Roman domination in a tree *

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Abstract

A Roman dominating function on a graph $G = (V, E)$ is a function $f: V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph G, denoted by $\gamma_R(G)$, is called the Roman domination number of G. In [E.J. Cockayne, P.A. Dreyer, Jr., S.M. Hedetniemi, S.T. Hedetniemi, Roman domination in graphs, Discrete Math. 278(2004) 11-22.], the authors stated a proposition which characterized trees which satisfy $\gamma_R(T) = \gamma(T) + 2$, where $\gamma(T)$ is the domination number of T. The authors thought the proof of the proposition was rather technical and chose to omit it's proof, however, the proposition is actually incorrect. In this paper, we will

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give a counterexample of this proposition and introduce the correct characterization of a tree T with $\gamma_R(T) = \gamma(T) + 2$.

Keywords: Roman dominating function, Roman domination number, Domination number, healthy spider, wounded spider.

1 Introduction

Graphs considered in this paper are finite and simple. For a graph G, $V = V(G)$ and $E = E(G)$ will denote its sets of vertices and edges, respectively. For $S \subseteq V$, set $N(S) = \{u \in V \setminus S$: There is $v \in S$ such that $uv \in E\}$, $N[S] = S \cup N(S)$. For any vertex $v \in V$, $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, the degree of a vertex v in G is denoted by $d(v)$. A set $S \subseteq V$ is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of order $\gamma(G)$ is called a γ–set of G. A set S of vertices is called a 2-packing of G, if for every pair of vertices u, $v \in S$, the distance of u and v in G, denoted by $d_G(u, v)$, is not smaller than 3.

Roman domination has been introduced in [1] as a new variety of the classical domination problem having both historical and mathematical interest. A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \mapsto \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a graph $G = (V, E)$, let f be a Roman domination function of G and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V : f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. We will write $f = (V_0, V_1, V_2)$. The weight of f is $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$. The minimum weight of an RDF of G is called the Roman domination number of G and is denoted by $\gamma_R(G)$. We say f is a $\gamma_R(G)-$ function if it is an RDF of G and $f(V) = \gamma_R(G)$. We refer to [1-3] for more background on the historical importance of the Roman domination problem and other results not mentioned here.

For a graph G, let $A(G) = \cup\{S \subseteq V(G) :\ S$ is a γ–set of $G\}$ and $B(G) = \cup\{V_2 \subseteq V(G) :\ f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$–function

}. That is, $A(G)$ is the set of vertices in some minimum dominating set and $B(G)$ is the set of vertices which receive a weight of 2 for some RDF.

For a positive integer t, a wounded spider is a star $K_{1,t}$ with at most $t-1$ of its edges subdivided. A star $K_{1,t}$ is a special case of a wounded spider. Similarly, for an integer $t \geq 2$, a healthy spider is a star $K_{1,t}$ with all of its edges subdivided. In a wounded spider, a vertex of degree t will be called a head vertex, and a vertex that is distance two from the head vertex will be called a foot vertex. The head and foot vertices are well defined except when the wounded spider is the path on two or four vertices. For P_2, we will consider both vertices to be head vertices, and in the case of P_4, we will consider both end vertices as foot vertices and both interior vertices as head vertices. Similarly, in a healthy spider, the vertex of degree t will be called the head vertex, and the vertices that are distance two from the head vertex will be the foot vertices. Note that, since $t \geq 2$, the head and foot vertices are well defined in a healthy spider. See Figure 1.

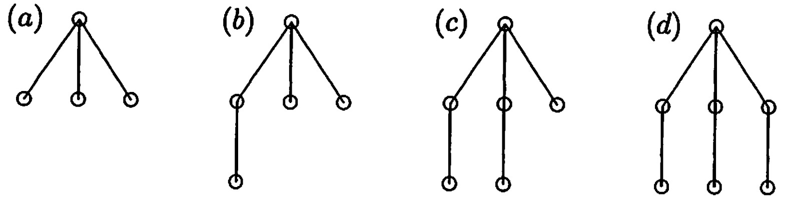

Figure 1: Wounded spiders (a)-(c) and a healthy spider (d)

In proposition 15 of [1] (reproduced as Proposition 3.1 in this paper) the authors obtained a characterization of trees for which $\gamma_R(T) = \gamma(T) + 2$. In this paper, we give a counterexample to this proposition and introduce the correct characterization of a tree T with $\gamma_R(T) = \gamma(T) + 2$.

2 Preliminaries

Firstly, we will summarize some useful facts.

Proposition 2.1 [1] For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Proposition 2.2 [1] For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.

Proposition 2.3 [1] If T is a tree on two or more vertices, then $\gamma_R(T) = \gamma(T) + 1$ if and only if T is a wounded spider.

Proposition 2.4 If T is a simple connected graph obtained from T_1 and T_2 by adding a new edge joining $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$, then we have the following conclusions.

(A) $\gamma_R(T_1) + \gamma_R(T_2) - \gamma_R(T) \in \{0,\ 1\}$. $\gamma(T_1) + \gamma(T_2) - \gamma(T) \in \{0,\ 1\}$.

(B) If $v_1 \in B(T_1)$ and $\gamma_R(T_2 - v_2) = \gamma_R(T_2) - 1$, then $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) - 1$.

If $v_1 \notin B(T_1)$ and $\gamma_R(T_1 - v_1) \geq \gamma_R(T_1)$, then $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2)$.

If $v_1 \notin B(T_1)$ and $v_2 \notin B(T_2)$, then $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2)$.

If $\gamma_R(T_1 - v_1) \geq \gamma_R(T_1)$ and $\gamma_R(T_2 - v_2) \geq \gamma_R(T_2)$, then $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2)$.

(C) If $v_1 \in A(T_1)$ and $\gamma(T_2 - v_2) = \gamma(T_2) - 1$, then $\gamma(T) = \gamma(T_1) + \gamma(T_2) - 1$.

If $v_1 \notin A(T_1)$ and $\gamma(T_1 - v_1) \geq \gamma_R(T_1)$, then $\gamma(T) = \gamma(T_1) + \gamma(T_2)$.

If $v_1 \notin A(T_1)$ and $v_2 \notin A(T_2)$, then $\gamma(T) = \gamma(T_1) + \gamma(T_2)$.

If $\gamma(T_1 - v_1) \geq \gamma(T_1)$ and $\gamma(T_2 - v_2) \geq \gamma(T_2)$, then $\gamma(T) = \gamma(T_1) + \gamma(T_2)$.

Proof (A)We need only to prove that $\gamma_R(T_1) + \gamma_R(T_2) - \gamma_R(T) \leq 1$. Otherwise, suppose that $h = (W_0,\ W_1,\ W_2)$ is a $\gamma_R(T)$-function such that $h(T) \leq \gamma_R(T_1) + \gamma_R(T_2) - 2$, let $f = (W_0 \cap V(T_1),\ W_1 \cap V(T_1),\ W_2 \cap V(T_1))$ and $g = (W_0 \cap V(T_2),\ W_1 \cap V(T_2),\ W_2 \cap V(T_2))$, then $f(V(T_1)) + g(V(T_2)) = h(V(T)) \leq \gamma_R(T_1) + \gamma_R(T_2) - 2$. We can suppose that $g(V(T_2)) \leq \gamma_R(T_2) - 1$, then $h(v_2) = 0$ and f is an RDF of T_1, and then $f(V(T_1)) \geq \gamma_R(T_1)$, and then we have $g(V(T_2)) \leq \gamma_R(T_2) - 2$, a contradiction.

(B) If $v_1 \in B(T_1)$ and $\gamma_R(T_2 - v_2) = \gamma_R(T_2) - 1$, let $f =$

$(U_0,\ U_1,\ U_2)$ be a $\gamma_R(T_1)$–function with $v_1 \in U_2$ and $g = (V_0,\ V_1,\ V_2)$ be a $\gamma_R(T_2-v_2)$–function, then $h = (U_0 \cup V_0 \cup \{v_2\},\ U_1 \cup V_1,\ U_2 \cup V_2)$ is an RDF of T, then $\gamma_R(T) \leq h(V(T)) = f(V(T_1)) + g(V(T_2)) = \gamma_R(T_1) + \gamma_R(T_2) - 1$. Therefore, $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) - 1$. On the other hand, if $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) - 1$, suppose that $h = (W_0,\ W_1,\ W_2)$ is a $\gamma_R(T)$–function such that $|W_1|$ is a minimum, let $f = (W_0 \cap V(T_1),\ W_1 \cap V(T_1),\ W_2 \cap V(T_1))$ and $g = (W_0 \cap V(T_2),\ W_1 \cap V(T_2),\ W_2 \cap V(T_2))$, then $f(V(T_1)) + g(V(T_2)) = h(V(T)) = \gamma_R(T_1) + \gamma_R(T_2) - 1$. We can suppose that $g(V(T_2)) = \gamma_R(T_2) - 1$, then f is an RDF of T_1 and $v_1 \in B(T_1)$. Moreover, $\gamma_R(T_2 - v_2) = g(V(T_2)) = \gamma_R(T_2) - 1$. (B) is proved.

(C) Similar to the proof of (B). Proposition 2.4 is proved.

The following three propositions are useful but easy to prove and we omit the details.

Proposition 2.5 If T is a wounded spider with a well defined head vertex v_0, and f is a $\gamma_R(T)$–function, then $f(v_0) = 2$.

Proposition 2.6 If T is a healthy spider with $|V(T)| \neq 5$ and f is a $\gamma_R(T)$–function, then $f(v_0) = 2$, where v_0 is the head vertex.

Proposition 2.7 T is a tree of order $n \geq 2$ with an RDF $f = (V_0,\ V_1,\ V_2)$ such that $|V_2| = 1$ and V_1 is a 2-packing. Then T is either a healthy spider or a wounded spider.

3 Counterexamples

First, let us consider the following Proposition.

Proposition 3.1 [1] If T is a tree on two or more vertices, then $\gamma_R(G) = \gamma(G) + 2$ if and only if either (A) T is a healthy spider or (B) T is a pair of wounded spiders T_1 and T_2, with a single edge joining $v \in V(T_1)$ and $w \in V(T_2)$, subject to the following conditions:

(1) If either tree is a P_2, then neither vertex in P_2 is joined to the head vertex of the other tree.

(2) v and w are not both foot vertices.

Here we will introduce a counterexample of Proposition 3.1.

Counterexample 3.2 Let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be a pair of wounded spiders with $T_1 \cong T_2$. For each $i = 1, 2$, $V_i = \{a_i, b_i, c_i, d_i, e_i, f_i\}$, $E_i = \{a_ib_i, b_ic_i, a_id_i, d_ie_i, a_if_i\}$. Let $G_0 = (V_0, E_0) = (V_1 \cup V_2, E_1 \cup E_2 \cup \{f_1f_2\})$. See Figure 2.

Figure 2: A tree G_0 with $\gamma_R(G_0) = \gamma(G_0) + 3$

G_0 is a tree which satisfies the conditions of Proposition 3.1, that is G_0 is formed from a pair of wounded spiders T_1 and T_2 with a single edge joining $f_1 \in V(T_1)$ and $f_2 \in V(T_2)$, further, neither f_1 nor f_2 is a foot vertex. The set $\{b_1, d_1, b_2, d_2, f_1\}$ is a γ−set of G_0 and the function which assigns 2 to the the vertices of degree three, 1 to the vertices of degree one, and zero otherwise, is a γ_R−function of G_0. Thus $\gamma_R(G_0) = \gamma(G_0) + 3 = 8$.

4 Main Results and Proof

Here we provide the correct characterization of trees T which satisfy $\gamma_R(T) = \gamma(T) + 2$.

Theorem 4.1 T is a tree with $\gamma_R(T) = \gamma(T) + 2$ if and only if at least one of the following cases is satisfied.

Case 1 T is a healthy spider.

Case 2 $T = (V(T), E(T)) = (V(T_1) \cup V(T_2) \cup V(F_i), E(T_1) \cup E(T_2) \cup E(F_i))$, where $1 \leq i \leq 7$ and T_j is a wounded spider with a head vertex u_j for each $j = 1, 2$. Moreover, T is not a wounded spider.

Case 3 $T = (V(T), E(T)) = (V(T_1) \cup V(T_2) \cup V(F_4), E(T_1) \cup E(T_2) \cup E(F_4))$, where T_1 is a wounded spider with a head vertex u_1 and T_2 is a healthy spider with a head vertex u_2.

where F_i, $1 \leq i \leq 7$, are joint graphs as follows. See Figure 3.

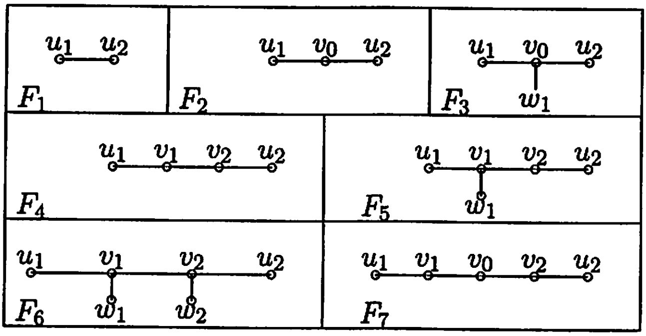

Figure 3: Joint graphs F_i, for $1 \leq i \leq 7$.

$F_1 = (V(F_1),\ E(F_1)) = (\{u_1,\ u_2\},\ \{u_1u_2\})$.

$F_2 = (V(F_2),\ E(F_2)) = (\{u_1,\ v_0,\ u_2\},\ \{u_1v_0,\ v_0u_2\})$.

$F_3 = (V(F_3),\ E(F_3)) = (\{u_1,\ u_2,\ v_0,\ w_1\},\ \{u_1v_0,\ v_0u_2,\ v_0w_1\})$.

$F_4 = (V(F_4),\ E(F_4)) = (\{u_1,\ v_1,\ v_2,\ u_2\},\ \{u_1v_1,\ v_1v_2,\ v_2u_2\})$.

$F_5 = (V(F_5),\ E(F_5)) = (\{u_1,\ v_1,\ v_2,\ u_2,\ w_1\},\ \{u_1v_1,\ v_1v_2,\ v_2u_2,\ v_1w_1\})$.

$F_6 = (V(F_6),\ E(F_6)) = (\{u_1,\ v_1,\ w_1,\ u_2,\ v_2,\ w_2\},\ \{u_1v_1,\ v_1w_1,\ u_2v_2,\ v_2w_2,\ v_1v_2\})$.

$F_7 = (V(F_7),\ E(F_7)) = (\{u_1,\ v_1,\ v_0,\ v_2,\ u_2\},\ \{u_1v_1,\ v_1v_0,\ v_0v_2,\ v_2u_2\})$.

Proof " $\Rightarrow$ " Let T be a tree of order $n \geq 2$ with $\gamma_R(T) = \gamma(T)+2$ such that T is not a healthy spider. Let $f = (V_0,\ V_1,\ V_2)$ be a $\gamma_R(T)-$ function such that $|V_1|$ is minimized. Then, by Proposition 4(c) of [1], V_1 is a 2-packing of T, and thus $\gamma(T) \geq |V_1|$. Moreover, $\gamma_R(T) - 2 = |V_1| + 2|V_2| - 2 = \gamma(T) \leq |V_1| + |V_2|$. Therefore, $1 \leq |V_2| \leq 2$. By Proposition 2.3, T is not a wounded spider and T is not a healthy spider by supposition, thus T is not a spider and we have $|V_2| > 1$.

Let $V_2 = \{u_1,\ u_2\}$, then $V_0 = N_T(V_2)$ and $V_1 = V(T) - V_0 - V_2$. Note that $1 \leq d_T(u_1,\ u_2) \leq 4$, let P be the path from u_1 to u_2 in T.

For each $j = 1,\ 2$, let T_j be the component of $T - E(P)$ containing u_j, let $f_j = (V_0 \cap V(T_j),\ V_1 \cap V(T_j),\ V_2 \cap V(T_j))$, then $f_j(u_j) = 2$, and then f_j is an RDF of T_j. Note that by Proposition 2.7, T_j is either a wounded spider or a healthy spider. Moreover, V_1 is a 2-packing. Therefore, $T = (V(T),\ E(T)) = (V(T_1) \cup V(T_2) \cup V(F_i),\ E(T_1) \cup E(T_2) \cup E(F_i))$, where $1 \leq i \leq 7$ and $T_1,\ T_2$ are spiders with head vertices $u_1,\ u_2$, respectively. For $1 \leq i \leq 7$, we claim that both T_1 and T_2 are wounded spiders, except for $i = 4$, where T_1 is a wounded spider and T_2 maybe a healthy spider or a wounded spider. Assume this claim is not true. There are two cases.

Case A. T_1 is a wounded spider and T_2 is a healthy spider. Moreover, $i \neq 4$. In this case, we have $u_1 \in A(T_1)$ and $u_2 \notin A(T_2)$.

If $i = 1$, then we have $\gamma(T) \leq \gamma(T_1) + \gamma(T_2)$ by Proposition 2.4(A). Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) \geq \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 3 \geq \gamma(T) + 3 = \gamma_R(T) + 1$, a contradiction.

If $i = 2$, then we have $\gamma(T) = \gamma(T_1) + \gamma(T_2)$. Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) \geq \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 3 = \gamma(T) + 3 = \gamma_R(T) + 1$, a contradiction.

If $i = 3,\ 5,\ 7$, then we have $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 1$. Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) + 1 \geq \gamma_R(T_1) + \gamma_R(T_2) + 1 = \gamma(T_1) + \gamma(T_2) + 4 = \gamma(T) + 3 = \gamma_R(T) + 1$, a contradiction.

If $i = 6$, then we have $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 2$. Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) + 2 \geq \gamma_R(T_1) + \gamma_R(T_2) + 2 = \gamma(T_1) + \gamma(T_2) + 5 = \gamma(T) + 3 = \gamma_R(T) + 1$, a contradiction.

Case B. Both T_1 and T_2 are healthy spiders. In this case, we have $u_1 \notin A(T_1)$ and $u_2 \notin A(T_2)$.

If $i = 1$, then we have $\gamma(T) \leq \gamma(T_1) + \gamma(T_2)$ by Proposition 2.4(A). Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) \geq \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 4 \geq \gamma(T) + 4 = \gamma_R(T) + 2$, a contradiction.

If $i = 2,\ 4$, then we have $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 1$. Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) \geq \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 4 = \gamma(T) + 3 = \gamma_R(T) + 1$, a contradiction.

If $i = 3,\ 5,\ 7$, then we have $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 1$. Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) + 1 \geq \gamma_R(T_1) + \gamma_R(T_2) + 1 = \gamma(T_1) + \gamma(T_2) + 5 = \gamma(T) + 4 = \gamma_R(T) + 2$, a contradiction.

If $i = 6$, then we have $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 2$. Therefore, $\gamma_R(T) = f(V(T)) = f_1(V(T_1)) + f_2(V(T_2)) + 2 \geq \gamma_R(T_1) + \gamma_R(T_2) + 2 = \gamma(T_1) + \gamma(T_2) + 6 = \gamma(T) + 4 = \gamma_R(T) + 2$, a contradiction.

" $\Leftarrow$ " The case that T is a healthy spider is trivial, we can suppose that T satisfied Case 2 or Case 3.

Let $V_2 = \{u_1,\ u_2\}$, $V_0 = N_T(V_2)$ and $V_1 = V(T) - V_0 - V_2$. Then $f = (V_0,\ V_1,\ V_2)$ is an RDF of T. For each $j = 1,\ 2$, $f_j = (V_0 \cap V(T_j),\ V_1 \cap V(T_j),\ \{u_j\})$ is a $\gamma_R(T_j)$–function. Moreover, by Proposition 2.5 and Proposition 2.6, for each $j = 1,\ 2$, we have $\gamma_R(T_j - u_j) = \gamma_R(T_j) - 1$ if and only if T_j is the path on two vertices.

Case 2. In this case, we have $u_1 \in A(T_1)$ and $u_2 \in A(T_2)$.

If $i = 1$, we have $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 2 = \gamma(T) + 2$ by Proposition 2.4. Note that we have $|V(T_1)| \geq 3$ and $|V(T_2)| \geq 3$. (Otherwise, T will be a wounded spider.)

If $i = 2,\ 4$, we can suppose that $|V(T)| \geq 6$. (Otherwise, T will be a healthy spider.) We also have $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 2 = \gamma(T) + 2$.

If $i = 3,\ 5,\ 7$, we have $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) + 1 = \gamma(T_1) + \gamma(T_2) + 3 = \gamma(T) + 2$.Note that we have $|V(T_1)| + |V(T_2)| \geq 5$ for $i = 3$. (Otherwise, T will be a wounded spider.)

If $i = 6$, we have $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) + 2 = \gamma(T_1) + \gamma(T_2) + 4 = \gamma(T) + 2$.

Case 3. In this case, we have $u_1 \in A(T_1)$ and $u_2 \notin A(T_2)$, and then $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 1$. Therefore, we have $\gamma_R(T) = \gamma_R(T_1) + \gamma_R(T_2) = \gamma(T_1) + \gamma(T_2) + 3 = \gamma(T) + 2$. Theorem 4.1 is proved.

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