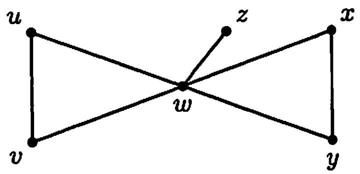


Figure 4.1: G

Remark 4.3 Every minimum weak edge detour set is a minimal weak edge detour set, but the converse is not true. For the graph G given in Figure 4.1, $S_5 = \{u, v, x, y, z\}$ is a minimal weak edge detour set of G but not a minimum weak edge detour set of G. Moreover, the minimal weak edge detour set $S_1 = \{u, w, x, z\}$ contains the cut-vertex w of G. Thus, a minimal weak edge detour set of a graph may contain a cut-vertex of G.

Definition 4.4 For a connected graph G, the *upper weak edge detour number* $dn_w^+(G)$ of G is defined to be the maximum cardinality of a minimal weak edge detour set of G.

Example 4.5 For the graph G given in Figure 4.1, clearly $dn_w(G) = 4$. On the other hand, $S = \{u, v, x, y, z\}$ is a minimal weak edge detour set of G. Since $|V| = 6$, it follows that $dn_w^+(G) = 5$.

Theorem 4.6 *For any connected graph G, $dn_w(G) \leq dn_w^+(G)$.*

Proof. Let S be any weak edge detour basis of G. Then S is also a minimal weak edge detour set of G and hence the result follows. ■

Remark 4.7 The bound in Theorem 4.6 is sharp. For any non-trivial path P, $dn_w(P) = dn_w^+(P) = 2$. Also for the graph G given in Figure 4.1, $dn_w(G) < dn_w^+(G)$.

In the following theorem, we give a class of graphs for which these two parameters are equal.

Theorem 4.8 *a) If G is the complete graph K_p $(p \geq 2)$ or the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$, then $dn_w(G) = dn_w^+(G) = 2$.*

b) If G is a tree with k end-vertices, then $dn_w(G) = dn_w^+(G) = k$.

Proof. a) Since any set of two vertices in K_p $(p \geq 2)$ or $K_{m,n}$ $(m, n \geq 2)$ is a weak edge detour set, it follows that $dn_w(G) = dn_w^+(G) = 2$.

b) By Corollary 2.12, the set of all end-vertices of G is the unique weak edge detour basis of G and so the result follows. ∎

Problem 4.9 *Characterize graphs G for which $dn_w(G) = dn_w^+(G)$.*

With the aid of next two results we prove the following theorem, which together with Theorem 4.8 gives a partial answer to the Problem 4.9.

Result 1. *Let G be an odd cycle of order $p \geq 7$. A set $S = \{u, v\}$ is a weak edge detour set of G if and only if u and v are adjacent.*

Proof. If u and v are adjacent, then every edge $e \neq uv$ of G lies on the u–v detour and the ends u and v of the edge uv belong to S. Hence S is a weak edge detour set of G.

Conversely, assume that S is a weak edge detour set of G. If u and v are not adjacent, then since G is an odd cycle, the edges of u–v geodesic do not lie on the u–v detour in G so that S is not a weak edge detour set of G, which is a contradiction. ∎

Result 2. *Let G be an even cycle of order $p \geq 6$. A set $S = \{u, v\}$ is a weak edge detour set of G if and only if u and v are adjacent or u and v are antipodal.*

Proof. If u and v are adjacent, then every edge $e \neq uv$ of G lies on the u–v detour and the ends u and v of the edge uv belong to S. If u and v are antipodal, then every edge e of G lies on a u–v detour in G. Thus S is a weak edge detour set of G.

Conversely, assume that S is a weak edge detour set of G. If u and v are not adjacent and u and v are not antipodal, then the edges of u–v geodesic do not lie on the u–v detour in G so that S is not a weak edge detour set of G, which is a contradiction. ∎

Theorem 4.10 *If G is the cycle C_p, then $dn_w(G) = 2$, $dn_w^+(G) = 2$ for $p = 3, 4, 5$ and $dn_w^+(G) = 3$ for $p \geq 6$.*

Proof. Any set of two adjacent vertices of $C_p\,(p \geq 3)$ is clearly a weak edge detour set of G so that $dn_w(G) = 2$. For $p = 3, 4, 5$, let $S \subseteq V$ be any set such that $|S| \geq 3$. Then there exists a pair of adjacent vertices u, v in S. Since $\{u, v\}$ is a weak edge detour set of G, it follows that S is not a minimal weak edge detour set of G and hence $dn_w^+(G) = 2$. Let $p \geq 6$. Now, we split into two cases:

Case 1. G is an odd cycle. Let $S \subseteq V$ be any set such that $|S| \geq 3$. If S contains two adjacent vertices, then by Result 1, S is not a minimal weak edge detour set of G. Hence any minimal weak edge detour set S of G with $|S| \geq 3$ must be an independent set. So, let S be any independent subset of V with $|S| = 3$. We claim that S is a minimal weak edge detour set of G. Let $S = \{u, v, w\}$. If w lies on the u–v detour, then all the edges of the cycle that constitute the u–v detour lie on the u–v detour and the edges on the u–v geodesic lie either on the w–u detour or w–v detour. Similarly, if w lies on the u–v geodesic, then all the edges of the cycle that constitute the u–v detour lie on the u–v detour and the edges on the u–v geodesic lie either on the w–u detour or w–v detour. Hence it follows from Result 1 that S is a minimal weak edge detour set of G. Thus we have shown that any independent subset S of V with $|S| = 3$ is a minimal weak edge detour set of G and so $dn_w^+(G) \geq 3$.

Now, if $dn_w^+(G) > 3$, let S_1 be a minimal weak edge detour set of G with $|S_1| \geq 4$. Since S_1 is an independent subset of V and since any set of three vertices of S_1 is a weak edge detour set of G, it follows that S_1 is not a minimal weak edge detour set of G, which is a contradiction. Therefore, $dn_w^+(G) = 3$.

Case 2. G is an even cycle. Let $S \subseteq V$ be any set such that $|S| \geq 3$. If S contains two adjacent vertices or two antipodal vertices, then by Result 2, S is not a minimal weak edge detour set of G. Hence any minimal weak edge detour set S of G with $|S| \geq 3$ must be an independent set and free from antipodal vertices. So, let S be any independent subset of V having no two antipodal vertices and $|S| = 3$. Then, as in Case 1, it follows from Result 2 that S is a minimal weak edge detour set of G and $dn_w^+(G) = 3$. ∎

Theorem 4.11 *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G with $dn_w(G) = a$ and $dn_w^+(G) = b$.*

Proof. Let $a = b$. Then for any tree T with a end-vertices $dn_w(G) = dn_w^+(G) = a$, by Theorem 4.8(b). So, assume that $2 \leq a < b$. Let $C : v_1, v_2, v_3, v_4, v_5, v_6, v_1$ be the cycle of length 6. The graph G is obtained from C by adding $b+1$ new vertices $z_1, z_2, \ldots, z_{a-1}, w, x_1, x_2, \ldots, x_{b-a+1}$

and joining each $z_i\,(1 \le i \le a-1)$ to v_2, w to v_1, v_3 and v_5 and each $x_i\,(1 \le i \le b-a+1)$ to both v_1 and v_3. The graph G is shown in Figure 4.2. Let $X = \{x_1, x_2, \ldots, x_{b-a+1}\}$, $Y = \{v_1, v_2, v_3\}$, $W = \{v_4, v_5, v_6, w\}$ and $Z = \{z_1, z_2, \ldots, z_{a-1}\}$.

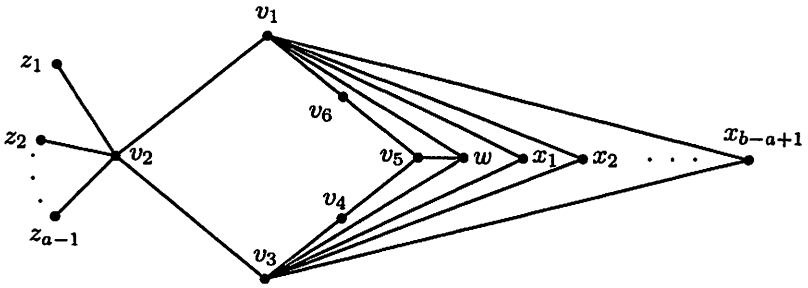


Figure 4.2: G

First, we show that $dn_w(G) = a$. By Theorem 2.11, every weak edge detour set of G contains Z. Clearly, Z is not a weak edge detour set of G and so $dn_w(G) \ge |Z| + 1 = a$. On the other hand, let $S = Z \cup \{v\}$ where $v \in W$. Then, for $v \in W$, $D(z_i, v) = 6$ if $v = v_5$ and $D(z_i, v) = 7$ if $v \ne v_5\,(1 \le i \le a-1)$. Since every edge of G lies on some z_i–v $(1 \le i \le a-1, v \in W)$ detour, S is a weak edge detour set of G and so $dn_w(G) \le |S| = a$. Therefore, $dn_w(G) = a$.

Now, we show that $dn_w^+(G) = b$. Let $S = X \cup Z$. Since $D(z_i, x_j) = 7\,(1 \le i \le a-1, 1 \le j \le b-a+1)$ and every edge of G lies on some z_i–x_j detour, S is a weak edge detour set of G. We claim that S is a minimal weak edge detour set of G. Assume, to the contrary, that S is not a minimal weak edge detour set of G. Then there is a proper subset T of S such that T is a weak edge detour set of G. Since T is a proper subset of S, there exists a vertex $s \in S$ and $s \notin T$. Since every weak edge detour set contains all end-vertices of G, we must have $s = x_i$ for $1 \le i \le b-a+1$, say $s = x_1$. Now, let $e = x_1v_1$. Since neither both the ends x_1 and v_1 of the edge e are in T nor the edge e lies on any x–y detour for $x, y \in T$, it follows that T is not a weak edge detour set of G, which is a contradiction. Thus S is a minimal weak edge detour set of G and so $dn_w^+(G) \ge |S| = a-1+b-a+1 = b$. Assume, to the contrary, that $dn_w^+(G) > b$. Let M be a minimal weak edge detour set of G with $|M| > b$. Then there exists at least one vertex, say $v \in M$ such that $v \notin S = X \cup Z$. Thus $v \in W \cup Y = \{v_1, v_2, v_3, v_4, v_5, v_6, w\}$.

Claim 1. $M \cap W = \phi$. Assume, to the contrary, that $M \cap W \ne \phi$. Then there exists a vertex $v \in M$ and $v \in W$. Clearly, $Z \cup \{v\}$ is a proper subset of M and a weak edge detour set of G by the first part of the proof of the theorem. This is a contradiction to the fact that M is a minimal weak edge detour set of G.

Claim 2. $X \not\subseteq M$. Assume, the contrary, that $X \subseteq M$. Then $X \cup Z$ is a proper subset of M and a weak edge detour set of G, which is a contradiction.

Claim 3. $M \cap X \neq \phi$. Assume, to the contrary, that $M \cap X = \phi$. Then $M = Z \cup T$, $T \subseteq Y$ and $T \neq \phi$. Then the edge v_1x_i (or v_3x_i) $(1 \leq i \leq b-a+1)$ neither lies on an x–y detour for $x, y \in M$ nor has both its ends in M. Hence M is not a weak edge detour set of G, which is a contradiction. Hence we conclude that $M = Z \cup T \cup X'$, where $T \subseteq Y$, $T \neq \phi$ and X' is a proper subset of X. Therefore, there exists a vertex $v \in X$ such that $v \notin M$, say $v = x_1$. Then the edge x_1v_1 neither lies on an x–y detour in G where $x, y \in M$ nor has both its ends x_1 and v_1 in M. Hence M is not a weak edge detour set of G, which is a contradiction. Therefore, $dn_w^+(G) = b$. ∎

Remark 4.12 The graph G in Figure 4.2 contains exactly five minimal weak edge detour sets namely $Z \cup \{v\}$, where $v \in \{v_4, v_5, v_6, w\}$ and $X \cup Z$. Hence this example shows that there is no "Intermediate Value Theorem" for minimal weak edge detour sets, that is, if k is an integer such that $dn_w(G) < k < dn_w^+(G)$, then there need not exist a minimal weak edge detour set of cardinality k in G.

Using the structure of the graph G constructed in the proof of Theorem 4.11, we can obtain a graph H_n of order n with $dn_w(G) = 2$ and $dn_w^+(G) = n - 7$ for all $n \geq 9$. Thus we have the following theorem.

Theorem 4.13 *There is an infinite sequence $\{H_n\}$ of connected graphs H_n of order $n \geq 9$ such that $dn_w(H_n) = 2$, $\lim_{n\to\infty} \frac{dn_w(H_n)}{n} = 0$ and $\lim_{n\to\infty} \frac{dn_w^+(H_n)}{n} = 1$.*

Proof. Let $n \geq 9$ and $C : v_1, v_2, v_3, v_4, v_5, v_6, v_1$ be the cycle of length 6. Then, the graph H_n is obtained from C by adding $n-6$ new vertices $z, w, x_1, x_2, \ldots, x_{n-8}$ and joining z to v_2, w to each v_1, v_3 and v_5 and each $x_i (1 \leq i \leq n-8)$ to both v_1 and v_3. The graph H_n is shown in Figure 4.3. Let $X = \{x_1, x_2, \ldots, x_{n-8}\}$, $Y = \{v_1, v_2, v_3\}$, $W = \{v_4, v_5, v_6, w\}$ and $Z = \{z\}$. It is clear from the proof of Theorem 4.11 that the graph G contains exactly five minimal weak edge detour sets namely $Z \cup \{v\}$, where $v \in \{v_4, v_5, v_6, w\}$ and $X \cup Z$ so that $dn_w(H_n) = 2$ and $dn_w^+(H_n) = n - 7$.

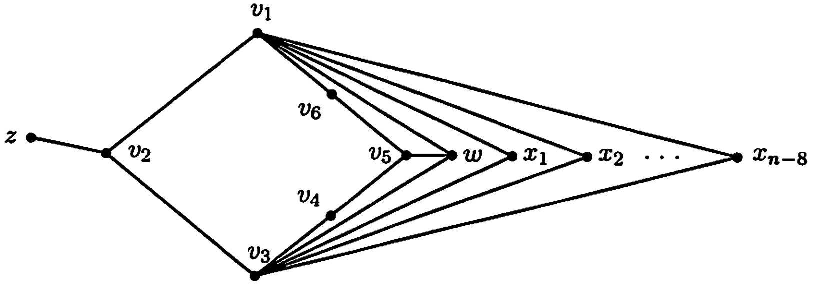


Figure 4.3: H_n

Hence the theorem follows. ∎

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