Grinberg's Criterion Applied to Some Non-Planar Graphs

G.L. Chia[a] and Carsten Thomassen[b]

[a]*Institute of Mathematical Sciences, University Malaya, 50603 Kuala Lumpur, Malaysia*
[b]*Department of Mathematics, Technical University of Denmark, DK-2800, Lyngby, Denmark*

Abstract

Robertson ([5]) and independently, Bondy ([1]) proved that the generalized Petersen graph $P(n, 2)$ is non-hamiltonian if $n \equiv 5 \pmod 6$, while Thomason [7] proved that it has precisely 3 hamiltonian cycles if $n \equiv 3 \pmod 6$. The hamiltonian cycles in the remaining generalized Petersen graphs were enumerated by Schwenk [6]. In this note we give a short unified proof of these results using Grinberg's theorem.

A celebrated result of Grinberg (see [4]) concerning planar hamiltonian graphs states that if a planar graph G has a hamiltonian cycle C which partitions its f_i faces of degree i into f_i' (respectively f_i'') faces of degree i in the interior (respectively exterior) of C, then

$$\sum_{i\geq 3}(i-2)(f_i' - f_i'') = 0.$$

If there is precisely one natural number i not congruent to 2(mod 3) such that $f_i > 0$, then Grinberg's equation cannot be satisfied, and hence the graph is non-hamiltonian. But even if the equation can be satisfied, it is still possible, in special cases, to use the criterion to prove that a graph is non-hamiltonian. Thus Thomassen [8] used the criterion to describe an infinite class of cubic planar hypohamiltonian graphs (all of which have a face partition that satisfies Grinberg's equation), and also the Tutte graph can be shown to be non-hamiltonian using the Grinberg criterion, see for example, [2] p.166 and [3] Chapter 6. In this note we apply the criterion to a class of non-hamiltonian graphs, namely some generalized Petersen graphs.

Suppose n and k are two integers such that $1 \leq k \leq n-1$ and $n \geq 5$. The *generalized Petersen graph* $P(n,k)$ is defined to have vertex-set $\{u_i, v_i : i = 0,1,\ldots,n-1\}$ and edge-set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : i = 0,1,\ldots,n-1$ with subscripts reduced modulo $n\}$.

Let F_m denote the m-th Fibonacci number defined by $F_1 = F_2 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for $m > 2$. It is easy to prove, by induction on n, that the number of matchings (including the empty matching) of the path with n vertices is F_{n+1}. Hence the number of matchings of the cycle with n vertices is $F_{n+1} + F_{n-1}$.

Theorem 1 *Let n be a natural number, $n \geq 5$. Then*

(i) ([1], [5]) $P(n,2)$ is non-hamiltonian if $n \equiv 5 \pmod 6$,

(ii) ([7]) $P(n,2)$ has precisely three hamiltonian cycles if $n \equiv 3 \pmod 6$, and

(iii) ([6]) the number of hamiltonian cycles in $P(n,2)$ is

$$\begin{cases} n & \text{if } n \equiv 1 \pmod 6 \\ 2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1) & \text{if } n \equiv 0,2 \pmod 6 \\ n + 2(F_{\frac{n}{2}+1} + F_{\frac{n}{2}-1} - 1) & \text{if } n \equiv 4 \pmod 6 \end{cases}$$

Proof: Assume first that $n \equiv 3,5 \pmod 6$ and $n \geq 5$.

If the two edges u_0u_1 and v_0v_2 are deleted from $P(n,2)$, the result is a planar graph whose face-degree sequence is $(5,\ldots,5,\frac{n+5}{2},\frac{n+7}{2})$ (see Figure 1) and hence is non-hamiltonian by Grinberg's criterion because when $n \equiv 3 \pmod 6$, $\frac{n+5}{2}$ and $\frac{n+7}{2}$ are $1,2 \pmod 3$ respectively while when $n \equiv 5 \pmod 6$, $\frac{n+5}{2}$ and $\frac{n+7}{2}$ are $2,0 \pmod 3$ respectively.

This means that

$(*)$ if $P(n,2)$ has a hamiltonian cycle C, then C must contain at least one of the edges u_0u_1, v_0v_2.

Assume now that C is a hamiltonian cycle of $P(n,2)$. Since C cannot contain all the edges of the inner cycle $v_0v_2v_4\cdots v_{n-2}v_0$, we may assume that v_0v_2 is not an edge in C. But then this implies that the paths $v_{n-2}v_0u_0$ and $u_2v_2v_4$ must be part of C.

Since v_0v_2 is not an edge in C, the observation $(*)$ implies that u_0u_1 is an edge in C. By symmetry, u_1u_2 is also an edge in C. This follows because there is an automorphism (the reflection fixing u_1, v_1) of $P(n,2)$ which interchanges between the edges u_1u_0, u_2u_1 and keeps v_0v_2 fixed. ("Reflection" here refers to the standard drawing of $P(n,k)$ where the vertices $u_0, u_1, \ldots$ and also the vertices $v_0, v_1, \ldots$ form convex n-gons.) Hence C contains the

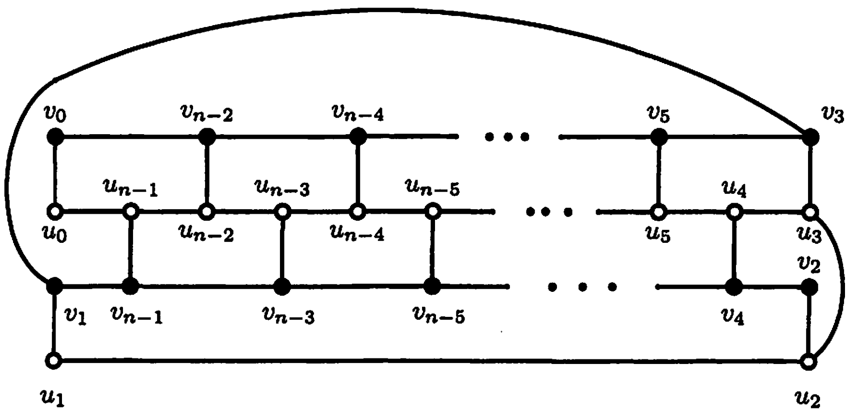

Figure 1: $P(n, 2)$ with u_0u_1 and v_0v_2 deleted, $n \equiv 3, 5 \pmod 6$

path $v_{n-2}v_0u_0u_1u_2v_2v_4$. So C also contains the path $v_{n-1}v_1v_3u_3u_4$ and therefore C does not contain the edge v_3v_5. Summarizing, we have proved that if a hamiltonian cycle C does not contain the edge v_0v_2, then C does not contain the edge v_3v_5 either. By repeating this argument with v_3v_5 instead of v_0v_2, we conclude that C does not contain the edge v_6v_8 either. In fact C does not contain any of the edges v_0v_2, v_3v_5, v_6v_8, v_9v_{11}, ... Since C must contain some edge of the inner cycle $v_0v_2v_4 \cdots v_{n-2}v_0$, we conclude that $n \equiv 0 \pmod 3$. Hence $P(n, 2)$ has no hamiltonian cycle if $n \equiv 5 \pmod 6$. In the case that $n \equiv 3 \pmod 6$, the argument eventually leads to a unique hamiltonian cycle which can be rotated to yield precisely three hamiltonian cycles.

Assume next that $n \equiv 1 \pmod 6$. Then Grinberg's equation is satisfied but only if the two faces of degrees $\frac{n+5}{2}$ and $\frac{n+7}{2}$ are both in the interior (or exterior) of the hamiltonian cycle on the resulting graph of Fig. 1. This is possible only if the edge v_1v_{n-1} is not contained in the hamiltonian cycle. Assume that C' is such a hamiltonian cycle. Then it is easy to see (from Fig. 1) that the paths $v_{n-3}v_{n-1}u_{n-1}u_0v_0v_{n-2}u_{n-2}u_{n-3}$ and $v_4v_2u_2u_1v_1v_3u_3u_4$ must be part of C'. There is a unique hamiltonian cycle containing these paths which can be rotated to yield n hamiltonian cycles. If it is not possible to obtain a hamiltonian cycle by deleting a pair u_iu_{i+1}, v_iv_{i+2}, then we get a contradiction as in the case when $n \equiv 5 \pmod 6$.

When $n \geq 6$ is even, $P(n, 2)$ is a planar graph. Again, the above method can be applied. Note that (*) cannot be satisfied for each pair u_iu_{i+1}, v_iv_{i+2} and also for each pair u_iu_{i-1}, v_iv_{i-2} because the argument in the case $n \equiv 3, 5 \pmod 6$ would lead to a 2-factor consisting of two cycles rather than a hamiltonian cycle. So there exists some pair of edges u_iu_{i+1}, v_iv_{i+2} (or some pair u_iu_{i-1}, v_iv_{i-2}), say u_0u_1, v_0v_2, which is avoided by some hamiltonian cycle.

Draw $P(n,2)$ in the plane such that the $\frac{n}{2}$-cycle $v_1v_3\dots v_{n-1}v_1$ is the outer face boundary. In this case, the face-degree sequence of the resulting graph (after deleting the edges u_0u_1 and v_0v_2) is $(5,\dots,5,\frac{n}{2},\frac{n}{2}+6)$ and Grinberg's equation is satisfied. For this to be possible, any hamiltonian cycle must contain the edge v_1v_{n-1} (which is common to both the $\frac{n}{2}$-face and the $(\frac{n}{2}+6)$-face) unless $n \equiv 4 \pmod 6$.

Assume first that $n \equiv 0,2 \pmod 6$. Then for any even integer $0 \le i \le \frac{n}{2}$, whenever the pair of edges v_iv_{i+2}, u_iu_{i+1} is deleted, then the paths $u_{i-1}u_iv_iv_{i-2}$, $v_{i-1}v_{i+1}u_{i+1}u_{i+2}v_{i+2}v_{i+4}$ and $u_{i+4}u_{i+3}v_{i+3}v_{i+5}$ must be part of the hamiltonian cycle. We now follow the hamiltonian cycle along the path $v_{i+2}v_{i+4}\dots$. If we never use an edge v_ju_j there is a unique way to continue the hamiltonian cycle. On the other hand, the first time we use an edge v_ju_j we repeat the previous configuration with j instead of i. Those edges of the cycle $v_0v_2\dots v_0$ which are not in the hamiltonian cycle clearly form a matching. On the other hand, whenever we specify a matching on this cycle, there is a unique hamiltonian cycle which avoids this matching and also avoids edges of the form u_iu_{i+1}, v_iv_{i+2}. Therefore the number of hamiltonian cycles in $P(n,2)$ avoiding some pairs of edges of the form u_iu_{i+1}, v_iv_{i+2} is $F_{\frac{n}{2}+1}+F_{\frac{n}{2}-1}-1$. Note that for any such hamiltonian cycle, $(*)$ fails for the pairs u_iu_{i+1}, v_iv_{i+2} but holds for the pairs u_iu_{i-1}, v_iv_{i-2}.

By symmetry (more precisely, by taking the cycle $u_1u_2...u_nu_1$ in its reverse order) the number of hamiltonian cycles in $P(n,2)$ avoiding pairs of edges of the form u_iu_{i-1}, v_iv_{i-2} is also $F_{\frac{n}{2}+1}+F_{\frac{n}{2}-1}-1$. Thus the total number of hamiltonian cycles in $P(n,2)$ is $2(F_{\frac{n}{2}+1}+F_{\frac{n}{2}-1}-1)$ in the case $n \equiv 0,2 \pmod 6$.

The reason that we have counted all hamiltonian cycles is that $(*)$ fails either for the pairs u_iu_{i+1}, v_iv_{i+2} or for the pairs u_iu_{i-1}, v_iv_{i-2}, as noted above.

For $n \equiv 4 \pmod 6$, there are two types of hamiltonian cycles, namely the $2(F_{\frac{n}{2}+1}+F_{\frac{n}{2}-1}-1)$ hamiltonian cycles which we have already counted, and also the unique hamiltonian cycle that avoids $u_0u_1, v_0v_2, v_1v_{n-1}$, and those which can be obtained from this hamiltonian cycle by rotation and reflection. There are n hamiltonian cycles of the latter type. This completes the proof. □

References

[1] J.A. Bondy, Variations on the hamiltonian theme, *Canad. Math. Bull.* **15** (1972) 57-62.

[2] M. Capobianco and J.C. Molluzzo, *"Examples and Counterexamples in Graph Theory"*, North-Holland, (1978) New York.

[3] G. Chartand and L.Lesniak, *"Graphs and Digraphs"*, Chapman & Hall/CRC, 4th Edition (2005), New York.

[4] E. Grinberg, Plane homogeneous graphs of degree three without Hamiltonian circuits, *Latvian Math. Yearbook, Izdat. Zinatne Riga* **4** (1968) 51-58 [Russian, Latvian and English summaries.]

[5] G.N. Robertson, *"Graphs under Girth, Valency, and Connectivity Constraints"*, Ph. D. Thesis, University of Waterloo, Ontario, Canada, 1968.

[6] A.J. Schwenk, Enumeration of Hamiltonian cycles in certain generalized Petersen graphs, *J. Combinat. Theory B* **47** (1989) 53-59.

[7] A.G. Thomason, Cubic graphs with three hamiltonian cycles are not always uniquely edge-colorable, *J. Graph Theory* **6** (1982) 219-221.

[8] C. Thomassen, Planar cubic hypohamiltonian and hypotraceable graphs, *J. Combinat. Theory B* **30** (1981) 36-44.

[9] W.T. Tutte, On Hamiltonian circuits, *J. London Math. Soc.* **21** (1946) 98-101.