Matrices related to the idempotent numbers and the numbers of planted forests

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Abstract

In this paper, we study the matrices related to the idempotent number and the number of planted forests with k components on the vertex set $[n]$. As a result, the factorizations of these two matrices are obtained. Furthermore, the discussion goes to the generalized case. Some identities and recurrences involving these two special sequences are also derived from the corresponding matrix representations.

Keywords: Matrices; Idempotent numbers; Planted forests; Combinatorial identities

1. Introduction

Recently, the Pascal matrix and several generalized Pascal matrices have catalyzed many investigations (see, e.g., [1, 2, 3, 8, 13, 14]). On the other hand, the Stirling matrices of the first kind and of the second kind as well as the Lah matrix are also received wide concern [4, 5, 12]. In the papers referred to above, we can see not only various properties satisfied by the corresponding matrices, especially the factorizations of them, but also some interesting and useful identities.

By the impetus of these works, in this paper, we will study the matrix related to the idempotent number $I(n, k)$ [6, Section 3.3] as well as the matrix related to the number $J(n, k)$ of planted forests with k components

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on the vertex set $[n]$ (cf., [11, Section 5.3]), where

$$I(n,k) = \binom{n}{k} k^{n-k}, \quad J(n,k) = \binom{n-1}{k-1} n^{n-k}.$$

It seems that the numbers $J(n,k)$ are more interesting for their explicit combinatorial meanings. Besides what we have just mentioned, $J(n,k)$ is also known as the number of labeled trees on $n+1$ nodes with maximal node degree k (see [10]).

Let's define IP_n and J_n to be the $n \times n$ matrices related to these two sequences:

$$(IP_n)_{i,j} = I(i,j), \quad (J_n)_{i,j} = J(i,j), \quad \text{for } i,j = 1,2,\cdots,n.$$

For example,

$$IP_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 6 & 1 & 0 \\ 4 & 24 & 12 & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 9 & 6 & 1 & 0 \\ 64 & 48 & 12 & 1 \end{pmatrix}.$$

As shown in [10], some well known combinatorial sequences, such as the number of labeled rooted trees with n nodes (n^{n-1}) and the number of labeled trees on n nodes (n^{n-2}), are all closely related to the matrix J_n. The readers are referred to [10] for more such sequences as well as the corresponding references.

In preparation for the study, we will first demonstrate the relationship between $I(n,k)$ and $J(n,k)$ (see [6, p. 164]), which is given by the lemma below.

Lemma 1.1. *We have*

$$\sum_{l=j}^{i} I(i,l) \cdot (-1)^{l-j} J(l,j) = \delta_{i,j}, \tag{1.1}$$

where $\delta_{i,j}$ is the Kronecker delta ($\delta_{i,i} = 1$, $\delta_{i,j} = 0$, $i \neq j$). As a consequence,

$$(IP_n^{-1})_{i,j} = (-1)^{i-j} J(i,j). \tag{1.2}$$

Proof. (1.1) is equivalent to the following identity:

$$\sum_{l=j}^{i} \binom{i}{l} \binom{l-1}{j-1} (-1)^{l-j} l^{i-j} = \delta_{i,j}.$$

To verify it, we need only consider the case when $i > j$. In fact, by appealing to the identity [7, p. 2, equation (1.13)]: $\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j = 0$ for $0 \leq j < n$, we have

$$\sum_{l=j}^{i}\binom{i}{l}\binom{l-1}{j-1}(-1)^{l-j}l^{i-j} = \sum_{l=j}^{i}\frac{i}{l}\binom{i-1}{l-1}\binom{l-1}{j-1}(-1)^{l-j}l^{i-j}$$

$$= i\binom{i-1}{j-1}\sum_{l=j}^{i}\binom{i-j}{l-j}(-1)^{l-j}l^{i-j-1}$$

$$= i\binom{i-1}{j-1}\sum_{k=0}^{i-j}\binom{i-j}{k}(-1)^k(k+j)^{i-j-1}$$

$$= i\binom{i-1}{j-1}\sum_{l=0}^{i-j-1}\binom{i-j-1}{l}j^{i-j-1-l}\left(\sum_{k=0}^{i-j}\binom{i-j}{k}(-1)^k k^l\right) = 0,$$

which completes the proof. □

In Section 2, the factorizations of IP_n will be given, and by making use of (1.2), J_n will also be factorized. Sections 3 is devoted to the generalized matrices of IP_n and J_n. Finally, in Section 4, some interesting identities as well as recurrence relations related to $I(n,k)$ and $J(n,k)$ are obtained with the method of matrix representation.

2. Factorizations of IP_n and J_n

Lemma 2.1. *The idempotent numbers $I(n,k)$ have the following generating functions:*

$$\Phi(t,u) = \sum_{n,k\geq 0} I(n,k)\frac{t^n}{n!}u^k = \exp(ute^t), \tag{2.1}$$

$$\Phi_k(t) = \sum_{n\geq k} I(n,k)\frac{t^n}{n!} = \frac{1}{k!}(te^t)^k. \tag{2.2}$$

Proof. By virtue of the explicit value of $I(n,k)$, we have

$$\Phi(t,u) = \sum_{n,k\geq 0}\binom{n}{k}k^{n-k}\frac{t^n}{n!}u^k = \sum_{k,l\geq 0}\frac{k^l}{k!l!}t^{k+l}u^k$$

$$= \sum_{k\geq 0}\frac{u^k t^k e^{kt}}{k!} = \exp(ute^t).$$

And (2.2) is a direct consequence of (2.1). □

Lemma 2.2. *The idempotent numbers $I(n,k)$ satisfy the following recurrence relation:*

$$I(n,k) = \sum_{m=k-1}^{n-1} (n-m)\binom{n-1}{m} I(m,k-1)\,.$$

Proof. Equate the coefficients of $t^{n-1}/(n-1)!$ in the first and last member of

$$\sum_{n\geq 0} I(n,k)\frac{t^{n-1}}{(n-1)!} = \frac{\mathrm{d}\Phi_k}{\mathrm{d}t} = (1+t)e^t\Phi_{k-1}(t)$$
$$= (1+t)\sum_{n\geq 0}\left(\sum_{m=0}^{n}\binom{n}{m} I(m,k-1)\right)\frac{t^n}{n!}\,,$$

and we will get the result. □

Now, defining A_n to be the $n\times n$ matrix by

$$(A_n)_{i,j} = (i-j+1)\binom{i-1}{j-1}\,, \quad \text{for } i,j = 1,2,\cdots,n\,,$$

and using the notation $\oplus$ for the direct sum of two matrices, we can obtain the factorization of the matrix IP_n from Lemma 2.2.

Theorem 2.3. *The matrix IP_n related to the idempotent numbers can be factorized as*

$$IP_n = A_n([1]\oplus IP_{n-1})\,. \tag{2.3}$$

For example, if $n=4$, we have

$$IP_4 =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 6 & 1 & 0 \\ 4 & 24 & 12 & 1 \end{pmatrix} = \begin{pmatrix} 1\binom{0}{0} & 0 & 0 & 0 \\ 2\binom{1}{0} & 1\binom{1}{1} & 0 & 0 \\ 3\binom{2}{0} & 2\binom{2}{1} & 1\binom{2}{2} & 0 \\ 4\binom{3}{0} & 3\binom{3}{1} & 2\binom{3}{2} & 1\binom{3}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 6 & 1 \end{pmatrix}.$$

Analogous to [2, 4, 13, 14], if we define the $n\times n$ matrix $\bar{A}_k$ by

$$\bar{A}_k = \begin{pmatrix} I_{n-k} & O \\ O & A_k \end{pmatrix},$$

we can further factorize the matrix IP_n. It is obvious that $\bar{A}_n = A_n$, and $\bar{A}_1$ is the identity matrix I_n.

Theorem 2.4. *The matrix IP_n can be factorized by the matrices $\bar{A}_k$'s:*

$$IP_n = \bar{A}_n \bar{A}_{n-1} \cdots \bar{A}_2 \bar{A}_1 \,. \tag{2.4}$$

For example,

$$IP_4 =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 6 & 1 & 0 \\ 4 & 24 & 12 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 9 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

Additionally, we can factorize the matrix J_n. In order to do it, the next lemma is required.

Lemma 2.5. *We have*

$$S_n \cdot L_n = A_n \,, \tag{2.5}$$

where S_n is a $n \times n$ matrix defined by

$$(S_n)_{i,j} = \begin{cases} 1\,, & \text{if } j = i\,, \\ i-1\,, & \text{if } j = i-1\,, \\ 0\,, & \text{else}\,, \end{cases}$$

and L_n is the general $n \times n$ Pascal matrix $(L_n)_{i,j} = \binom{i-1}{j-1}$. Moreover, the inverse matrices of S_n and A_n are given by:

$$(S_n^{-1})_{i,j} = \begin{cases} \frac{(-1)^{i+j}(i-1)!}{(j-1)!}\,, & \text{if } i \geq j\,, \\ 0\,, & \text{else}\,, \end{cases}$$

$$(A_n^{-1})_{i,j} = (-1)^{i+j} \binom{i-1}{j-1} \sum_{l=j}^{i} \frac{(i-j)!}{(i-l)!} \,.$$

Proof. By means of the definition of the matrix product, we can easily verify (2.5) as well as the values of the elements of S_n^{-1}. And in light of the well known fact that $(L_n^{-1})_{i,j} = (-1)^{i-j}\binom{i-1}{j-1}$, we can also deduce $(A_n^{-1})_{i,j}$ for $i, j = 1, 2, \cdots, n$. □

With Lemma 2.5, we can prove a recurrence relation satisfied by $J(n,k)$. Actually, equation (2.3) implies that

$$IP_n^{-1} = ([1] \oplus IP_{n-1})^{-1} A_n^{-1} \,.$$

And in view of (1.2), we have

$$J(i,j) = \sum_{l=j}^{i} J(i-1,l-1)\binom{l-1}{j-1}\sum_{k=j}^{l}\frac{(l-j)!}{(l-k)!}, \tag{2.6}$$

Thus, according to this recurrence, the factorizations of the matrix J_n can be deduced.

Theorem 2.6. *The matrix J_n can be factorized as*

$$J_n = ([1]\oplus J_{n-1})B_n = \bar{B}_1\bar{B}_2\cdots\bar{B}_{n-1}\bar{B}_n, \tag{2.7}$$

where the $n\times n$ matrix B_n is defined by

$$(B_n)_{i,j} = \binom{i-1}{j-1}\sum_{l=j}^{i}\frac{(i-j)!}{(i-l)!}, \quad \text{for } i,j=1,2,\cdots,n,$$

and $\bar{B}_k = \begin{pmatrix} I_{n-k} & O \\ O & B_k \end{pmatrix}$, *with the special cases $\bar{B}_n = B_n$ and $\bar{B}_1 = I_n$.*

For example,

$$J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 9 & 6 & 1 & 0 \\ 64 & 48 & 12 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 9 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 16 & 15 & 6 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 5 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 4 & 1 & 0 \\ 16 & 15 & 6 & 1 \end{pmatrix}.$$

3. Generalized numbers and matrices

It is obvious that an inverse relation can be obtained from Lemma 1.1. That is, for two sequences $\{a_n\}$ and $\{b_n\}$,

$$a_n = \sum_{k=0}^{n}\binom{n}{k}k^{n-k}b_k,$$

if and only if

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n-1}{k-1} n^{n-k} a_k .$$

This inverse relation as well as some generalizations can be found in [9, Chapter 3]. And in the present paper, we will consider one of these generalizations: for two sequences $\{a_n\}$ and $\{b_n\}$,

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{n-k} b_k ,$$

if and only if

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (x+k)(x+n)^{n-k-1} a_k .$$

Naturally, we will regard $I(x;n,k) = \binom{n}{k}(x+k)^{n-k}$ and $J(x;n,k) = \binom{n}{k}(x+k)(x+n)^{n-k-1}$ as the generalizations of $I(n,k)$ and $J(n,k)$, respectively. And the corresponding $n \times n$ matrices $IP_n[x]$ and $J_n[x]$, which are defined as follows:

$$(IP_n[x])_{i,j} = I(x;i,j), \quad (J_n[x])_{i,j} = J(x;i,j), \\ \text{for } i,j = 1,2,\cdots,n,$$

can also be viewed as the generalizations of IP_n and J_n, respectively. For instance,

$$IP_4[x] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2(x+1) & 1 & 0 & 0 \\ 3(x+1)^2 & 3(x+2) & 1 & 0 \\ 4(x+1)^3 & 6(x+2)^2 & 4(x+3) & 1 \end{pmatrix},$$

$$J_4[x] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2(x+1) & 1 & 0 & 0 \\ 3(x+1)(x+3) & 3(x+2) & 1 & 0 \\ 4(x+1)(x+4)^2 & 6(x+2)(x+4) & 4(x+3) & 1 \end{pmatrix}.$$

In this section, we will give the factorizations of these two matrices.

Lemma 3.1. *The generalized idempotent numbers $I(x;n,k)$ have the following generating functions:*

$$\Phi(x;t,u) = \sum_{n,k\geq 0} I(x;n,k)\frac{t^n}{n!}u^k = e^{xt}\exp(ute^t)\,, \tag{3.1}$$

$$\Phi_k(x;t) = \sum_{n\geq k} I(x;n,k)\frac{t^n}{n!} = \frac{1}{k!}t^k e^{(x+k)t}\,. \tag{3.2}$$

Proof. In fact, we have

$$\begin{aligned}
&\sum_{n,k\geq 0}\binom{n}{k}(x+k)^{n-k}\frac{t^n}{n!}u^k \\
&= \sum_{k,l\geq 0}\frac{u^k t^{k+l}(x+k)^l}{k!l!} = \sum_{k\geq 0}\frac{u^k t^k}{k!}\sum_{l\geq 0}\frac{(x+k)^l t^l}{l!} = \sum_{k\geq 0}\frac{u^k t^k e^{(x+k)t}}{k!} \\
&= e^{xt}\sum_{k\geq 0}\frac{u^k t^k e^{kt}}{k!} = e^{xt}\exp(ute^t)\,,
\end{aligned}$$

from which (3.1) holds. And by identifying the coefficients of u^k in (3.1), we can obtain the vertical generating function (3.2). □

Lemma 3.2. *The generalized idempotent numbers $I(x;n,k)$ satisfy the following recurrence relations:*

$$(1-\frac{k}{n})I(x;n,k) = (k+x)I(x;n-1,k)\,, \tag{3.3}$$

$$\frac{k}{n}I(x;n,k) = \sum_{m=k-1}^{n-1}\binom{n-1}{m}I(x;m,k-1)\,. \tag{3.4}$$

Proof. (3.3) can be verified directly. For (3.4), we differentiate (3.2) with respect to t first:

$$\sum_{n} I(x;n,k)\frac{t^{n-1}}{(n-1)!} = \frac{\partial\Phi_k(x,t)}{\partial t} = e^t\Phi_{k-1}(x,t) + (k+x)\Phi_k(x,t)\,.$$

And then, by equating the coefficients of $t^{n-1}/(n-1)!$ in the formula above, we have

$$I(x;n,k) = \sum_{m=0}^{n-1}\binom{n-1}{m}I(x;m,k-1) + (k+x)I(x;n-1,k)\,,$$

which leads us at once to the result in light of (3.3). □

Corollary 3.3. *We have*

$$\frac{k}{n}I(n,k) = \sum_{m=k-1}^{n-1} \binom{n-1}{m} I(m,k-1)\,. \tag{3.5}$$

Proof. This is a direct consequence by letting $x = 0$ in (3.4). □

If we define the $n \times n$ matrix $\widetilde{IP}_n$ by $(\widetilde{IP}_n)_{i,j} = jI(i,j)/i$, we can get its factorizations immediately from (3.5):

$$\widetilde{IP}_n = L_n([1] \oplus IP_{n-1}) = L_n\bar{A}_{n-1}\cdots\bar{A}_2\bar{A}_1\,. \tag{3.6}$$

Moreover, according to (2.3) and (2.5), we have $IP_n = S_nL_n([1] \oplus IP_{n-1})$, which yields the relationship between IP_n and $\widetilde{IP}_n$:

$$\widetilde{IP}_n = S_n^{-1} IP_n\,. \tag{3.7}$$

In order to get the factorizations of $IP_n[x]$ and $J_n[x]$, we should define some new matrices, which will be shown below. For $i, j = 1, 2, \cdots, n$,

$$(\widetilde{IP}_n[x])_{i,j} = \frac{j}{i}I(x;i,j)\,, \quad (L_n[x])_{i,j} = \begin{cases} \binom{i-1}{j-1}\,, & \text{if } j \neq 1\,, \\ (x+1)^{i-1}\,, & \text{if } j = 1\,, \end{cases}$$

$$(S_n[x])_{i,j} = \begin{cases} x^{i-j-1}(x+j)\,, & \text{if } j < i\,, \\ 1\,, & \text{if } j = i\,, \\ 0\,, & \text{if } j > i\,. \end{cases}$$

The inverse matrices of $L_n[x]$ and $S_n[x]$ can be easily found.

$$(L_n^{-1}[x])_{i,j} = \begin{cases} (-1)^{i-j}\binom{i-1}{j-1}\,, & \text{if } j \neq 1\,, \\ -x^{i-1} + (-1)^{i-1}\,, & \text{if } j = 1\,, \end{cases}$$

$$(S_n^{-1}[x])_{i,j} = \begin{cases} \frac{(-1)^{i+j}(i-1)!}{j!}(x+j)\,, & \text{if } j < i\,, \\ 1\,, & \text{if } j = i\,, \\ 0\,, & \text{if } j > i\,. \end{cases}$$

Thus, from (3.4), we get the relationship between $IP_n[x]$ and $\widetilde{IP}_n[x]$:

$$\widetilde{IP}_n[x] = L_n[x]([1] \oplus IP_{n-1}[x])\,. \tag{3.8}$$

On the other hand, analogous to (3.7), we obtain another one:

$$\widetilde{IP}_n[x] = S_n^{-1}[x] IP_n[x]\,. \tag{3.9}$$

Making use of (3.8) and (3.9), the factorization of $IP_n[x]$ can be finally achieved.

Theorem 3.4. *The matrix $IP_n[x]$ related to the generalized idempotent numbers can be factorized as*

$$IP_n[x] = S_n[x]L_n[x]([1] \oplus IP_{n-1}[x]).$$

For example, we have

$$IP_4[x] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x+1 & 1 & 0 & 0 \\ x(x+1) & x+2 & 1 & 0 \\ x^2(x+1) & x(x+2) & x+3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x+1 & 1 & 0 & 0 \\ (x+1)^2 & 2 & 1 & 0 \\ (x+1)^3 & 3 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2(x+1) & 1 & 0 \\ 0 & 3(x+1)^2 & 3(x+2) & 1 \end{pmatrix}.$$

By defining the $n \times n$ matrix $A_n[x] = S_n[x]L_n[x]$, we can obtain the factorization of $J_n[x]$.

Theorem 3.5. *The matrix $J_n[x]$ can be factorized as*

$$J_n[x] = ([1] \oplus J_{n-1}[x])B_n[x],$$

where the $n \times n$ matrix $B_n[x]$ is defined by $(B_n[x])_{i,j} = (-1)^{i-j}A(x;i,j)$ for $i,j = 1,2,\cdots,n$, and $A(x;i,j)$ is the (i,j)-entry of $A_n[x]$.

4. Identities and recurrence relations

According to the matrix representations obtained in Section 2 and Section 3, a number of interesting identities and recurrence relations related to the numbers $I(n,k)$ and $J(n,k)$ can be achieved.

Theorem 4.1. *We have*

$$\sum_{l=k}^{n}(-1)^{l-k}I(n,l)J(l-1,k-1) = (n-k+1)\binom{n-1}{k-1}, \tag{4.1}$$

$$\sum_{l=k}^{n}(-1)^{l-k}\binom{n-k}{l-k}(l-1)^{l-k-1}l^{n-l-1} = \frac{n-k+1}{n(k-1)}, \quad \text{for } k \geq 2, \tag{4.2}$$

and

$$\sum_{l=k}^{n}(n-l+1)\binom{n-k}{l-k}(k-1)^{l-k} = nk^{n-k-1}, \quad \text{for } k \geq 1. \tag{4.3}$$

Proof. (4.1) and (4.2) are direct consequences of the fact that $IP_n([1] \oplus IP_{n-1})^{-1} = A_n$. And (4.3) is a equivalent form of (2.3). □

Theorem 4.2. *We have*

$$\sum_{l=k}^{n}(-1)^{n-l}I(n-1,l-1)J(l,k) = \binom{n-1}{k-1}\sum_{l=k}^{n}\frac{(n-k)!}{(n-l)!}, \tag{4.4}$$

$$\sum_{l=k}^{n}(-1)^{n-l}\binom{n-k}{l-k}(l-1)^{n-l}l^{l-k} = \sum_{l=k}^{n}\frac{(n-k)!}{(n-l)!}, \tag{4.5}$$

and

$$\sum_{l=k}^{n}(-1)^{l-k}(l-k+1)\binom{l-1}{k-1}J(n,l) = J(n-1,k-1), \tag{4.6}$$

$$\sum_{l=k}^{n}(-1)^{l-k}(l-k+1)\binom{n-k}{l-k}n^{n-l} = (k-1)(n-1)^{n-k-1}, \quad \text{for } n \geq 2. \tag{4.7}$$

Proof. (4.4) and (4.5) follow from $([1]\oplus J_{n-1})^{-1}J_n = B_n$; while (4.6) and (4.7) follow from $J_nB_n^{-1} = ([1]\oplus J_{n-1})$. □

Equation (3.6) $\widetilde{IP_n} = L_n([1]\oplus IP_{n-1})$ will lead us at once to the next three theorems.

Theorem 4.3. *We have*

$$\sum_{l=k}^{n}\binom{l-1}{k-1}J(n-1,l-1) = \frac{k}{n}J(n,k), \tag{4.8}$$

$$\sum_{l=k}^{n}(l-1)\binom{n-k}{l-k}(n-1)^{n-l-1} = kn^{n-k-1}, \quad \text{for } n \geq 2, \tag{4.9}$$

and

$$\sum_{l=k}^{n}\binom{n-k}{l-k}(k-1)^{l-k} = k^{n-k}. \tag{4.10}$$

Proof. We can get (4.8) and (4.9) from the factorization $\widetilde{IP_n}^{-1} = ([1]\oplus IP_{n-1})^{-1}L_n^{-1}$, and get (4.10) from (3.6). □

Theorem 4.4. *We have*

$$\sum_{l=k}^{n}(-1)^{l-k}\frac{l+1}{n+1}I(n+1,l+1)J(l,k) = \binom{n}{k}, \tag{4.11}$$

$$\sum_{l=k}^{n}(-1)^{l-k}\binom{n-k}{l-k}(l+1)^{n-l}l^{l-k-1}=\frac{1}{k}, \quad \text{for } k\geq 1, \tag{4.12}$$

and

$$\sum_{l=k}^{n}(-1)^{l-k}\frac{k+1}{l+1}I(n,l)J(l+1,k+1)=(-1)^{n-k}\binom{n}{k}, \tag{4.13}$$

$$\sum_{l=k}^{n}(-1)^{l-k}\frac{k+1}{l+1}\binom{n-k}{l-k}(l+1)^{l-k}l^{n-l}=(-1)^{n-k}. \tag{4.14}$$

Proof. These four identities follow from $\widetilde{IP}_n([1]\oplus IP_{n-1})^{-1}=L_n$ and $([1]\oplus IP_{n-1})\widetilde{IP}_n^{-1}=L_n^{-1}$, respectively. □

Theorem 4.5. *We have*

$$\frac{k}{n}\sum_{l=k}^{n}(-1)^{n-l}\binom{n}{l}I(l,k)=I(n-1,k-1), \tag{4.15}$$

$$\sum_{l=k}^{n}(-1)^{n-l}\binom{n-k}{l-k}k^{l-k}=(k-1)^{n-k}. \tag{4.16}$$

and

$$\frac{k}{n}\sum_{l=k}^{n}(-1)^{l-k}\binom{l}{k}J(n,l)=J(n-1,k-1), \tag{4.17}$$

$$\sum_{l=k}^{n}(-1)^{l-k}\binom{n-k}{l-k}l\cdot n^{n-l-1}=(k-1)(n-1)^{n-k-1}, \quad \text{for } n\geq 2. \tag{4.18}$$

Proof. These four identities follow from $L_n^{-1}\widetilde{IP}_n=([1]\oplus IP_{n-1})$ and $\widetilde{IP}_n^{-1}L_n=([1]\oplus IP_{n-1})^{-1}$, respectively. □

Finally, by appealing instead to (3.7) $\widetilde{IP}_n=S_n^{-1}IP_n$, we will achieve the following theorems.

Theorem 4.6. *We have*

$$I(n,k)=\frac{n!}{n-k}\sum_{l=k}^{n-1}\frac{(-1)^{n-l-1}}{(l-1)!}I(l,k), \tag{4.19}$$

$$k^n=(n-k)!\sum_{l=k}^{n}\frac{(-1)^{n-l}l}{(l-k)!}k^{l-1}=(n-k-1)!\sum_{l=k}^{n-1}\frac{(-1)^{n-l-1}l}{(l-k)!}k^{l}, \tag{4.20}$$

and

$$J(n,k+1) = (\frac{1}{k} - \frac{1}{n})J(n,k). \tag{4.21}$$

Proof. They follow from $\widetilde{IP}_n = S_n^{-1} IP_n$ and $\widetilde{IP}_n^{-1} = IP_n^{-1} S_n$, respectively. □

Theorem 4.7. *We have*

$$I(n-1,k) = (\frac{1}{k} - \frac{1}{n})I(n,k), \tag{4.22}$$

and

$$J(n,k) = \frac{1}{n-k} \sum_{l=k+1}^{n} \frac{l!}{(k-1)!} J(n,l), \tag{4.23}$$

$$\frac{1}{n^k} = (n-k-1)! \sum_{l=k+1}^{n} \frac{l}{(n-l)! n^l} = (n-k)! \sum_{l=k}^{n} \frac{l}{(n-l)! n^{l+1}}. \tag{4.24}$$

Proof. They follow from $S_n \widetilde{IP}_n = IP_n$ and $\widetilde{IP}_n^{-1} S_n^{-1} = IP_n^{-1}$, respectively. Moreover, we can see that (4.22) is just a special case of (3.3). □

Theorem 4.8. *We have*

$$\sum_{l=k}^{n} (-1)^{l-k} \frac{l}{n} I(n,l) J(l,k) = \frac{(-1)^{n+k}(n-1)!}{(k-1)!}, \tag{4.25}$$

$$\sum_{l=k}^{n} (-1)^{l-k} \binom{n-k}{l-k} l^{n-k} = (-1)^{n-k}(n-k)!, \tag{4.26}$$

and

$$\sum_{l=k}^{n} (-1)^{l-k} \frac{k}{l} I(n,l) J(l,k) = \begin{cases} 1, & \text{if } n = k, \\ k, & \text{if } n = k+1, \\ 0, & \text{else}, \end{cases} \tag{4.27}$$

$$k^2 \binom{n}{k} \sum_{l=k}^{n} (-1)^{l-k} \binom{n-k}{l-k} l^{n-k-2} = \begin{cases} 1, & \text{if } n = k, \\ k, & \text{if } n = k+1, \\ 0, & \text{else}. \end{cases} \tag{4.28}$$

Proof. These four identities follow from $\widetilde{IP}_n IP_n^{-1} = S_n^{-1}$ and $IP_n \widetilde{IP}_n^{-1} = S_n$, respectively. □

Remark. The identities in this section are all verified by the method of matrix representation. However, we should notice that some of them can also be obtained by means of inverse relations. For instance, (4.16) can also be deduced from (4.10) according to the famous inverse relation:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k , \quad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k .$$

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