Vertex PI indices of some sums of graphs *

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Abstract: The vertex Padmakar-Ivan (PI_v) index of a graph G is defined as the summation of the sums of $[m_{eu}(e|G) + m_{ev}(e|G)]$ over all edges $e = uv$ of a connected graph G, where $m_{eu}(e|G)$ is the number of vertices of G lying closer to u than to v, and $m_{ev}(e|G)$ is the number of vertices of G lying closer to v than to u. In this paper, we give the explicit expressions of the vertex PI indices of some sums of graphs.
Key words: Vertex PI index; Sums of graphs

1 Introduction

Molecular structure descriptors, frequently called topological indices, are used in theoretical chemistry for the design of chemistry compounds with given physico-chemical properties or given pharmacologic and biological activities. There are several topological indices have been defined and many of them have found applications as means for modeling chemical, pharmaceutical and other properties of molecules, while the Wiener index [1, 2, 3, 4, 5] is the oldest and most thoroughly examined. The Szeged index [6, 7, 8, 9] is closely related to the Wiener index and is a vertex-multiplicative type that takes into account how the vertices of a given molecular graph are distributed and coincides with the Wiener index on trees. It has been considered from many points of view. Since the Szeged index takes into account how the vertices are distributed, it is natural to introduce an index that takes into account the distribution of edges. The Padmakar-Ivan (PI) index [10, 11, 12, 13] is an additive index that takes into account the distribution of edges and, therefore, complements

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the Szeged index in a certain sense. It is useful to mention that the PI index is a unique topological index related to parallelism of edges. All the indices mentioned have many chemical applications and it was shown that the PI index correlates well with the Wiener and Szeged indices and that they all correlate with the physico-chemical properties and biological activities of a large number of diverse and complex compounds. Very recently, a new topological index, the vertex PI index, was introduced [11, 12] and some of its properties were derived [14, 15, 16, 17, 18]. Its definition is similar to that of the (edge) PI index, in that it is additive, but now the distances of vertices (instead of edges) from edges is considered.

In [3], four new sums of graphs and their Wiener indices have been studied. In this paper we give the explicit expressions of the vertex PI indices of the four sums of graphs.

2 Preliminaries

We first recall some operations on graphs.

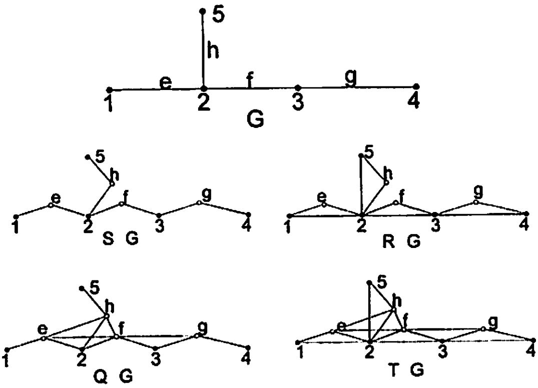

Fig.1: A graph G and $S(G)$,$R(G)$,$Q(G)$,$T(G)$

Definition 2.1. For a connected graph G, define four related graphs as follows (see Fig.1):

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.

(b) $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G, and then joining each new vertex to the end vertices of the corresponding edge.

(c) $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G, and then joining with edges those pairs of new vertices on adjacent edge of G.

(d) $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graphs $S(G)$ and $T(G)$ are also called the subdivision and total graph of G, respectively. For more details on these operations we refer the reader to [19].

Let G_1 and G_2 be two connected graphs. For convenience, in what follows we denote $V(G_i)$ and $E(G_i)$ by V_i and E_i, $i = 1, 2$, respectively. Next we make further operations on these graphs.

Definition 2.2. *Let F be one of the symbols S, R, Q or T. We denote by $G_1 +_F G_2$ the F-sum of G_1 and G_2 for which the set of vertices $V(G_1 +_F G_2) = (V_1 \bigcup E_1) \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $u_1 = v_1 \in V_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E(F(G_1))$.*

Note that $G_1 +_F G_2$ has $|V_2|$ copies of the graph $F(G_1)$, and we may label these copies by vertices of G_2. The vertices in each copy have two situations: The vertices in V_1 (we refer to these vertices as black vertices) and the vertices in E_1 (we refer to these vertices as white vertices). Now we join only black vertices with the same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2.

Let $e = uv$ be an edge of a graph G, denote by $m_{eu}(e|G)$ (or $m_{ev}(e|G)$) the number of vertices lying closer to the vertex u (or v) than to v (or u). The *vertex PI index* of G, PI_v, is defined as the summation of the sums of $m_{eu}(e|G) + m_{ev}(e|G)$ over all edges e of G. Note that in these definitions the vertices equidistant from both ends of the edge e are not counted. This implies that we can write $PI_v = \sum_{e \in E(G)} m_e(G)$, where $m_e(G) = m_{eu}(e|G) + m_{ev}(e|G)$ is the number of vertices of G that are not equidistant from both ends of the edge e.

The following three lemmas are from [3] and will be used repeatedly in the proof of our main results.

Lemma 2.3. *Let G_1 and G_2 be two connected graphs and $v = (v_1, v_2)$ be a vertex of $G_1 +_F G_2$. Then:*
(a) If $v_1 \notin E_1$ (that is v is a black vertex), then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$ we have $d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2)$.
(b) If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$, with $u_2 \neq v_2, u = u_1^1v_1^1 \in E_1$ and $u_1^1, v_1^1 \in V_1$ (that is v and u are white vertices in different copies of $F(G_1)$), we have $d(u, v|G_1 +_F G_2) = 1 + d(u_2, v_2|G_2) + min\{d(u_1^1, v_1|F(G_1)), d(v_1^1, v_1|F(G_1))\}$.
(c) If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$, where $u_2 = v_2$ and $u_1 \in E_1$ (that is v and u are white vertices in the same copy of $F(G_1)$), we have $d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) = d(u_1, v_1|F(G_1))$.

Lemma 2.4. *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E_1$, $u_2, v_2 \in V_2$ and $F = S$ or R. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G_1 +_F G_2$, with $u_2 \neq v_2$, we have*

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}$$

Lemma 2.5. *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E_1$, $u_2, v_2 \in V_2$ and $F = Q$ or T. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G_1 +_F G_2$, with $u_2 \neq v_2$, we have*

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ 1 + d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } {}^{u_1 \neq v_1}_{u_2 \neq v_2}. \end{cases}$$

3 Main results

For convenience, we introduce the following notations.

$A := \sum\{m_e(G_1 +_F G_2) : e = uv, u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2\}$,

$B := \sum\{m_e(G_1 +_F G_2) : e = uv, u = (u_1, u_2) \in V_1 \times V_2, v = (v_1, v_2) \in E_1 \times V_2\}$,

$C := \sum\{m_e(G_1 +_F G_2) : e = uv, u = (u_1, u_2), v = (v_1, v_2) \in E_1 \times V_2\}$.

Let $e = uv$ be an edge of a graph G. We denote by $N_{(u,v)}(G)$ the set of all vertices u' of G satisfying $d(u, u') = d(v, u')$, and by $n_{(u,v)}(G)$ the cardinality of $N_{(u,v)}(G)$ and let $n(G) = \sum_{uv \in E(G)} n_{(u,v)}(G)$.

Theorem 3.1. *Let G_1 and G_2 be two connected graphs. Then $PI_v(G_1 +_S G_2) = (|V_1| + |E_1|)(|V_1||V_2||E_2| - |V_1|n(G_2) + 2|V_2|^2|E_1|)$.*

Proof. (1) Suppose that $e = uv$ is an edge of $G_1 +_S G_2$, where $u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2$. Then, by the definition of S, we know $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$. For any $w = (w_1, w_2) \in V(G_1 +_s G_2)$, by Lemma 2.3 (a), we have $d(w, u|G_1 +_S G_2) = d(w_1, u_1|S(G_1)) + d(w_2, u_2|G_2)$, $d(w, v|G_1 +_S G_2) = d(w_1, v_1|S(G_1)) + d(w_2, v_2|G_2) = d(w_1, u_1|S(G_1)) + d(w_2, v_2|G_2)$. While $d(w_2, u_2|G_2) = d(w_2, v_2|G_2)$ if $d(w, u|G_1 +_S G_2) = d(w, v|G_1 +_S G_2)$. Therefore, $m_{e=uv}(G_1 +_S G_2) = (|V_1| + |E_1|)(|V_2| - n_{(u_2,v_2)}(G_2))$. This implies that $A = |V_1|(|V_1| + |E_1|)(|E_2||V_2| - n(G_2))$.

(2) Suppose that $e = uv$ is an edge of $G_1 +_S G_2$, where $u = (u_1, u_2) \in V_1 \times V_2$, $v = (v_1, v_2) \in E_1 \times V_2$. Then by the definition of S, we have $u_2 = v_2$ and u_1 is an end of v_1. For any $w = (w_1, w_2) \in V(G_1 +_s G_2)$, we consider two cases:

(i) Suppose that $w \in V_1 \times V_2$; by Lemma 2.3(a), as in the proof of (1), we have $d(w_1, u_1|S(G_1)) = d(w_1, v_1|S(G_1))$. It is clear that there exist no

vertex satisfying the above equation.
(ii) Suppose that $w \in E_1 \times V_2$; by Lemma 2.4, we have

$$d(w,v|G_1 +_S G_2) = \begin{cases} 2 + d(w_2, v_2|G_2) & \text{if } v_1 = w_1, \\ d(v_1, w_1|S(G_1)) + d(v_2, w_2|G_2) & \text{if } v_1 \neq w_1. \end{cases}$$

If $v_1 = w_1$, for $d(w,u|G_1 +_S G_2) = d(w,v|G_1 +_S G_2)$, $d(w_1, u_1|S(G_1)) = 2$. By the definition of S, there exist no vertex such that the above equation.

If $v_1 \neq w_1$, by (i), there exist no vertex satisfying the above equation. Therefore, $B = |V_2|(|V_1| + |E_1|) \cdot 2|E_1||V_2| = 2|E_1||V_2|^2(|V_1| + |E_1|)$.
(3) By the definition of S, we have $C = 0$. Using the above result we can compute the vertex Padmakar-Ivan index (PI_v) of $G_1 +_S G_2$.

$PI_v(G_1 +_S G_2) = A + B + C$

$= |V_1|(|V_1| + |E_1|)(|E_2||V_2| - n(G_2)) + |V_2|^2(|V_1| + |E_1|) \cdot 2|E_1|$

$= (|V_1| + |E_1|)(|V_1||V_2||E_2| - |V_1|n(G_2) + 2|V_2|^2|E_1|)$. □

Theorem 3.2. *Let G_1 and G_2 be two connected graphs. Then* $PI_v(G_1 +_R G_2) = (|V_1| + |E_1|)[(|V_1||V_2||E_2| - |V_1|n(G_2 + 3|V_2|^2|E_1|)] - |V_2|^2 n(R(G_1))$.

Proof. (1)We break down the values A into two sums $A = A_1 + A_2$, where
$A_1 = \sum\{m_e(G_1 +_R G_2) : e = uv, u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2, u_1 = v_1\}$
$A_2 = \sum\{m_e(G_1 +_R G_2) : e = uv, u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2, u_2 = v_2\}$.

As in the proof of Theorem 3.1 (1), $A_1 = |V_1|(|V_1| + |E_1|)(|E_2||V_2| - n(G_2))$.

$\forall u, v \in V_1 \times V_2, u_2 = v_2$, $\forall w = (w_1, w_2) \in V(G_1 +_R G_2)$, by Lemma 2.3(a), we have
$d(w,u|G_1 +_R G_2) = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2)$,
$d(w,v|G_1 +_R G_2) = d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2)$.
Thus $d(w_1, u_1|R(G_1)) = d(w_1, v_1|R(G_1))$if $d(w,u|G_1 +_R G_2) = d(w,v|G_1 +_R G_2)$. Therefore, $m_{e=uv}(G_1 +_R G_2) = |V_2|(|V_1| + |E_1|) - |V_2| n_{(u_1,v_1)}(R(G_1))$. This implies that

$$A_2 = |E_1||V_2|^2(|V_1| + |E_1|) - |V_2|^2 \sum_{(x,y)\in E(R(G_1)), x,y \in V_1} n_{(x,y)}(R(G_1)).$$ So,

$$A = (|V_1| + |E_1|)[(|V_1||V_2||E_2| - |V_1|n(G_2) + |V_2|^2|E_1|] - |V_2|^2 \sum_{(x,y)\in E(R(G_1)), x,y \in V_1} n_{(x,y)}(R(G_1)).$$

(2) Suppose that $e = uv$ is an edge of $G_1 +_R G_2$, where $u = (u_1, u_2) \in V_1 \times V_2$, $v = (v_1, v_2) \in E_1 \times V_2$. Then by the definition of R, we have $u_2 = v_2$ and u_1 is an end of v_1 in G_1. For any $w = (w_1, w_2) \in V(G_1 +_R G_2)$, we consider the following two cases:
(i) Suppose that $w \in V_1 \times V_2$; by Lemma 2.3(a), similar to the proof of Theorem 3.1 (2) (i), we have $d(u_1, w_1|R(G_1)) = d(v_1, w_1|R(G_1))$.

(ii) Suppose that $w \in E_1 \times V_2$; by Lemma 2.4, we have

$$d(w, v|G_1 +_R G_2) = \begin{cases} 2 + d(w_2, v_2|G_2) & \text{if } w_1 = v_1, \\ d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2) & \text{if } w_1 \neq v_1. \end{cases}$$

If $w_1 = v_1$, by the equality $d(w, u|G_1 +_R G_2) = d(w, v|G_1 +_R G_2)$, we have $d(w_1, u_1|R(G_1)) = 2$. According to the definition of R, there does not exist such vertex satisfying the equation.

If $w_1 \neq v_1$, as in the proof of Theorem 3.2(1), we have $d(w_1, u_1|R(G_1)) = d(w_1, v_1|R(G_1))$. Hence,
$m_{e=uv}(G_1 +_R G_2) = |V_2|(|V_1| + |E_1|) - |V_2|n_{(u_1,v_1)}(R(G_1))$, which implies that $B = 2|E_1||V_2|^2(|V_1| + |E_1|) - |V_2|^2 \sum_{\substack{(x,y)\in E(R(G_1)) \\ x\in V_1, y\in E_1}} n_{(x,y)}(R(G_1))$.

(3) By the definition of R, we have $C = 0$. Using the above results we can obtain: $PI_v(G_1 +_R G_2) = A + B + C = (|V_1| + |E_1|)(|V_1||V_2||E_2| - |V_1|n(G_2) + 3|V_2|^2|E_1|) - |V_2|^2 n(R(G_1))$. □

Recall that the *Padmakar-Ivan (PI) index* of a graph G, $PI(G)$, is the summation of the sums of $n_{eu}(e|G) + n_{ev}(e|G)$ over all the edges $e = uv$ of G, where $n_{eu}(e|G)$ (or $n_{ev}(e|G)$) is the number of edges lying closer to the vertex u (or v) than the vertex v (or u). In this definition, edges equidistant from both ends of the edge $e = uv$ are not counted. One of the oldest graph invariants is the *first Zagreb index*, which was introduced by Gutman and Trinajstić [20], and it is defined as $M_1(G) = \sum_{v\in V(G)} deg(v)^2$ for a graph G. The graphs with a fixed number of edges and vertices, with smallest Zagreb index are completely characterized in [21]. For more details and related results see [22, 23, 24]. Now we compute the PI_v index of $G_1 +_F G_2$, where $F = Q$ or T.

Theorem 3.3. *Let G_1 and G_2 be two connected graphs. Then* $PI_v(G_1 +_Q G_2) = (|V_1| + |E_1|)[|V_1||V_2||E_2| - |V_1|n(G_2) + |V_2|^2(\frac{1}{2}M_1(G_1) + |E_1|)] - 2|E_1||V_2|(|V_2|-1)(|E_1|-1) - |V_2|^2(PI_v(G_1) - 2|E_1|) + (|V_2|^2 - |V_2|)PI(G_1) - |V_2|^2 \sum_{xy\in E(Q(G_1)), x,y\in E_1} n_{(x,y)}(Q(G_1)) - 2|V_2|(|V_2| - 1)(\frac{1}{2}M_1(G_1) - |E_1|)$.

Proof. (1)As in the proof of Theorem 3.1(1), we have $A = |V_1|(|V_1| + |E_1|)(|E_2||V_2| - n(G_2))$.

(2) Suppose that $e = uv$ is an edge of $G_1 +_Q G_2$, where $u = (u_1, u_2) \in V_1 \times V_2$, $v = (v_1, v_2) \in E_1 \times V_2$. Then by the definition of Q, we have $u_2 = v_2$ and u_1 is an end of v_1 in G_1. For any $w = (w_1, w_2) \in V(G_1 +_Q G_2)$, we consider two cases:

(i) Suppose that $w \in V_1 \times V_2$; by Lemma 2.3 (a), as in the proof of Theorem 3.1 (2) (i), we have $d(w_1, u_1|Q(G_1)) = d(w_1, v_1|Q(G_1))$. Suppose that $v_1 = (u_1, u_1')$, by the definition of Q, w_1 is a vertex lying closer to the vertex u_1 than to the vertex u_1' in G_1, but $w_1 \neq u_1$.

(ii) Suppose that $w \in E_1 \times V_2$; by Lemma 2.5, we have

$$d(w,v|G_1+_Q G_2)=\begin{cases}2+d(w_2,v_2|G_2) & \text{if } w_1=v_1,\\ 1+d(w_1,v_1|Q(G_1))+d(w_2,v_2|G_2) & \text{if } wv_1\neq v_1.\end{cases}$$

If $w_1=v_1$, for $d(w,u|G_1+_Q G_2)=d(w,v|G_1+_Q G_2)$, $d(w_1,u_1|Q(G_1))=2$. There does not exist such vertex satisfying the equation.

If $w_2=v_2$, for $d(w,u|G_1+_Q G_2)=d(w,v|G_1+_Q G_2)$, $d(w_1,u_1|Q(G_1))=d(w_1,v_1|Q(G_1))$. Suppose that $v_1=(u_1,u_1')$, by the definition of Q, w_1 is an edge lying closer to the vertex u_1 than to the vertex u_1' in G_1.

If $w_1\neq v_1$ and $w_2\neq v_2$, for $d(w,u|G_1+_Q G_2)=d(w,v|G_1+_Q G_2)$, we have $d(w_1,u_1|Q(G_1))=1+d(w_1,v_1|Q(G_1))$. This implies that $B=2|E_1||V_2|^2(|V_1|+|E_1|)-2|E_1||V_2|(|V_2|-1)(|E_1|-1)-|V_2|^2(PI_v(G_1)-2|E_1|)+(|V_2|^2-|V_2|)PI(G_1)$.

(3) Suppose that $e=uv$ is an edge of $G_1+_Q G_2$, where $u=(u_1,u_2),v=(v_1,v_2)\in E_1\times V_2$. Then by the definition of Q, we have $u_2=v_2$.

(i)Suppose that $w=(w_1,w_2)\in V_1\times V_2$, we have $d(w,u|G_1+_Q G_2)=d(w_1,u_1|Q(G_1))+d(w_2,u_2|G_2)$, $d(w,v|G_1+_Q G_2)=d(w_1,v_1|Q(G_1))+d(w_2,v_2|G_2)$. and $d(w,u|G_1+_Q G_2)=d(w,v|G_1+_Q G_2)$, if and only if $d(w_1,u_1|Q(G_1))=d(w_1,v_1|Q(G_1))$.

(ii)Suppose that $w=(w_1,w_2)\in E_1\times V_2$, if $w_1=u_1$, by Lemma 2.5, we have $d(w,u|G_1+_Q G_2)=2+d(w_2,u_2|G_2)$, $d(w,v|G_1+_Q G_2)=1+d(w_1,v_1|Q(G_1))+d(w_2,v_2|G_2)$, and $d(w,u|G_1+_Q G_2)=d(w,v|G_1+_Q G_2)$ if and only if $d(v_1,w_1|Q(G_1))=1$, which is obvious.

If $v_1=w_1$, $d(v,w|G_1+_Q G_2)=2+d(v_2,w_2|G_2)$, $d(u,w|G_1+_Q G_2)=1+d(u_1,w_1|Q(G_1))+d(u_2,w_2|G_2)$. Thus $d(u_1,w_1|Q(G_1))=1$ if $d(u,w|G_1+_Q G_2)=d(v,w|G_1+_Q G_2)$. It is obvious that $d(u_1,w_1|Q(G_1))=1$.

If $v_1\neq w_1$ and $u_1\neq w_1$, we have $d(u,w|G_1+_Q G_2)=1+d(u_1,w_1|Q(G_1))+d(u_2,w_2|G_2)$, $d(v,w|G_1+_Q G_2)=1+d(v_1,w_1|Q(G_1))+d(v_2,w_2|G_2)$, and if $d(u,w|G_1+_Q G_2)=d(v,w|G_1+_Q G_2)$, then $d(u_1,w_1|Q(G_1))=d(v_1,w_1|Q(G_1))$. So, $m_{(u,v)}(G_1+_Q G_2)=|V_2|(|V_1|+|E_1|)-2(|V_2|-1)-|V_2|n_{(u_1,v_1)}(Q(G_1))$. This implies that:

$$\begin{aligned}C &= \sum_{uv\in E(G_1+_Q G_2),u,v\in E_1\times V_2}[|V_2|(|V_1|+|E_1|)-2(|V_2|-1)]\\ &\quad -|V_2|^2\sum_{xy\in E(Q(G_1)),x,y\in E_1}n_{(x,y)}(Q(G_1))\\ &= |V_2|(\tfrac{1}{2}M_1(G_1)-|E_1|)[|V_2|(|V_1|+|E_1|)-2(|V_2|-1)]\\ &\quad -|V_2|^2\sum_{xy\in E(Q(G_1)),x,y\in E_1}n_{(x,y)}(Q(G_1)).\end{aligned}$$

By using the above results, we obtain the desired expression: $PI_v(G_1+_Q G_2)=A+B+C=(|V_1|+|E_1|)[|V_1||V_2||E_2|-|V_1|n(G_2)+|V_2|^2(\frac{1}{2}M_1(G_1)+|E_1|)]-2|E_1||V_2|(|V_2|-1)(|E_1|-1)-|V_2|^2(PI_v(G_1)-2|E_1|)+(|V_2|^2-|V_2|)PI(G_1)-|V_2|^2\sum_{\substack{xy\in E(Q(G_1))\\ x,y\in E_1}}n_{(x,y)}(Q(G_1))-2|V_2|(|V_2|-1)(\frac{1}{2}M_1(G_1)-|E_1|)$. □

Theorem 3.4. *Let G_1 and G_2 be two connected graphs. Then*

$$PI_v(G_1 +_T G_2) = (|V_1|+|E_1|)[|V_1||V_2||E_2|-|V_1|n(G_2)+|V_2|^2(\frac{1}{2}M_1(G_1)+|E_1|)+3|V_2|^2|E_1|]-2|E_1||V_2|(|V_2|-1)(|E_1|-1)-|V_2|^2(PI_v(G_1)-2|E_1|)+(|V_2|^2-|V_2|)PI(G_1)-|V_2|^2 \sum_{\substack{xy\in E(T(G_1))\\ x,y\in E_1 \, or \, V_1}} n_{(x,y)}(T(G_1)) - 2|V_2|(|V_2|-1)(\frac{1}{2}M_1(G_1)-|E_1|).$$

Proof. (1)As in the proof of the Theorem 3.2(1), we have

$$A = A_1 + A_2 = (|V_1|+|E_1|)[(|V_1||V_2||E_2| - |V_1|n(G_2) + |V_2|^2|E_1|] - |V_2|^2 \sum_{\substack{(x,y)\in E(T(G_1))\\ x,y\in V_1}} n_{(x,y)}(T(G_1)).$$

(2) As in the proof of Theorem 3.3 (2), we have

$$B = 2|E_1||V_2|^2(|V_1|+|E_1|) - 2|E_1||V_2|(|V_2|-1)(|E_1|-1) - |V_2|^2(PI_v(G_1) - 2|E_1|) + (|V_2|^2 - |V_2|)PI(G_1).$$

(3)As in the proof of Theorem 3.3 (3), we have

$$C = \sum_{\substack{uv\in E(G_1+_TG_2)\\ u,v\in E_1\times V_2}} [|V_2|(|V_1|+|E_1|)-2|V_2|+2]-|V_2|^2 \sum_{\substack{xy\in E(T(G_1))\\ x,y\in E_1}} n_{(x,y)}(T(G_1))$$

$$= |V_2|(\frac{1}{2}M_1(G_1) - |E_1|)[|V_2|(|V_1|+|E_1|) - 2(|V_2|-1)] - |V_2|^2 \sum_{\substack{xy\in E(T(G_1))\\ x,y\in E_1}} n_{(x,y)}(T(G_1)).$$

By using the above results, we obtain the desired expression of the vertex Padmakar-Ivan index (PI_v) of $G_1 +_T G_2$. $PI_v(G_1 +_T G_2) = A + B + C = (|V_1| + |E_1|)[|V_1||V_2||E_2| - |V_1|n(G_2) + |V_2|^2(\frac{1}{2}M_1(G_1) + |E_1|) + 3|V_2|^2|E_1|] - 2|E_1||V_2|(|V_2|-1)(|E_1|-1) - |V_2|^2(PI_v(G_1) - 2|E_1|) + (|V_2|^2 - |V_2|)PI(G_1) - |V_2|^2 \sum_{\substack{xy\in E(T(G_1))\\ x,y\in E_1 \, or \, V_1}} n_{(x,y)}(T(G_1)) - 2|V_2|(|V_2|-1)(\frac{1}{2}M_1(G_1) - |E_1|)$. □

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