Connectivity of direct products of graphs

Wei Wang * Ni-Ni Xue
College of Information Engineering, Tarim University,
Alar, Xinjiang, 843300, P.R.China

Abstract

Let $\kappa(G)$ be the connectivity of G and $G \times H$ the direct product of G and H. We prove that for any graphs G and K_n with $n \geq 3$, $\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\}$, which was conjectured by Guji and Vumar.

Keywords: Connectivity; Direct product; Minimum degree

1 Introduction

Throughout this paper we consider only finite undirected graphs without loops and multiple edges.

Let $G = (V(G), E(G))$ be a graph. The connectivity of G is the number, denoted as $\kappa(G)$, equal to the fewest number of vertices whose removal from G results in a disconnected or trivial graph. The direct (or Kronecker) product $G \times H$ of graph G and H has vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)\}$.

The connectivity of direct products of graphs has been studied recently. Unlike the case of Cartesian products where the general formula was obtained [4, 5], results for direct products have been given only in special cases. Mamut and Vumar [3] considered product of two complete graphs and proved for any K_m and K_n with $n \geq m \geq 2$ and $n \geq 3$,

$$\kappa(K_m \times K_n) = (m-1)(n-1). \tag{1}$$

Later, Guji and Vumar [2] proved for any bipartite graph G and K_n with $n \geq 3$,

$$\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\}, \tag{2}$$

*Corresponding author: wangwei.math@gmail.com

where $\delta(G)$ denoted the minimum degree of G. In the same paper, Guji and Vumar conjectured (2) holds even without the assumption of bipartiteness of G.

In the next section we shall prove the conjecture.

2 The result

Theorem 1. *$\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\}$ for $n \geq 3$.*

The proof of the theorem will be postponed to the end of this section. We first give some properties on direct products of graphs [1].

Lemma 1. (1) *The direct product of nontrivial graphs G and H is connected if and only if both factors are connected and at least one factor contains an odd cycle.*

(2) *$\delta(G \times H) = \delta(G)\delta(H)$, and in particular, $\delta(G \times K_n) = (n-1)\delta(G)$.*

We shall always label $V(G) = \{u_1, \ldots, u_m\}, V(K_n) = \{v_1, \ldots, v_n\}$ and set $S_i = \{u_i\} \times V(K_n)$. Let $S \subseteq V(G \times K_n)$ satisfy the following two conditions:

(1). $|S| < \min\{n\kappa(G), (n-1)\delta(G)\}$, and

(2). $S'_i := S_i - S \neq \emptyset$, for $i = 1, 2, \ldots, m$.

Associated with G, K_n and S, we define a new graph G^* as follows:

(1). $V(G^*) = \{S'_1, S'_2, \ldots, S'_m\}$, and

(2). $E(G^*) = \{S'_iS'_j : E(S'_i, S'_j) \neq \emptyset\}$, where $E(S'_i, S'_j)$ denotes the collection of all edges in $(G \times K_n - S)$ with one end in S'_i and the other in S'_j.

Notice G^* can be defined only if $\kappa(G) > 0$ since otherwise condition (1) is meaningless.

Lemma 2. *If G is connected then G^* is connected.*

Proof. Suppose G^* is not connected. Then the vertices of G^* can be partitioned into two parts, X^* and Y^*, such that there are no edges joining a vertex in X^* and a vertex in Y^*. Let $r = |X^*|$. Without loss of generality, we may assume $X^* = \{S'_1, \ldots, S'_r\}$ and $Y^* = \{S'_{r+1}, \ldots, S'_m\}$.

Let $X = \{u_1, \ldots, u_r\}$ and $Y = \{u_{r+1}, \ldots, u_m\}$. Since G is connected, there is at least one edge joining a vertex in X and a vertex in Y. Let Z be the collection of ends of all edges in $E(X, Y)$.

Let $Z^* = \{S'_j : j \in \{1, \ldots, m\}$ and $|S'_j| = 1\}$. For each $u_i \in Z$, by the definition of Z, there is an edge $u_iu_j \in E(X, Y)$. It follows that both S'_i and S'_j contains exactly one element since otherwise $E(S'_i, S'_j)$ contains at least one edge by the definition of $G \times K_n$. Therefore $S'_i \in Z^*$ and we have $|Z| \leq |Z^*|$. We need to consider two cases:

Case 1: Either $X \subseteq Z$ or $Y \subseteq Z$. We may assume $X \subseteq Z$, then the degree of any vertex $u_i \in X$ can not exceed $|Z|-1$. Therefore $\delta(G) \leq |Z|-1$. By a simple calculation, we have

$$|S| \geq (n-1)|Z^*| \geq (n-1)|Z| > (n-1)\delta(G) \geq \min\{n\kappa(G), (n-1)\delta(G)\},$$

a contradiction.

Case 2: $X \not\subseteq Z$ and $Y \not\subseteq Z$. Either of $X \cap Z$ and $Y \cap Z$ is a separating set. Therefore, $\kappa(G) \leq \min\{|X \cap Z|, |Y \cap Z|\} \leq |Z|/2$. Similarly, we have

$$|S| \geq (n-1)|Z^*| \geq (n-1)|Z| > |Z|n/2 \geq n\kappa(G) \geq \min\{n\kappa(G), (n-1)\delta(G)\},$$

again a contradiction. □

While the above lemma tells us that the new graph G^* is connected, what we most concern is the connectedness of $G \times K_n - S$. We need the following lemma.

Lemma 3. *Any vertex of G^*, S_i', as a subset of $V(G \times K_n - S)$, is contained in the vertex set of some component of $G \times K_n - S$.*

Proof. It suffices to prove the lemma for $i = 1$.

If $|S_1'| = 1$, then the assertion holds trivially. We need to consider two cases:

Case 1: $|S_1'| \geq 3$. Assume S_1' is not contained in any component of $G \times K_n - S$. Then there must exist a component C such that $0 < |S_1' \cap V(C)| \leq |S_1'|/2 < |S_1'| - 1$. Let $(u_1, v_s) \in S_1' \cap V(C)$ be any vertex. Since $|S| < (n-1)\delta(G)$, it follows by (2) of lemma 1 that (u_1, v_s) has at least one adjacent vertex in $G \times K_n - S$. Let (u_j, v_p) be an adjacent vertex of (u_1, v_s). Clearly, $(u_j, v_p) \in V(C)$ and $S_1' - \{(u_1, v_p)\} \subseteq V(C)$ since every vertex in $S_1' - \{(u_1, v_p)\}$ is adjacent to (u_j, v_p). It follows $|S_1' \cap V(C)| \geq |S_1'| - 1$, a contradiction.

Case 2: $|S_1'| = 2$. Let $Z^* = \{S_j' : j \in \{1, \ldots, m\}$ and $|S_j'| = 1\}$ and C^* be a component of $G^* - Z^*$ containing S_1'. Let r be the order of C^*. Without loss of generality, we may assume $V(C^*) = \{S_1', \ldots, S_r'\}$.

Since each $S_j' \in V(C^*)$ contains at least two elements, any edge $S_k'S_j'$ in C^* implies every vertex in S_k' has at least one adjacent vertex in S_j'. Therefore, if there is a vertex S_j' in C^* contained in the vertex set of some component C of $G \times K_n - S$, then every S_k' is contained in $V(C)$ provided $S_k'S_j' \in E(C^*)$. It follows by the connectedness of C^* that $\cup_{i=1}^r S_i' \subseteq V(C)$ and hence $S_1' \subseteq V(C)$.

By case1, we may assume each $S_j' \in V(C^*)$ contains exactly two elements. Let $S_j' = \{u_j\} \times F_j, j = 1, \ldots, r$.

Subcase 2.1: There exists an edge $S'_jS'_k$ in C^* with $F_j \neq F_k$. One easily verify that $S'_j \cup S'_k$ induces a connected subgraph of $G \times K_n - S$. The lemma follows.

Subcase 2.2: There exists no edge $S'_jS'_k$ in C^* with $F_j \neq F_k$. By the connectedness of C^*, all F_j in C^* are equal. Notice that C^* and the subgraph induced by $\cup_{i=1}^{r} S'_i$ are isomorphic to $G[u_1, \dots, u_r]$ and $G[u_1, \dots, u_r] \times K_2$, respectively. We claim $G[u_1, \dots, u_r]$ must contain an odd cycle, which will finish our proof by (1) of lemma 1.

Suppose $G[u_1, \dots, u_r]$ does not contain an odd cycle. Then either $r = 1$ or $G[u_1, \dots, u_r]$ is bipartite. Either of the two cases implies $\delta(G[u_1, \dots, u_r]) \leq r/2$. Let $j \in \{1, \dots, r\}$ such that $\deg_{G[u_1,\dots,u_r]}(u_j) = \delta(G[u_1, \dots, u_r])$.

Let u_k be any adjacent vertex of u_j in G, then either $S'_k \in |Z^*|$, or S'_k is an adjacent vertex of S'_j in C^*. Therefore,

$$\delta(G) \leq \deg_G(u_j) = \deg_{C^*}(S'_j) + |Z^*| = \deg_{G[u_1,\dots,u_r]}(u_j) + |Z^*| \leq r/2 + |Z^*|. \tag{3}$$

By a simple calculation,

$$|S| \geq (n-2)r + (n-1)|Z^*| \geq (n-1)(\frac{r}{2} + |Z^*|). \tag{4}$$

From (3) and (4), we obtain

$$|S| \geq (n-1)\delta(G), \tag{5}$$

a contradiction.

□

Lemma 4. *Let $m = |G| \geq 2$ and u_i be any vertex of G. Then*
(1). $\delta(G - u_i) \geq \delta(G) - 1$, and
(2). $\kappa(G - u_i) \geq \kappa(G) - 1$.

Proof of Theorem 1. We apply induction on $m = |G|$. It trivially holds when $m = 1$. We therefore assume $m \geq 2$ and that the result holds for all graphs of order $m - 1$.

It is clear $\kappa(G \times K_n) \leq \min\{n\kappa(G), (n-1)\delta(G)\}$ by lemma 1. The nontrivial part of the proof is hence to show the other inequality. We may assume $\kappa(G) > 0$. Let $S \subseteq V(G \times K_n)$ satisfy condition (1), i.e., $|S| < \min\{n\kappa(G), (n-1)\delta(G)\}$.

Case 1: S satisfies condition (2). It follows by lemma 2 and lemma 3 that $(G \times K_n - S)$ is connected.

Case 2: S does not satisfy condition (2). Then there exists an S_i contained in S. Therefore, $S - S_i \subseteq V((G - u_i) \times K_n)$ and

$$\begin{aligned} |S - S_i| &= |S| - n \\ &< \min\{n\kappa(G), (n-1)\delta(G)\} - n \\ &\leq \min\{n(\kappa(G) - 1), (n-1)(\delta(G) - 1)\} \\ &\leq \min\{n\kappa(G - u_i), (n-1)\delta(G - u_i)\}. \end{aligned}$$

the last inequality above follows from lemma 4.

By the induction assumption,

$$\kappa((G - u_i) \times K_n) = \min\{n\kappa(G - u_i), (n-1)\delta(G - u_i)\}.$$

Hence, $(G - u_i) \times K_n - (S - S_i)$ is connected. It follows by isomorphism that $G \times K_n - S$ is connected.

Either of the two cases implies $(G \times K_n - S)$ is connected. Thus,

$$\kappa(G \times K_n) \geq \min\{n\kappa(G), (n-1)\delta(G)\}.$$

The proof of the theorem is completed by induction.

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