Simple Kirkman Packing Designs SKPD($\{3, 4\}, v$) with index two

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Abstract

A simple Kirkman packing design SKPD($\{w, w+1\}, v$) with index λ is a resolvable packing with distinct blocks and maximum possible number of parallel classes, each containing $u = v - w\lfloor \frac{v}{w} \rfloor$ blocks of size $w+1$ and $(v-u(w+1))/w$ blocks of size w, such that each pair of distinct elements occurs in at most λ blocks. In this paper, we solve the spectrum of simple Kirkman packing designs SKPD($\{3, 4\}, v$) with index 2 completely.

Key words: simple; Kirkman packing design; group-divisible design; frame

1 Introduction

Let X be a set of v points. A *packing* of X with index λ is a collection of subsets of X (called *blocks*) such that any pair of distinct points from X occurs together in at most λ blocks in the collection. Let K be a set of positive integers. Denote by $P_\lambda(K, v)$ a packing on v points with block sizes all in K. A packing is called *simple* if all its blocks are distinct. A packing is called *resolvable* if its block set admits a partition into *parallel classes*, each parallel class being a partition of the point set X. A simple *Kirkman packing*, denoted $\mathrm{SKP}_\lambda(K, v)$, is a simple resolvable $P_\lambda(K, v)$. It is obvious

*Research supported by the National Natural Science Youth Foundation of China under Grant 10501023. E-mail: caohaitao@njnu.edu.cn that a simple Kirkman packing SKP is equivalent to a resolvable packing KP when $\lambda = 1$. We usually write KP(K, v) instead of SKP$_1(K, v)$.

A simple *Kirkman packing design* with block size w and $w+1$, denoted SKPD$_\lambda(\{w, w+1\}, v)$, is an SKP$_\lambda(\{w, w+1\}, v)$ with maximum possible number $m(v)$ of parallel classes, each containing $u = v - w\lfloor \frac{v}{w} \rfloor$ blocks of size $w+1$ and $(v - u(w+1))/w$ blocks of size w, such that each pair of distinct elements occurs in at most λ blocks. In [2, 7, 11], a KPD$(\{3,4\}, v)$ is also called Kirkman school project design.

Kirkman packing designs have been studied by many researchers and found to have a number of applications. Cao et al. [2, 3, 4] have given some applications in threshold schemes, and Fang et al. [8, 9] have shown that these designs can be used in the construction of uniform designs in statistics. There are also some known results on the existence of simple Kirkman packing designs SKPD$_\lambda(\{w, w+1\}, v)$. When $\lambda = 1$, $w = 3$ and $v \equiv 1 \ (mod\ 3)$, the spectrum problem for KPD$(\{3,4\}, v)$ has been almost completely solved by Černý, Horák and Wallis [5], Phillips, Wallis and Rees [11], Colbourn and Ling [7], Cao and Zhu [4], finally Cao and Du [2]. When $\lambda = 1$, $w = 3$ and $v \equiv 2 \ (mod\ 3)$, the spectrum problem for KPD$(\{3,4\}, v)$ has been completely solved by Cao and Tang [3] and Cao [1]. The spectrum problem for SKPD$_\lambda(\{4,5\}, v)$ has been partly solved by Cao [1], Cao and Du [2]. In this article, we shall be restricting our attention to simple Kirkman packing designs SKPD$_2(\{3,4\}, v)$, $v \equiv 1, 2 \ (mod\ 3)$. Some simple computation shows:

Lemma 1.1 *If there exists an SKPD$_2(\{3,4\}, v)$, $v \equiv 1$ or 2 $(mod\ 3)$, then $m(v) \leq n(v)$, where*

$$n(v) = \begin{cases} v-3, & v \geq 7,\ v \equiv 1 \ (mod\ 3), \\ v-5, & v \geq 17,\ v \equiv 2 \ (mod\ 3), \\ 2, & v = 4, \\ v-4, & v \in \{8, 11, 14\}. \end{cases}$$

It is easy to see that $m(4) = 1$. In this paper we shall prove that $m(v) = n(v)$ for all $v \geq 7$, $v \neq 11$, and $m(11) = 6$.

Theorem 1.2 (i) *There is an* $SKPD_2(\{3,4\},v)$ *with* $n(v)$ *parallel classes for every* $v \equiv 1,2 \pmod 3$, $v \geq 7$ *and* $v \neq 11$; (ii) *There exists an* $SKPD_2(\{3,4\},11)$ *with* 6 *parallel classes.*

2 Preliminaries

In this section we shall define some of the auxiliary designs and establish some of the fundamental results which will be used later. The reader is referred to [6] for more information on designs, and, in particular, group divisible designs and frames.

Let K and M be sets of positive integers. A *group divisible design* (GDD) $GD(K,\lambda,M;v)$ is a triple $(X,\mathcal{G},\mathcal{B})$ where

1. X is a v-set (of points),
2. $\mathcal{G}$ is a collection of nonempty subsets of X (called groups) with cardinality in M and which partition X,
3. $\mathcal{B}$ is a collection of subsets of X (called blocks) with cardinality at least two in K,
4. no block intersects any group in more than one point,
5. each pair of distinct points not contained in a group is contained in exactly λ blocks.

The type of the GDD $(X,\mathcal{G},\mathcal{B})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description: type $1^i 2^j 3^k \cdots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on.

A GDD $(K,\lambda,M;v)$ is resolvable if the blocks of $\mathcal{B}$ can be partitioned into parallel classes. We shall denote by $GD(k,\lambda,m;v)$ a $GDD(\{k\},\lambda,\{m\};v)$. A *transversal design* $TD(k,n)$ is a $(k,1)$-GDD of type n^k, it is *idempotent* if it contains a parallel class of blocks. It is well known that a $TD(k,n)$ is equivalent to $k-2$ mutually orthogonal Latin squares of order n. For the transversal design, we need the following result.

Theorem 2.1 ([6]) (i) *There exists an idempotent $TD(5,v)$ for any positive integer $v \geq 5$ and $v \notin \{6,10\}$.* (ii) *There exists an idempotent $TD(8,v)$ for any positive integer $v \geq 99$ and $v \in E = \{8,11,13,16,19,23,25,31,37, 41,49,57,67,79,87\}$.*

We shall use the following Wilson's Fundamental Construction (WFC) for GDDs.

Lemma 2.2 ([14]) *Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a GDD and let $w : X \to Z^+ \cup \{0\}$ be a weighting of the GDD. For every $x \in X$, let S_x be the multiset of $w(x)$ copies of x. For each block $B \in \mathcal{B}$, assume a k-GDD of type $\{S_x : x \in B\}$ is given. Then there is a k-GDD of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

A GDD $(X, \mathcal{G}, \mathcal{B})$ is called *frame resolvable* if its block set $\mathcal{B}$ admits a partition into holey parallel classes, each holey parallel class being a partition of $X - G_j$ for some $G_j \in \mathcal{G}$. We shall denote by (K,λ)-frame a frame resolvable (K,λ)-GDD. The groups in a (K,λ)-frame are often referred to as holes. In a $(3,2)$-frame, it is well known that to each group G_j there are exactly $|G_j|$ holey parallel classes that partition $X - G_j$. From [15], we have the following result for simple $(3,2)$-frames.

Theorem 2.3 ([15]) *There exists a simple (3,2)-frame of type h^u if and only if $u \geq 4$ and $h(u-1) \equiv 0 \pmod 3$.*

The main technique that we will be using throughout the remainder of the article is a variant of Stinson's "Filling in Holes" construction. In applying the "Filling in Holes" construction, we will require simple $(3,2)$-frames in which the blocks are not necessarily all of the same size. To get these, we shall use the following recursive construction.

Lemma 2.4 ([13]) *Suppose that there is a K-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_m^{t_m}$ and that for each $k \in K$ there is a simple $(3,2)$-frame of type h^k. Then there is a simple $(3,2)$-frame of type $(hg_1)^{t_1}(hg_2)^{t_2} \cdots (hg_m)^{t_m}$.*

In order to use the 'Filling in Holes' construction, we need the notion of an *incomplete* simple Kirkman packing design (ISKPD). Let $a \in \{1, 2\}$ and $v \equiv a \ (mod\ 3)$. For $h \equiv v \ (mod\ 3)$, $h \geq 4$ or $h = 3$ when $a = 1$, an ISKPD $_2(\{3,4\}, v, h)$ is defined to be a triple $(V, H, \mathcal{B})$ which satisfies the following properties:

1. V is a v-set of points, H is an h-subset of V (called a *hole*) and $\mathcal{B}$ is a collection of subsets of V (called *blocks*), each block having size 3 or 4;

2. $|H \cap B| \leq 1$ for all $B \in \mathcal{B}$;

3. any two points of V appear either in H or in at most two blocks of $\mathcal{B}$;

4. $\mathcal{B}$ admits a partition into $v - h$ parallel classes, each consists of a blocks of size 4 and $(v - 4a)/3$ triples on V, and $h - 3$ when $a = 1$ or $h - 5$ when $a = 2$ *auxiliary parallel classes*, each consists of $(v - h)/3$ triples on $V \backslash H$.

It is obvious that an $ISKPD_2(\{3,4\}, v, h)$ is an SKPD$_2(\{3,4\}, v)$ for $h = 3$, $a = 1$ or $h = 5$, $a = 2$. If the missing subdesign in an incomplete simple Kirkman packing design exists, then one can "fill in hole" in the ISKPD with a copy of that design.

Lemma 2.5 *Let $a \in \{1, 2\}$ and $v \equiv a \ (mod\ 3)$. If there exists an ISKPD$_2$ $(\{3,4\}, v, h)$ and an SKP$_2(\{3,4\}, h)$ with $h - 3$ when $a = 1$ or $h - 5$ when $a = 2$ parallel classes, then there is an SKPD$_2(\{3,4\}, v)$.*

Lemma 2.6 *There is an ISKPD$_2(\{3,4\}, 13, 3)$.*

Proof: Take the point set $X = Z_{10} \cup \{\infty_1, \infty_2, \infty_3\}$. The 10 parallel classes will be generated from the initial parallel class P_0 by +1 modulo 10. The blocks of P_0 are listed below.

$0, 1, 2, 5 \quad 3, 6, \infty_1 \quad 4, 8, \infty_2 \quad 7, 9, \infty_3$

□

Lemma 2.7 *There is an* $ISKPD_2(\{3,4\},v,4)$ *for* $v \in \{16,19,22,25,28,31,34,37,40,46,49\}$.

Proof: Take the point set $X = Z_{v-4} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$. Let $B = \{0,(v-4)/3,2(v-4)/3\}$. The required holey parallel class can be obtained from B by $+1$ modulo $v-4$. The $v-4$ parallel classes will be generated modulo $v-4$ from the initial parallel class P_0. For each v, the blocks of the initial parallel class are listed below.

$v = 16:\quad 0,1,2,6 \quad 3,5,\infty_1 \quad 4,9,\infty_2 \quad 7,10,\infty_3 \quad 8,11,\infty_4$

$v = 19:\quad 0,1,2,4 \quad 5,9,12 \quad 3,10,\infty_1 \quad 6,11,\infty_2 \quad 7,13,\infty_3 \quad 8,14,\infty_4$

$v = 22:\quad 0,1,2,4 \quad 3,6,13 \quad 7,11,16 \quad 5,12,\infty_1 \quad 9,17,\infty_2 \quad 10,15,\infty_3 \quad 8,14,\infty_4$

$v = 25:\quad 0,1,2,4 \quad 3,6,12 \quad 7,14,18 \quad 5,10,15 \quad 5,12,\infty_1 \quad 11,20,\infty_2 \quad 13,19,\infty_3 \quad 8,16,\infty_4$

$v = 28:\quad 0,1,2,4 \quad 3,6,11 \quad 8,15,21 \quad 5,12,17 \quad 10,16,20 \quad 14,23,\infty_1 \quad 13,22,\infty_2 \quad 9,19,\infty_3 \quad 7,18,\infty_4$

$v = 31:\quad 0,1,2,4 \quad 3,6,10 \quad 8,18,23 \quad 5,11,19 \quad 12,17,24 \quad 7,15,21 \quad 14,25,\infty_1 \quad 13,22,\infty_2 \quad 9,20,\infty_3 \quad 16,26,\infty_4$

$v = 34:\quad 0,1,2,4 \quad 3,6,10 \quad 9,19,24 \quad 5,11,17 \quad 12,20,28 \quad 13,22,27 \quad 7,18,25 \quad 14,23,\infty_1 \quad 15,26,\infty_2 \quad 8,21,\infty_3 \quad 16,29,\infty_4$

$v = 37:\quad 0,1,2,4 \quad 3,6,10 \quad 8,20,25 \quad 5,11,16 \quad 12,21,30 \quad 14,22,32 \quad 7,17,24 \quad 9,23,29 \quad 13,27,\infty_1 \quad 15,28,\infty_2 \quad 18,26,\infty_3 \quad 19,31,\infty_4$

$v = 40:\quad 0,1,2,4 \quad 3,6,10 \quad 8,21,27 \quad 5,11,16 \quad 14,26,34 \quad 13,22,32 \quad 7,12,25 \quad 15,24,35 \quad 9,23,30 \quad 17,31,\infty_1 \quad 20,28,\infty_2 \quad 18,33,\infty_3 \quad 19,29,\infty_4$

$v = 46:\quad 0,1,2,4 \quad 3,6,10 \quad 20,32,41 \quad 21,26,37 \quad 8,15,23 \quad 9,17,29 \quad 13,27,40 \quad 19,25,38 \quad 12,30,35 \quad 14,24,34 \quad 22,33,39 \quad 5,31,\infty_1 \quad 7,16,\infty_2 \quad 18,36,\infty_3 \quad 11,28,\infty_4$

$v = 49:\quad 0,1,2,4 \quad 20,30,40 \quad 24,38,42 \quad 8,15,23 \quad 9,17,26 \quad 7,12,18 \quad 13,22,35 \quad 10,21,34 \quad 25,28,44 \quad 19,31,43 \quad 14,32,39 \quad 5,11,27 \quad 3,29,\infty_1 \quad 6,37,\infty_2 \quad 16,33,\infty_3 \quad 36,41,\infty_4$

□

Lemma 2.8 *There is an* $ISKPD_2(\{3,4\},v,5)$ *for* $v \in \{23,26,29,32,35\}$.

Proof: Take the point set $X = Z_{v-5} \cup \{\infty_1,\infty_2,\cdots,\infty_5\}$. The $v-5$ parallel classes will be generated modulo $v-5$ from the initial parallel class P_0. For each v, the blocks of the initial parallel class are listed below.

$v = 23:\quad 0,1,2,4 \quad 3,6,11,15 \quad 5,12,\infty_1 \quad 14,9,\infty_2 \quad 7,13,\infty_3 \quad 8,16,\infty_4 \quad 10,17,\infty_5$

$v = 26:$	$0,1,2,4$	$3,6,10,15$	$5,11,16$	$20,12,\infty_1$	$13,19,\infty_2$	$7,14,\infty_3$
	$8,18,\infty_4$	$9,17,\infty_5$				
$v = 29:$	$0,1,2,4$	$3,6,10,15$	$5,11,18$	$7,17,22$	$23,12,\infty_1$	$13,21,\infty_2$
	$14,20,\infty_3$	$8,16,\infty_4$	$9,19,\infty_5$			
$v = 32:$	$0,1,2,4$	$3,6,10,15$	$5,11,18$	$7,16,21$	$9,20,26$	$24,12,\infty_1$
	$13,23,\infty_2$	$14,22,\infty_3$	$8,19,\infty_4$	$25,17,\infty_5$		
$v = 35:$	$0,1,2,4$	$3,6,10,15$	$5,11,18$	$7,16,22$	$8,21,26$	$12,20,28$
	$24,13,\infty_1$	$9,23,\infty_2$	$14,25,\infty_3$	$29,19,\infty_4$	$27,17,\infty_5$	

□

Lemma 2.9 *There is an* $ISKPD_2(\{3,4\},v,8)$ *for* $v \in \{32,38,50\}$.

Proof: Take the point set $X = Z_{v-8} \cup \{\infty_1, \infty_2, \cdots, \infty_8\}$. Let $B = \{0,1,5\}$. The required three holey parallel classes can be obtained from B by $+1$ modulo $v-8$. The $v-8$ parallel classes will be generated modulo $v-8$ from the initial parallel class P_0. For each v, the blocks of the initial parallel class are listed below.

$v = 32:$	$0,1,3,12$	$2,4,7,11$	$13,5,\infty_1$	$6,16,\infty_2$	$8,21,\infty_3$	$9,19,\infty_4$
	$10,18,\infty_5$	$14,20,\infty_6$	$15,22,\infty_7$	$17,23,\infty_8$		
$v = 38:$	$0,1,3,5$	$2,8,11,17$	$4,12,20$	$6,13,23$	$7,19,\infty_1$	$9,22,\infty_2$
	$10,24,\infty_3$	$14,25,\infty_4$	$15,27,\infty_5$	$16,26,\infty_6$	$18,29,\infty_7$	$21,28,\infty_8$
$v = 50:$	$0,1,3,5$	$2,8,11,17$	$9,29,39$	$6,13,23$	$7,15,33$	$4,16,27$
	$10,24,31$	$12,26,34$	$14,30,\infty_1$	$18,35,\infty_2$	$19,32,\infty_3$	$20,38,\infty_4$
	$21,40,\infty_5$	$22,37,\infty_6$	$25,36,\infty_7$	$28,41,\infty_8$		

□

3 Direct constructions for small orders

In this section, we construct SKPDs with small orders, some of which will be used as input designs in recursive constructions of Section 4. For most of these orders, instead of listing all the blocks and the parallel classes of the desired designs, we only list the blocks of the initial parallel class P_0, the other parallel classes can be generated under some additive group G.

Lemma 3.1 *There is an* $SKPD_2(\{3,4\},v)$ *with* $n(v)$ *parallel classes for each* $v \in \{7,10\}$.

Proof: For each v, let the point set $X = Z_v$. We list the required $n(v)$ parallel classes as below.

$v=7$:	P_1: 0,1,2,3	4,5,6	P_2: 0,1,4,5	2,3,6				
	P_3: 0,2,4,6	1,3,5	P_4: 0,3,5,6	1,2,4				
$v=10$:	P_1: 0,1,2,3	4,5,6	7,8,9	P_2: 0,2,5,8	1,4,6	3,7,9		
	P_3: 0,4,7,8	1,5,9	2,3,6	P_4: 0,5,6,9	1,2,7	3,4,8		
	P_5: 1,3,5,8	2,4,9	0,6,7	P_6: 2,4,5,7	1,6,8	0,3,9		
	P_7: 2,6,8,9	0,1,4	3,5,7					

□

Lemma 3.2 *There is an* $SKPD_2(\{3,4\},8)$ *with 4 parallel classes.*

Proof: From Lemma 3.1, we have an $\text{SKPD}_2(\{3,4\},7)$ with 4 parallel classes. Add a new point ∞ to all the triples of this SKPD, it is easy to check that the resulting design is an $\text{SKPD}_2(\{3,4\},8)$ with 4 parallel classes. □

Lemma 3.3 *There is an* $SKPD_2(\{3,4\},11)$ *with* 6 *parallel classes.*

Proof: We first construct a simple Kirkman packing $\text{SKP}_2(\{3,4\},11)$ with 6 parallel classes. The blocks are as follows.

P_1: 1,2,3,4	5,6,7,8	0,9,10	P_2: 1,2,5,6	3,4,7,9	0,8,10
P_3: 1,3,5,8	4,6,7,10	0,2,9	P_4: 1,4,6,9	0,3,5,7	2,8,10
P_5: 0,1,7,8	2,3,6,9	4,5,10	P_6: 1,7,9,10	0,2,4,5	3,6,8

A computer exhaustive search shows that there does not exist an SKP_2 $(\{3,4\},11)$ with $n(11) = 7$ parallel classes. Therefore, the above SKP is indeed an SKPD. □

Lemma 3.4 *There is an* $SKPD_2(\{3,4\},14)$ *with* 10 *parallel classes.*

Proof: Let $G = Z_{12}$. We take the point set $Z_{12} \cup \{\infty_1, \infty_2\}$. The following three initial parallel classes P_1, P_2 and P_3 will generate nine parallel classes by +4 modulo 12. The blocks of the three initial parallel classes and the last parallel class Q are listed below.

P_1:	1,2,3,5,6	4,6,8,11	0,9,∞_1	7,10,∞_2
P_2:	1,2,6,8	3,4,5,11	7,10,∞_1	0,9,∞_2
P_3:	0,5,11,∞_1	4,6,7,∞_2	1,3,10	2,8,9
Q:	1,5,9,∞_2	2,6,10,∞_1	3,7,11	0,4,8

□

Lemma 3.5 *There is an $SKPD_2(\{3,4\},17)$ with 12 parallel classes.*

Proof: Let $G = Z_{16}$. We take the point set $Z_{16} \cup \{\infty\}$. The required 12 parallel classes will be generated from three initial parallel classes P_1, P_2 and P_3 by +4 modulo 16. The blocks of the three initial parallel classes are listed below.

P_1 :	0,1,2,3	4,5,6,8	7,9,12	10,11,14	13,15,∞
P_2 :	0,3,5,9	1,7,10,14	2,8,12	4,11,15	6,13,∞
P_3 :	0,5,10,15	1,6,8,14	2,7,11	3,4,12	9,13,∞

□

Lemma 3.6 *There is an $SKPD_2(\{3,4\},20)$ with 15 parallel classes.*

Proof: Let $G = Z_{20}$. We take the point set Z_{20}. The required 15 parallel classes will be generated from three initial parallel classes P_1, P_2 and P_3 by +4 modulo 20. The blocks of the three initial parallel classes are listed below.

P_1 :	0,1,2,3	4,5,6,8	7,9,11	10,12,17	14,15,18	13,16,19
P_2 :	0,4,9,13	1,5,10,15	2,11,17	3,8,16	6,12,18	7,14,19
P_3 :	0,7,11,12	4,10,14,17	1,9,18	2,8,19	3,5,13	6,15,16

□

Combining Lemmas 3.2-3.6, we have the following.

Lemma 3.7 (i) *There is an $SKPD_2(\{3,4\},v)$ with $n(v)$ parallel classes for every $v \in \{8,14,17,20\}$.* (ii) *There is an $SKPD_2(\{3,4\},11)$ with 6 parallel classes.*

Lemma 3.8 *There exist an $SKPD_2(\{3,4\},v)$ and an $ISKPD_2(\{3,4\},v,5)$ with $v-5$ parallel classes for every $v \equiv 2 \pmod 3$, $41 \le v \le 74$ and $v \ne 50$.*

Proof: Take the point set $Z_{v-5} \cup \{\infty_1, \cdots, \infty_5\}$. we only list the blocks of the initial parallel class P_0, the other parallel classes can be generated under the additive group Z_{v-5}. For each v, P_0 contains two blocks in common, $B_1 = \{0,1,2,4\}$ and $B_2 = \{3,6,10,15\}$. The other blocks are as follow.

$v = 41$:	5,11,16	7,20,27	8,22,28	12,23,31	13,25,34	14,24,32
	9,26,∞_1	17,30,∞_2	18,33,∞_3	19,29,∞_4	21,35,∞_5	

$v = 44:$	12, 26, 34	7, 13, 21	8, 17, 27	9, 19, 30	11, 24, 31	16, 32, 38
	18, 23, 36	5, 28, ∞_1	14, 29, ∞_2	20, 35, ∞_3	22, 33, ∞_4	25, 37, ∞_5
$v = 47:$	14, 22, 38	21, 36, 41	8, 17, 25	9, 19, 31	12, 33, 39	7, 18, 35
	11, 24, 37	16, 27, 34	5, 28, ∞_1	13, 32, ∞_2	20, 30, ∞_3	23, 29, ∞_4
	26, 40, ∞_5					
$v = 53:$	18, 33, 47	7, 13, 20	16, 29, 44	17, 25, 34	12, 22, 36	14, 26, 46
	9, 27, 38	11, 32, 37	23, 31, 41	28, 39, 45	8, 35, ∞_1	5, 30, ∞_2
	19, 42, ∞_3	21, 43, ∞_4	24, 40, ∞_5			
$v = 56:$	23, 35, 48	7, 13, 20	26, 40, 46	9, 19, 27	12, 22, 33	14, 28, 44
	18, 37, 42	8, 30, 39	16, 31, 50	21, 38, 49	24, 32, 47	5, 34, ∞_1
	11, 36, ∞_2	17, 41, ∞_3	25, 43, ∞_4	29, 45, ∞_5		
$v = 59:$	25, 34, 53	7, 13, 20	23, 40, 46	9, 19, 27	8, 42, 47	14, 26, 39
	11, 30, 41	21, 35, 52	16, 32, 43	12, 37, 45	18, 36, 51	24, 38, 48
	5, 31, ∞_1	17, 33, ∞_2	22, 44, ∞_3	28, 50, ∞_4	29, 49, ∞_5	
$v = 62:$	30, 46, 52	21, 40, 45	8, 17, 25	9, 19, 27	28, 42, 49	14, 26, 39
	18, 32, 48	23, 38, 53	5, 16, 36	11, 44, 55	7, 33, 43	12, 35, 41
	13, 31, 50	20, 37, ∞_1	22, 51, ∞_2	24, 47, ∞_3	29, 54, ∞_4	34, 56, ∞_5
$v = 65:$	11, 30, 51	7, 13, 20	8, 17, 25	23, 41, 56	12, 22, 33	14, 26, 39
	18, 32, 46	21, 37, 52	24, 44, 50	9, 40, 45	5, 35, 43	16, 42, 53
	19, 38, 54	29, 47, 57	27, 49, ∞_1	28, 55, ∞_2	31, 48, ∞_3	34, 58, ∞_4
	36, 59, ∞_5					
$v = 68:$	23, 41, 62	29, 40, 59	9, 19, 27	20, 44, 49	12, 22, 33	14, 26, 39
	18, 32, 46	21, 36, 51	24, 43, 60	5, 11, 31	7, 13, 42	8, 16, 48
	17, 37, 53	25, 50, 57	34, 47, 56	28, 54, ∞_1	35, 58, ∞_2	38, 55, ∞_3
	45, 61, ∞_4	30, 52, ∞_5				
$v = 71:$	30, 48, 64	28, 55, 60	9, 19, 27	13, 35, 41	12, 22, 33	14, 26, 39
	18, 32, 46	21, 36, 51	23, 40, 59	24, 45, 65	5, 11, 34	16, 47, 58
	7, 42, 50	8, 52, 61	20, 37, 53	29, 49, 56	17, 43, ∞_1	25, 54, ∞_2
	31, 57, ∞_3	38, 62, ∞_4	44, 63, ∞_5			
$v = 74:$	29, 48, 64	30, 52, 63	8, 13, 45	20, 55, 61	12, 22, 33	14, 26, 39
	18, 32, 46	21, 36, 51	23, 40, 56	24, 42, 66	28, 49, 68	5, 11, 34
	16, 38, 65	17, 37, 62	7, 50, 59	19, 27, 58	25, 35, 43	9, 47, ∞_1
	31, 54, ∞_2	41, 67, ∞_3	44, 57, ∞_4	53, 60, ∞_5		

□

Lemma 3.9 *There is an* $SKPD_2(\{3,4\}, v)$ *with* $v - 5$ *parallel classes for each* $v \in \{32, 38, 50\}$.

Proof: From Lemma 3.2, we have an $\mathrm{SKPD}_2(\{3,4\}, 8)$ with 4 parallel classes. Thus we can use Lemma 2.5 with $h = 8$ to get the required SKPDs, where the needed ISKPDs come from Lemma 2.9. □

Combining Lemmas 2.8, 3.7- 3.9, we get the following.

Lemma 3.10 (i) *There is an* $SKPD_2(\{3,4\},v)$ *with* $n(v)$ *parallel classes for every* $v \equiv 2 \pmod 3$, $8 \le v \le 74$ *and* $v \ne 11$; (ii) *There is an* $SKPD_2(\{3,4\},11)$ *with 6 parallel classes.*

Lemma 3.11 *There is an* $SKPD_2(\{3,4\},v)$ *with* $v-3$ *parallel classes for every* $v \equiv 1 \pmod 3$, $7 \le v \le 49$ *and* $v \ne 43$.

Proof: We use Lemma 2.5 with $h = 4$ to get the required SKPDs for $16 \le v \le 49$ and $v \ne 43$, where the needed ISKPDs come from Lemma 2.7. The left three values come from Lemma 3.1 and Lemma 2.6. □

4 Main results

In this section, we shall prove our main results. For our purpose we need the following "Filling in Holes" construction.

Lemma 4.1 *Suppose*

1. *there is a simple (3,2)-frame of type* $g_1 g_2 \cdots g_m$;
2. *there is an* $ISKPD_2(\{3,4\}, g_i + w, w)$ *for every* $1 \le i < m$;
3. *there is an* $SKPD_2(\{3,4\}, g_m + w)$.

Then there is an $SKPD_2(\{3,4\}, w + \sum_{1 \le i \le m} g_i)$.

Proof: Suppose $(X, \mathcal{G}, \mathcal{B})$ is a simple (3,2)-frame of type $g_1 g_2 \cdots g_m$, where $\mathcal{G} = \{G_1, \cdots, G_m\}$ and $|G_i| = g_i$ $(1 \le i \le m)$. For $1 \le i < m$, there are g_i frame parallel classes missing the group G_i, and the same number of parallel classes in the $ISKPD_2(\{3,4\}, g_i + w, w)$ which contains one or two blocks of size 4; match them up arbitrarily, placing the g_i points of the $ISKPD_2(\{3,4\}, g_i + w, w)$ on the i-th group of the frame and the w points in its hole on w new points.

Next, each $ISKPD_2(\{3,4\}, g_i + w, w)$ contains $w-3$ or $w-5$ auxiliary parallel classes of triples. Form unions of these with the same parallel classes of the $SKPD_2(\{3,4\}, g_m + w)$, to form $w-3$ or $w-5$ additional parallel classes. There remain g_m parallel classes of the SKPD$_2$($\{3,4\}, g_m + w$), which can be match arbitrarily with the g_m frame parallel classes of the m-th group to complete the construction.

It is easy to check that this construction gives an SKPD$_2$($\{3,4\}, w + \sum_{1\le i\le m} g_i$). □

Lemma 4.2 *For each v, $v \equiv 0 \pmod 3$, $v \ge 117$, there is a simple (3,2)-frame of type $24^a 21^b (3u)^1$, where $v = 24a + 21b + 3u$, $a \ge 3$, $b \ge 0$, $0 \le u \le 8$.*

Proof: Let $(n_0, n_1, \cdots)$ be the infinite sequence of integers defined as follows. The initial sequence $(n_0, \cdots, n_{15})$ is $(8, 11, 13, 16, 19, 23, 25, 31, 37, 41, 49, 57, 67, 79, 87, 99)$. Let $n_{15+j} = 99 + j$ for $j \ge 1$. From Theorem 2.1, an idempotent TD $(8, n_i)$ exists for each n_i, $i \ge 0$. Let $t = v/3$. Since $v \ge 117$, we have $t \ge 39$. There exists an integer n_i from the sequence so that $7n_i - 17 \le t \le 8n_i$. This can always be done because $8n_i \ge 7n_{i+1} - 18$ for all $i \ge 0$. Let $t = 8a + 7b + u$, where $a = n_i - 1 - b - \omega$, $0 \le b \le n_i - 5 - \omega$, $0 \le \omega \le 2$ and $0 \le u \le 8$. Form the idempotent TD $(8, n_i)$ with groups $G_1, \cdots, G_8$ and blocks $B_1, \cdots, B_{n_i}$ in one parallel class. Delete all the points in blocks $B_1, \cdots, B_\omega$ when $\omega \ge 1$ and $8 - u$ points in block $B_{\omega+1}$ that lie in $G_1, \cdots, G_{8-u}$. Furthermore, delete b points in G_8 that lie in $B_{n_i-b+1}, \cdots, B_{n_i}$ when $b \ge 1$. Taking the truncated blocks $B_1, \cdots, B_{n_i}$ as groups, we have formed a GDD of type $8^a 7^b u^1$ having all blocks of size at least four. In fact, for values of t in different sub-intervals below we can always find the corresponding parameters b, w, u and a as follows:

$t \in [7n_i - 3, 8n_i]$,	$b \in [0, n_i - 5]$,	$w = 0$,	$u \in [0, 8]$,	$a = n_i - 1 - b$;
$t \in [7n_i - 10, 8n_i - 8]$,	$b \in [0, n_i - 6]$,	$w = 1$,	$u \in [0, 8]$,	$a = n_i - 2 - b$;
$t \in [7n_i - 17, 8n_i - 16]$,	$b \in [0, n_i - 7]$,	$w = 2$,	$u \in [0, 8]$,	$a = n_i - 3 - b$;

Now we may apply Lemma 2.4 with weight $h = 3$ to obtain a simple (3,2)-frame of type $24^a 21^b (3u)^1$, where the required simple (3,2)-frames of types 3^m, $m \ge 4$, come from Theorem 2.3. It is easy to check that $a \ge 3$. □

To get more frames we also need the following results on 4-GDD of type $g^4 m^1$. From Rees [12] and [10] we have:

Theorem 4.3 *There exists a 4-GDD of type g^4m^1 with $m > 0$ if and only if $g \equiv m \equiv 0$ (mod 3) and $0 < m \leq 3g/2$.*

Lemma 4.4 *There exists a simple (3,2)-frame of type g^4m^1 with $m > 0$ if and only if $g \equiv m \equiv 0$ (mod 3) and $0 < m \leq 3g/2$.*

Proof: From Theorem 4.3 there exists a 4-GDD of type g^4m^1. Applying Lemma 2.4 with $h = 1$ and $k = 4$ we get the required simple (3,2)-frame, as a simple (3,2)- frame of type 1^4 comes from Theorem 2.3. □

Lemma 4.5 *There is an $SKPD_2(\{3,4\},v)$ for $v \equiv 1$ (mod 3), $v \geq 121$.*

Proof: By Lemma 4.2, there is a simple (3,2)-frame of type $24^a21^b(3u)^1$ on v' points, $v' \equiv 0$ (*mod* 3) and $v' \geq 117$, where $a \geq 3$, $b \geq 0$, and $0 \leq u \leq 8$. For $v \equiv 1$ (*mod* 3), apply Lemma 4.1 with $w = 4$. Use an $ISKPD_2(\{3,4\},28,4)$ and an $ISKPD_2(\{3,4\},25,4)$ from Lemma 2.7 to fill all but one hole of size $3u$ if $u > 0$ or size 24 if $u = 0$, and fill the final hole with $3u + 1$ parallel classes of an $SKPD_2(\{3,4\},3u + 4)$ which comes from Lemma 3.11. Then we obtain an $SKPD_2(\{3,4\},v)$, where $v = v' + 4$. □

Lemma 4.6 *There is an $SKPD_2(\{3,4\},v)$ for $v \equiv 1$ (mod 3), $52 \leq v \leq 118$.*

Proof: From Lemma 4.4 we have frames of types $24^4x_1^1(0 \leq x_1 \leq 18)$, $18^4x_2^1(0 \leq x_2 \leq 21)$, $15^4x_3^1(0 \leq x_1 \leq 9)$ and $12^4x_4^1(0 \leq x_4 \leq 9)$, where $x_i \equiv 0$ (*mod* 3) for $1 \leq i \leq 4$. Thus we may obtain the required SKPDs by a similar construction as above with hole size $w = 4$, where all the needed SKPDs and ISKPDs come from Lemma 3.11 and Lemma 2.7. □

Lemma 4.7 *There is an $SKPD_2(\{3,4\},43)$.*

Proof: From Theorem 2.3 we have a simple (3,2)-frame of type 10^4. We apply Lemma 4.1 with $w = 3$ to fill in holes using an $ISKP_2(\{3,4\},13,3)$ from Lemma 2.6. □

Combining Lemmas 3.11, 4.5- 4.7, we have the following.

Theorem 4.8 *There exists an* $SKPD_2(\{3,4\},v)$ *with* $n(v)$ *parallel classes for every* $v \equiv 1 \ (mod\ 3)$.

Similarly, for $v \equiv 2 \ (mod\ 3)$, apply Lemma 4.1 with $w = 5$. Use an $ISKPD_2(\{3,4\},29,5)$ and an $ISKPD_2(\{3,4\},26,5)$ from Lemma 2.8 to fill all but one hole of size $3u$ if $u > 0$ or size 24 if $u = 0$, and fill the final hole with $3u$ parallel classes of an $SKPD_2(\{3,4\},3u+5)$ which comes from Lemma 3.10. Thus we can obtain the following lemma.

Lemma 4.9 *There is an* $SKPD_2(\{3,4\},v)$ *for* $v \equiv 2 \ (mod\ 3)$, $v \geq 122$.

Lemma 4.10 *There is an* $SKPD_2(\{3,4\},v)$ *for* $v \equiv 2 \ (mod\ 3)$, $77 \leq v \leq 119$.

Proof: From Lemma 4.4 we have frames of types $24^4 x_1^1 (0 \leq x_1 \leq 18)$ and $18^4 x_2^1 (0 \leq x_2 \leq 21)$, where $x_i \equiv 0 \ (mod\ 3)$ for $i = 1, 2$. Thus we may obtain the required SKPDs by a similar construction as Lemma 4.6 with hole size $w = 5$, where all the needed SKPDs and ISKPDs come from Lemma 3.10 and Lemma 2.8. □

Combining Theorem 4.8, Lemmas 3.10, 4.9 and 4.10, we have proved our main results as below.

Theorem 4.11 (i) *There is an* $SKPD_2(\{3,4\},v)$ *with* $n(v)$ *parallel classes for every* $v \equiv 1, 2 \ (mod\ 3)$ *and* $v \neq 11$; (ii) *There is an* $SKPD_2(\{3,4\},11)$ *with* 6 *parallel classes.*

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