Computation of the edge Wiener indices of the sum of graphs

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Abstract
The edge versions of Wiener index, which were based on distance between two edges in a connected graph G, were introduced by Iranmanesh et al. in 2008. In this paper, we find the edge Wiener indices of the sum of graphs. Then as an application of our results, we find the edge Wiener indices of graphene, C_4-nanotubes and C_4- nanotori.

Keywords. Distance, Wiener index, Edge Wiener indices, Sum of graphs, Graphene, C_4- nanotubes, C_4- nanotori.

1. Introduction

Through out this paper, we consider only simple, undirected, connected and finite graphs. A simple graph is a graph without any loops or multiple edges. We write $G = (V(G), E(G))$, for a graph G with the set of vertices $V(G)$ and the set of edges $E(G)$. We denote by $e = [u, v]$, the edge connecting vertices u and v of G. The distance between u and v, is denoted by $d(u, v|G)$ and it is defined as the number of edges in a shortest path connecting u and v.
The ordinary (vertex) version of Wiener index (or Wiener number) of G, is the defined as the sum of distances between all pairs of vertices of G. That is: $W(G) = W_v(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v|G)$. This index was introduced by the Chemist, Harold Wiener [17] about 60 years ago, within the study of relations between the structure of organic compounds and their properties. This index is the first and most important topological index in Chemistry. So many interesting works have been done on it, in both Chemistry and Mathematics [1, 3-9, 12, 14, 16].
The edge versions of Wiener index of G, which were based on distance between all pairs of edges of G, were introduced by Iranmanesh et al. in 2008 [10].
They are as follows:

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Definition 1.1. [10] The first edge Wiener index of G, is defined as:

$$W_{e_0}(G) = \sum_{\{e,f\}\subseteq E(G)} d_0(e,f|G), \text{ where } d_0(e,f|G) = \begin{cases} d_1(e,f|G)+1 & \text{if } e \neq f \\ 0 & \text{if } e = f \end{cases}$$

and $d_1(e,f|G) = \min\{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\}$, such that $e = [u,v]$, $f = [z,t]$. In fact, the first edge Wiener index of G is equal to the ordinary Wiener index of the line graph of G, i.e., $W_{e_0}(G) = W_v(L(G))$.

Definition 1.2. [10] The second edge Wiener index of G, is defined as:

$$W_{e_4}(G) = \sum_{\{e,f\}\subseteq E(G)} d_4(e,f|G), \text{ where } d_4(e,f|G) = \begin{cases} d_2(e,f|G) & \text{if } e \neq f \\ 0 & \text{if } e = f \end{cases}$$

and $d_2(e,f|G) = \max\{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\}$, such that $e = [u,v]$, $f = [z,t]$.

Let us recall the definition of the sum of two graphs.

Definition 1.3. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be connected graphs. The sum of G_1 and G_2 is denoted by $G_1 + G_2$, that is a graph with the vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 + G_2$, are adjacent if and only if $[u_1 = v_1 \text{ and } [u_2, v_2] \in E(G_2)]$ or $[u_2 = v_2 \text{ and } [u_1, v_1] \in E(G_1)]$.

Theorem 1.4. [15] Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two connected graphs. The distance between two vertices (u_1, u_2) and (v_1, v_2) of $G_1 + G_2$, is equal to $d((u_1,u_2),(v_1,v_2)|G_1 + G_2) = d(u_1, v_1|G_1) + d(u_2, v_2|G_2)$.

Proof. See Theorem 1 in [15]. □

In this paper, we find the edge Wiener indices of the sum of graphs. Then, as an application of our results, we obtain the edge Wiener indices of graphene, C_4-nanotubes and C_4- nanotori.

Carbon nanotubes, carbon nanotori and graphene are three important types of carbon structures. (See figures 1, 2 and 3)

Carbon nanotubes (CNTs) are allotropes of carbon with molecular structures that are tubular in shape, having diameters on the order of a few nanometers and lengths that can be as much as several millimeters.

Nanotubes are categorized as single-walled (SWNTs) and multi-walled (MWNTs) nanotubes. If a nanotube is bent so that its ends meet, a nanotorus is produced. These types of carbon structures, form the strongest and stiffest materials yet discovered on Earth. Their novel properties make them potentially useful in many applications in materials science, nanotechnology, electronics, optics and architecture.

Graphene is a one-atom-thick planer sheet of carbon atoms that are densely packed in a two-dimensional (2D) honeycomb crystal lattice and is a basic building block for graphitic materials of all other dimensionalities. It can be wrapped up into 0D fullerenes, rolled into 1D nanotubes or stacked into 3D graphite (figure 4). The term graphene was coined as a combination of graphite and the suffix –ene by Hannes-Peter Boehm, who described single-layer carbon foils in 1962.

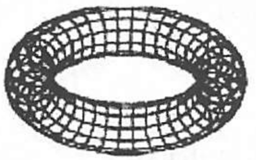

Figure 1. A C_4-Nanotube.

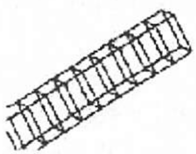

Figure 2. A C_4-Nanotorus.

Figure 3. Ghraphene

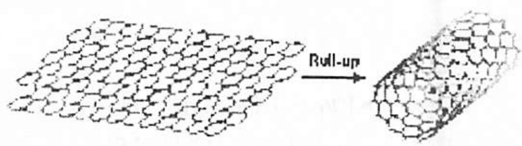

Figure 4. Ghraphene sheet can be rolled in to a single walled nanotube

2. Computation of the edge Wiener indices of the sum of graphs

In order to find the edge Wiener indices of $G_1 + G_2$, first we define the sets A and B as follows:

$$A = \{[(u_1,u_2),(u_1,v_2)] \in E(G_1+G_2) : u_1 \in V(G_1),\ [u_2,v_2] \in E(G_2)\}$$

$$B = \{[(u_1,u_2),(v_1,u_2)] \in E(G_1+G_2) : u_2 \in V(G_2),\ [u_1,v_1] \in E(G_1)\}$$

It is easy to see that: $A \cup B = E(G_1+G_2)$, $A \cap B = \phi$, $|A| = |V(G_1)|\ |E(G_2)|$ and $|B| = |E(G_1)|\ |V(G_2)|$.

Set;

$A_1 = \{\{e,f\} \subseteq A : e \neq f, e = [(u_1,u_2),(u_1,v_2)], f = [(a_1,u_2),(a_1,v_2)],\ u_1, a_1 \in V(G_1),$ $u_2, v_2 \in V(G_2)\}$

$A_2 = \{\{e,f\} \subseteq A : e \neq f, e = [(u_1,u_2),(u_1,v_2)], f = [(a_1,a_2),(a_1,b_2)],\ u_1, a_1 \in V(G_1),$ $u_2, v_2, a_2, b_2 \in V(G_2),\ [u_2,v_2] \neq [a_2,b_2]\}$

$B_1 = \{\{e,f\} \subseteq B : e \neq f, e = [(u_1,u_2),(v_1,u_2)], f = [(u_1,a_2),(v_1,a_2)],\ u_1,v_1 \in V(G_1), u_2,a_2 \in V(G_2)\}$

$B_2 = \{\{e,f\} \subseteq B : e \neq f, e = [(u_1,u_2),(v_1,u_2)], f = [(a_1,a_2),(b_1,a_2)],\ u_2,a_2 \in V(G_2), u_1,v_1,a_1,b_1 \in V(G_1),\ [u_1,v_1] \neq [a_1,b_1]\}$

Obviously, $A_1 \cap A_2 = B_1 \cap B_2 = \phi$ and $A_1 \cup A_2$, $B_1 \cup B_2$ are the sets of all two element subsets of A and B, respectively. Also,

$$|A_1| = \binom{|V(G_1)|}{2}|E(G_2)|,\ |B_1| = \binom{|V(G_2)|}{2}|E(G_1)|.$$

In the first Proposition, we find $d_0(e,f|G_1+G_2)$ and $d_4(e,f|G_1+G_2)$ for all $\{e,f\} \subseteq A$.

Proposition 2.1. Let $\{e,f\} \subseteq A$ and $e \neq f$.

(i) If $\{e,f\} \in A_1$ and $e = [(u_1,u_2),(u_1,v_2)], f = [(a_1,u_2),(a_1,v_2)]$, then

$d_0(e,f|G_1+G_2) = d_4(e,f|G_1+G_2) = d(u_1,a_1|G_1)+1$

(ii) If $\{e,f\} \in A_2$ and $e = [(u_1,u_2),(u_1,v_2)], f = [(a_1,a_2),(a_1,b_2)]$, then for $i \in \{0,4\}$, $d_i(e,f|G_1+G_2) = d_i([u_2,v_2],[a_2,b_2]|G_2) + d(u_1,a_1|G_1)$

Proof. (i) Let $\{e,f\} \in A_1$ and $e = [(u_1,u_2),(u_1,v_2)], f = [(a_1,u_2),(a_1,v_2)]$. Using the definition of $d_0(e,f)$, $d_4(e,f)$ and the formula of distance between two vertices of G_1+G_2, we have:

$d_0(e,f|G_1+G_2) = 1 + \min\{d((u_1,u_2),(a_1,u_2)|G_1+G_2), d((u_1,u_2),(a_1,v_2)|G_1+G_2), d((u_1,v_2),(a_1,u_2)|G_1+G_2), d((u_1,v_2),(a_1,v_2)|G_1+G_2)\} = 1 + \min\{d(u_1,a_1|G_1) + d(u_2,u_2|G_2), d(u_1,a_1|G_1) + d(u_2,v_2|G_2), d(u_1,a_1|G_1) + d(v_2,u_2|G_2), d(u_1,a_1|G_1) + d(v_2,v_2|G_2)\} = 1 + \min\{d(u_2,u_2|G_2), d(u_2,v_2|G_2), d(v_2,u_2|G_2), d(v_2,v_2|G_2)\} + d(u_1,a_1|G_1) = 1 + \min\{0,1,1,0\} + d(u_1,a_1|G_1) = d(u_1,a_1|G_1)+1$ and

$d_4(e,f|G_1+G_2) = \max\{d((u_1,u_2),(a_1,u_2)|G_1+G_2), d((u_1,u_2),(a_1,v_2)|G_1+G_2), d((u_1,v_2),(a_1,u_2)|G_1+G_2), d((u_1,v_2),(a_1,v_2)|G_1+G_2)\} = \max\{d(u_1,a_1|G_1) + d(u_2,u_2|G_2), d(u_1,a_1|G_1) + d(u_2,v_2|G_2), d(u_1,a_1|G_1) + d(v_2,u_2|G_2), d(u_1,a_1|G_1) + d(v_2,v_2|G_2)\} = \max\{d(u_2,u_2|G_2), d(u_2,v_2|G_2), d(v_2,u_2|G_2), d(v_2,v_2|G_2)\} + d(u_1,a_1|G_1) = \max\{0,1,1,0\} + d(u_1,a_1|G_1) = d(u_1,a_1|G_1)+1.$

Therefore $d_0(e,f|G_1+G_2) = d_4(e,f|G_1+G_2) = d(u_1,a_1|G_1)+1$ and the equality in part (i) of Proposition 2.1, holds.

(ii) Let $\{e,f\}\in A_2$ and $e=[(u_1,u_2),(u_1,v_2)], f=[(a_1,a_2),(a_1,b_2)]$. In this case $[u_2,v_2]\neq[a_2,b_2]$ and we have:

$d_0(e,f|G_1+G_2)=1+\min\{d((u_1,u_2),(a_1,a_2)|G_1+G_2), d((u_1,u_2),(a_1,b_2)|G_1+G_2), d((u_1,v_2),(a_1,a_2)|G_1+G_2), d((u_1,v_2),(a_1,b_2)|G_1+G_2)\}=1+\min\{d(u_1,a_1|G_1)+d(u_2,a_2|G_2), d(u_1,a_1|G_1)+d(u_2,b_2|G_2), d(u_1,a_1|G_1)+d(v_2,a_2|G_2), d(u_1,a_1|G_1)+d(v_2,b_2|G_2)\}=1+\min\{d(u_2,a_2|G_2), d(u_2,b_2|G_2), d(v_2,a_2|G_2), d(v_2,b_2|G_2)\}+d(u_1,a_1|G_1)=d_0([u_2,v_2],[a_2,b_2]|G_2)+d(u_1,a_1|G_1)$ and

$d_4(e,f|G_1+G_2)=\max\{d((u_1,u_2),(a_1,a_2)|G_1+G_2), d((u_1,u_2),(a_1,b_2)|G_1+G_2), d((u_1,v_2),(a_1,a_2)|G_1+G_2), d((u_1,v_2),(a_1,b_2)|G_1+G_2)\}=\max\{d(u_1,a_1|G_1)+d(u_2,a_2|G_2), d(u_1,a_1|G_1)+d(u_2,b_2|G_2), d(u_1,a_1|G_1)+d(v_2,a_2|G_2), d(u_1,a_1|G_1)+d(v_2,b_2|G_2)\}=\max\{d(u_2,a_2|G_2), d(u_2,b_2|G_2), d(v_2,a_2|G_2), d(v_2,b_2|G_2)\}+d(u_1,a_1|G_1)=d_4([u_2,v_2],[a_2,b_2]|G_2)+d(u_1,a_1|G_1)$.

Therefore, for $i\in\{0,4\}$, $d_i(e,f|G_1+G_2)=d_i([u_2,v_2],[a_2,b_2]|G_2)+d(u_1,a_1|G_1)$, which completes the proof. □

In the next Proposition, we find $d_0(e,f|G_1+G_2)$ and $d_4(e,f|G_1+G_2)$ for all $\{e,f\}\subseteq B$.

Proposition 2.2. Let $\{e,f\}\subseteq B$ and $e\neq f$.

(i) If $\{e,f\}\in B_1$ and $e=[(u_1,u_2),(v_1,u_2)], f=[(u_1,a_2),(v_1,a_2)]$, then $d_0(e,f|G_1+G_2)=d_4(e,f|G_1+G_2)=d(u_2,a_2|G_2)+1$

(ii) If $\{e,f\}\in B_2$ and $e=[(u_1,u_2),(v_1,u_2)], f=[(a_1,a_2),(b_1,a_2)]$, then for $i\in\{0,4\}$, $d_i(e,f|G_1+G_2)=d_i([u_1,v_1],[a_1,b_1]|G_1)+d(u_2,a_2|G_2)$.

Proof. The proof is similar to the proof of Proposition 2.1. □

In the next Proposition, we find $d_0(e,f|G_1+G_2)$ and $d_4(e,f|G_1+G_2)$ for all $e\in A$, $f\in B$.

Proposition 2.3. Let $e\in A$ and $f\in B$, such that $e=[(u_1,u_2),(u_1,v_2)]$ and $f=[(a_1,a_2),(b_1,a_2)]$, then

(i) $d_0(e,f|G_1+G_2)=1+\min\{d(u_1,a_1|G_1), d(u_1,b_1|G_1)\}+\min\{d(u_2,a_2|G_2), d(v_2,a_2|G_2)\}$

(ii) $d_4(e,f|G_1+G_2)=\max\{d(u_1,a_1|G_1), d(u_1,b_1|G_1)\}+\max\{d(u_2,a_2|G_2), d(v_2,a_2|G_2)\}$

Proof. Let $e \in A$ and $f \in B$, such that $e = [(u_1, u_2), (u_1, v_2)]$ and $f = [(a_1, a_2), (b_1, a_2)]$, then

(i) $d_0(e, f|G_1 + G_2) = 1 + \min\{d((u_1, u_2), (a_1, a_2)|G_1 + G_2), d((u_1, u_2), (b_1, a_2)|G_1 + G_2), d((u_1, v_2), (a_1, a_2)|G_1 + G_2), d((u_1, v_2), (b_1, a_2)|G_1 + G_2)\} = 1 + \min\{d(u_1, a_1|G_1) + d(u_2, a_2|G_2), d(u_1, b_1|G_1) + d(u_2, a_2|G_2), d(u_1, a_1|G_1) + d(v_2, a_2|G_2), d(u_1, b_1|G_1) + d(v_2, a_2|G_2)\} = 1 + \min\{\min\{d(u_1, a_1|G_1) + d(u_1, b_1|G_1)\} + d(u_2, a_2|G_2), \min\{d(u_1, a_1|G_1) + d(u_1, b_1|G_1)\} + d(v_2, a_2|G_2)\} = 1 + \min\{d(u_1, a_1|G_1), d(u_1, b_1|G_1)\} + \min\{d(u_2, a_2|G_2), d(v_2, a_2|G_2)\}$ and part (i) of Proposition 2.3, holds.

(ii) $d_4(e, f|G_1 + G_2) = \max\{d((u_1, u_2), (a_1, a_2)|G_1 + G_2), d((u_1, u_2), (b_1, a_2)|G_1 + G_2), d((u_1, v_2), (a_1, a_2)|G_1 + G_2), d((u_1, v_2), (b_1, a_2)|G_1 + G_2)\} = \max\{d(u_1, a_1|G_1) + d(u_2, a_2|G_2), d(u_1, b_1|G_1) + d(u_2, a_2|G_2), d(u_1, a_1|G_1) + d(v_2, a_2|G_2), d(u_1, b_1|G_1) + d(v_2, a_2|G_2)\} = \max\{\max\{d(u_1, a_1|G_1) + d(u_1, b_1|G_1)\} + d(u_2, a_2|G_2), \max\{d(u_1, a_1|G_1) + d(u_1, b_1|G_1)\} + d(v_2, a_2|G_2)\} = \max\{d(u_1, a_1|G_1), d(u_1, b_1|G_1)\} + \max\{d(u_2, a_2|G_2), d(v_2, a_2|G_2)\}$, so part (ii) of Proposition 2.3, holds. □

Using Proposition 2.1, we conclude two following Lemmas.

Lemma 2.4. $\sum_{\{e,f\} \in A_1} d_0(e, f|G_1 + G_2) = \sum_{\{e,f\} \in A_1} d_4(e, f|G_1 + G_2) = |E(G_2)|W(G_1) + \binom{|V(G_1)|}{2}|E(G_2)|$

Proof. By part (i) of Proposition 2.1, we have:

$\sum_{\{e,f\} \in A_1} d_0(e, f|G_1 + G_2) = \sum_{\{e,f\} \in A_1} d_4(e, f|G_1 + G_2) = \sum\{d(u_1, a_1|G_1) + 1 : \{e, f\} \in A_1, e = [(u_1, u_2), (u_1, v_2)], f = [(a_1, u_2), (a_1, v_2)]\} = \sum\{d(u_1, a_1|G_1) : \{e, f\} \in A_1, e = [(u_1, u_2), (u_1, v_2)], f = [(a_1, u_2), (a_1, v_2)]\} + |A_1| = |E(G_2)| \sum_{\{u_1, a_1\} \subseteq V(G_1)} d(u_1, a_1|G_1) + \binom{|V(G_1)|}{2}|E(G_2)| =$

$$|E(G_2)|W(G_1)+\binom{|V(G_1)|}{2}|E(G_2)|. \qquad \square$$

Lemma 2.5. For $i\in\{0,4\}$, we have:

$$\sum_{\{e,f\}\in A_2} d_i(e,f|G_1+G_2)=|V(G_1)|^2W_{e_i}(G_2)+2\binom{|E(G_2)|}{2}W(G_1)$$

Proof. By part (ii) of Proposition 2.1, for $i\in\{0,4\}$ we have:

$$\sum_{\{e,f\}\in A_2} d_i(e,f|G_1+G_2)=\sum\{d_i([u_2,v_2],[a_2,b_2]|G_2)+d(u_1,a_1|G_1):\{e,f\}\in A_2,$$
$$e=[(u_1,u_2),(u_1,v_2)],f=[(a_1,a_2),(a_1,b_2)]\}=\sum\{d_i([u_2,v_2],[a_2,b_2]|G_2):$$
$$\{e,f\}\in A_2,e=[(u_1,u_2),(u_1,v_2)],f=[(a_1,a_2),(a_1,b_2)]\}+$$
$$\sum\{d(u_1,a_1|G_1):\{e,f\}\in A_2,\ e=[(u_1,u_2),(u_1,v_2)],f=[(a_1,a_2),(a_1,b_2)]\}=$$
$$|V(G_1)|^2\sum_{\{[u_2,v_2],[a_2,b_2]\}\subseteq E(G_2)}([u_2,v_2],[a_2,b_2]|G_2)+2\binom{|E(G_2)|}{2}\sum_{\{u_1,a_1\}\subseteq V(G_1)}d(u_1,a_1|G_1)=$$
$$|V(G_1)|^2W_{e_i}(G_2)+2\binom{|E(G_2)|}{2}W(G_1). \qquad \square$$

By Lemmas 2.4 and 2.5, we have the following result.

Corollary 2.6. For $i\in\{0,4\}$, we have:

$$\sum_{\{e,f\}\subseteq A} d_i(e,f|G_1+G_2)=|E(G_2)|^2W(G_1)+|V(G_1)|^2W_{e_i}(G_2)+\binom{|V(G_1)|}{2}|E(G_2)|$$

Proof. Since d_0 and d_4 are distances, so for every $e\in E(G_1+G_2)$, we have: $d_0(e,e|G_1+G_2)=d_4(e,e|G_1+G_2)=0$. Now by Lemmas 2.4 and 2.5, for $i\in\{0,4\}$ we have:

$$\sum_{\{e,f\}\subseteq A} d_i(e,f|G_1+G_2)=\sum_{\{e,f\}\subseteq A,e\neq f} d_i(e,f|G_1+G_2)=$$
$$\sum_{\{e,f\}\in A_1} d_i(e,f|G_1+G_2)+\sum_{\{e,f\}\in A_2} d_i(e,f|G_1+G_2)=|E(G_2)|W(G_1)+$$
$$\binom{|V(G_1)|}{2}|E(G_2)|+|V(G_1)|^2W_{e_i}(G_2)+2\binom{|E(G_2)|}{2}W(G_1)=$$
$$|E(G_2)|^2W(G_1)+|V(G_1)|^2W_{e_i}(G_2)+\binom{|V(G_1)|}{2}|E(G_2)|. \qquad \square$$

Using Proposition 2.2, we conclude two next Lemmas.

Lemma 2.7. $\sum_{\{e,f\}\in B_1} d_0(e,f|G_1+G_2) = \sum_{\{e,f\}\in B_1} d_4(e,f|G_1+G_2) =$

$$|E(G_1)|W(G_2)+\binom{|V(G_2)|}{2}|E(G_1)|$$

Proof. The proof is similar to the proof of Lemma 2.4. □

Lemma 2.8. For $i\in\{0,4\}$, we have:

$$\sum_{\{e,f\}\in B_2} d_i(e,f|G_1+G_2) = |V(G_2)|^2 W_{e_i}(G_1)+2\binom{|E(G_1)|}{2}W(G_2)$$

Proof. The proof is similar to the proof of Lemma 2.5. □

Lemmas 2.7 and 2.8, indicate the following Corollary.

Corollary 2.9. For $i\in\{0,4\}$, we have:

$$\sum_{\{e,f\}\subseteq B} d_i(e,f|G_1+G_2) = |E(G_1)|^2 W(G_2)+|V(G_2)|^2 W_{e_i}(G_1)+\binom{|V(G_2)|}{2}|E(G_1)|$$

Proof. Similar to the proof of Corollary 2.6, we can obtain the desire result. □

Here, we introduce two topological indices of a graph G as follows:

$$Min(G) = \sum_{u\in V(G)}\sum_{[a,b]\in E(G)} \min\{d(u,a|G), d(u,b|G)\}$$

$$Max(G) = \sum_{u\in V(G)}\sum_{[a,b]\in E(G)} \max\{d(u,a|G), d(u,b|G)\}$$

Using Proposition 2.3, we have the following Lemma.

Lemma 2.10.

(i) $\sum_{e\in A, f\in B} d_0(e,f|G_1+G_2) = |V(G_1)|\,|V(G_2)|\,|E(G_1)|\,|E(G_2)|+$

$|V(G_2)|\,|E(G_2)|Min(G_1)+|V(G_1)|\,|E(G_1)|Min(G_2)$

(ii) $\sum_{e\in A, f\in B} d_4(e,f|G_1+G_2) =$

$|V(G_2)|\,|E(G_2)|Max(G_1)+|V(G_1)|\,|E(G_1)|Max(G_2)$

Proof. (i) By part (i) of Proposition 2.3, we have:

$\sum_{e\in A, f\in B} d_0(e,f|G_1+G_2) = \sum\{1+\min\{d(u_1,a_1|G_1), d(u_1,b_1|G_1)\}+\min\{d(u_2,a_2|G_2),$

$d(v_2,a_2|G_2)\} : e\in A, f\in B,\ e=[(u_1,u_2),(u_1,v_2)],\ f=[(a_1,a_2),(b_1,a_2)]\} =$

$|A|\ |B| + \sum\{\min\{d(u_1,a_1|G_1),d(u_1,b_1|G_1)\}: e\in A, f\in B,\ e=[(u_1,u_2),(u_1,v_2)],$
$f=[(a_1,a_2),(b_1,a_2)]\} + \sum\{\min\{d(u_2,a_2|G_2),d(v_2,a_2|G_2)\}: e\in A, f\in B,$
$e=[(u_1,u_2),(u_1,v_2)], f=[(a_1,a_2),(b_1,a_2)]\} = |V(G_1)|\ |V(G_2)|\ |E(G_1)|\ |E(G_2)| +$
$|V(G_2)|\ |E(G_2)| \sum_{u_1\in V(G_1)} \sum_{[a_1,b_1]\in E(G_1)} \min\{d(u_1,a_1|G_1),d(u_1,b_1|G_1)\} +$
$|V(G_1)|\ |E(G_1)| \sum_{a_2\in V(G_2)} \sum_{[u_2,v_2]\in E(G_2)} \min\{d(u_2,a_2|G_2),d(v_2,a_2|G_2)\} =$
$|V(G_1)|\ |V(G_2)|\ |E(G_1)|\ |E(G_2)| + |V(G_2)|\ |E(G_2)|Min(G_1) +$
$|V(G_1)|\ |E(G_1)|Min(G_2)$.

(ii) By part (ii) of Proposition 2.3, we have:

$\sum_{e\in A, f\in B} d_4(e,f|G_1+G_2) = \sum\{\max\{d(u_1,a_1|G_1),d(u_1,b_1|G_1)\} + \max\{d(u_2,a_2|G_2),$
$d(v_2,a_2|G_2)\}: e\in A, f\in B, e=[(u_1,u_2),(u_1,v_2)], f=[(a_1,a_2),(b_1,a_2)]\} =$
$\sum\{\max\{d(u_1,a_1|G_1),d(u_1,b_1|G_1)\}: e\in A, f\in B, e=[(u_1,u_2),(u_1,v_2)],$
$f=[(a_1,a_2),(b_1,a_2)]\} + \sum\{\max\{d(u_2,a_2|G_2),d(v_2,a_2|G_2)\}: e\in A, f\in B,$
$e=[(u_1,u_2),(u_1,v_2)], f=[(a_1,a_2),(b_1,a_2)]\} =$
$|V(G_2)|\ |E(G_2)| \sum_{u_1\in V(G_1)} \sum_{[a_1,b_1]\in E(G_1)} \max\{d(u_1,a_1|G_1),d(u_1,b_1|G_1)\} +$
$|V(G_1)|\ |E(G_1)| \sum_{a_2\in V(G_2)} \sum_{[u_2,v_2]\in E(G_2)} \max\{d(u_2,a_2|G_2),d(v_2,a_2|G_2)\} =$
$|V(G_2)|\ |E(G_2)|Max(G_1) + |V(G_1)|\ |E(G_1)|Max(G_2)$. □

Finally, as the main purpose of this paper, we express Theorem 2.11, which characterizes the edge Wiener indices of the sum of two graphs.

Theorem 2.11. Let $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$ be two simple undirected connected finite graphs, then

(i) $W_{e_0}(G_1+G_2) = |E(G_2)|^2 W(G_1) + |E(G_1)|^2 W(G_2) + |V(G_2)|^2 W_{e_0}(G_1) +$
$|V(G_1)|^2 W_{e_0}(G_2) + \binom{|V(G_1)|}{2}|E(G_2)| + \binom{|V(G_2)|}{2}|E(G_1)| +$
$|V(G_1)|\ |V(G_2)|\ |E(G_1)|\ |E(G_2)| + |V(G_2)|\ |E(G_2)|Min(G_1) +$
$|V(G_1)|\ |E(G_1)|Min(G_2)$

(ii) $W_{e_4}(G_1+G_2) = |E(G_2)|^2 W(G_1) + |E(G_1)|^2 W(G_2) + |V(G_2)|^2 W_{e_4}(G_1) +$

$$|V(G_1)|^2 W_{e_4}(G_2) + \binom{|V(G_1)|}{2}|E(G_2)| + \binom{|V(G_2)|}{2}|E(G_1)| +$$

$$|V(G_2)|\ |E(G_2)| Max(G_1) + |V(G_1)|\ |E(G_1)| Max(G_2)$$

Proof. Since $E(G_1 + G_2) = A \cup B$, $A \cap B = \phi$, for $i \in \{0,4\}$ we have:

$$W_{e_i}(G_1+G_2) = \sum_{\{e,f\} \subseteq E(G_1+G_2)} d_i(e,f|G_1+G_2) = \sum_{\{e,f\} \subseteq A} d_i(e,f|G_1+G_2) +$$

$$\sum_{\{e,f\} \subseteq B} d_i(e,f|G_1+G_2) + \sum_{e \in A, f \in B} d_i(e,f|G_1+G_2).$$

Now by Corollaries 2.6, 2.9 and Lemma 2.10, the proof is clear. □

Corollary 2.12. For every two simple undirected connected finite graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, we have:

$$W_{e4}(G_1+G_2) - W_{e0}(G_1+G_2) = |V(G_2)|^2 (W_{e4}(G_1) - W_{e0}(G_1)) +$$

$$|V(G_1)|^2 (W_{e4}(G_2) - W_{e0}(G_2)) + |V(G_2)|\ |E(G_2)|(Max(G_1) - Min(G_1)) +$$

$$|V(G_1)|\ |E(G_1)|(Max(G_2) - Min(G_2)) - |V(G_1)|\ |V(G_2)|\ |E(G_1)|\ |E(G_2)|$$

Proof. The proof is straightforward. □

3. Computation of *Min*(*G*) and *Max*(*G*) for some familiar graphs

Let P_n, C_n, S_n and K_n, denote the n-vertex path, cycle, star and complete graph, respectively. Let $K_{a,b}$ be complete bipartite graph on $a+b$ vertices. In this section, we want to compute $Min(G)$ and $Max(G)$ for these familiar graphs. This can be done as follows:

(i) Computation of $Min(P_n)$ and $Max(P_n)$

$$Min(P_n) = \sum_{u \in V(P_n)} \sum_{\{a,b\} \in E(P_n)} \min\{d(u,a|P_n), d(u,b|P_n)\} =$$

$$2\sum_{k=1}^{n-1}\binom{k}{2} = 2\sum_{k=1}^{n-1}\frac{k^2-k}{2} = \sum_{k=1}^{n-1}k^2 - \sum_{k=1}^{n-1}k = \frac{(n-1)n(2n-1)}{6} - \frac{n(n-1)}{2} =$$

$$\frac{n(n-1)(n-2)}{3} = 2\binom{n}{3},$$

$$Max(P_n) = \sum_{u \in V(P_n)} \sum_{\{a,b\} \in E(P_n)} \max\{d(u,a|P_n), d(u,b|P_n)\} =$$

$$2\sum_{k=1}^{n}\binom{k}{2} = 2\sum_{k=1}^{n}\frac{k^2-k}{2} = \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} =$$
$$\frac{n(n^2-1)}{3} = 2\binom{n+1}{3}.$$

(ii) Computation of $Min(C_n)$ and $Max(C_n)$

We consider two following cases:

Case 1. If n is odd, then

$$Min(C_n) = \sum_{u\in V(C_n)}\sum_{[a,b]\in E(C_n)} \min\{d(u,a|C_n), d(u,b|C_n)\} =$$
$$n(0+1+2+...+(\frac{n-1}{2}-1)+\frac{n-1}{2}+(\frac{n-1}{2}-1)+...+2+1+0) =$$
$$n(2(0+1+2+...+(\frac{n-1}{2}-1))+\frac{n-1}{2}) = \frac{n(n-1)^2}{4},$$
$$Max(C_n) = \sum_{u\in V(C_n)}\sum_{[a,b]\in E(C_n)} \max\{d(u,a|C_n), d(u,b|C_n)\} =$$
$$n(1+2+3+...+\frac{n-1}{2}+\frac{n-1}{2}+\frac{n-1}{2}+...+3+2+1) =$$
$$n(2(1+2+3+...+\frac{n-1}{2})+\frac{n-1}{2}) = \frac{n(n-1)(n+3)}{4}.$$

Case 2. If n is even, then

$$Min(C_n) = \sum_{u\in V(C_n)}\sum_{[a,b]\in E(C_n)} \min\{d(u,a|C_n), d(u,b|C_n)\} =$$
$$n(0+1+2+...+(\frac{n}{2}-1)+(\frac{n}{2}-1)+...+2+1+0) = 2n(0+1+2+...+(\frac{n}{2}-1)) =$$
$$\frac{n^2(n-2)}{4},$$
$$Max(C_n) = \sum_{u\in V(C_n)}\sum_{[a,b]\in E(C_n)} \max\{d(u,a|C_n), d(u,b|C_n)\} =$$
$$n(1+2+3+...+\frac{n}{2}+\frac{n}{2}+...+3+2+1) = 2n(1+2+3+...+\frac{n}{2}) = \frac{n^2(n+2)}{4}.$$

(iii) Computation of $Min(S_n)$ and $Max(S_n)$

$$Min(S_n) = \sum_{u\in V(S_n)}\sum_{[a,b]\in E(S_n)} \min\{d(u,a|S_n), d(u,b|S_n)\} =$$
$$0\times|E(S_n)| + (|V(S_n)|-1)(0\times 1 + (|E(S_n)|-1)\times 1) = (n-2)(n-1) = 2\binom{n-1}{2},$$

$$Max(S_n) = \sum_{u \in V(S_n)} \sum_{[a,b] \in E(S_n)} \max\{d(u,a|S_n), d(u,b|S_n)\} =$$
$$|E(S_n)| + (|V(S_n)| - 1)(1 + 2(|E(S_n)| - 1)) = 2(n-1)^2.$$

(iv) Computation of $Min(K_n)$ and $Max(K_n)$

$$Min(K_n) = \sum_{u \in V(K_n)} \sum_{[a,b] \in E(K_n)} \min\{d(u,a|K_n), d(u,b|K_n)\} =$$
$$\sum_{u \in V(K_n)} (\deg(u) \times 0 + (|E(K_n)| - \deg(u)) \times 1) = \sum_{u \in V(K_n)} (|E(K_n)| - \deg(u)) =$$
$$|E(K_n)|\ |V(K_n)| - 2|E(K_n)| = |E(K_n)|\ (|V(K_n)| - 2) = \binom{n}{2}(n-2) =$$
$$\frac{n}{2}(n-1)(n-2) = 3\binom{n}{3},$$
$$Max(K_n) = \sum_{u \in V(K_n)} \sum_{[a,b] \in E(K_n)} \max\{d(u,a|K_n), d(u,b|K_n)\} =$$
$$\sum_{u \in V(K_n)} \sum_{[a,b] \in E(K_n)} 1 = |V(K_n)|\ |E(K_n)| = n\binom{n}{2} = \frac{n^2(n-1)}{2}.$$

(v) Computation of $Min(K_{a,b})$ and $Max(K_{a,b})$

$$Min(K_{a,b}) = \sum_{u \in V(K_{a,b})} \sum_{[v,z] \in E(K_{a,b})} \min\{d(u,v|K_{a,b}), d(u,z|K_{a,b})\} =$$
$$a((0 \times b) + (1 \times (ab - b))) + b((0 \times a) + (1 \times (ab - a))) = a(ab - b) + b(ab - a) =$$
$$a^2b - ab + ab^2 - ab = ab(a + b - 2),$$
$$Max(K_{a,b}) = \sum_{u \in V(K_{a,b})} \sum_{[v,z] \in E(K_{a,b})} \max\{d(u,v|K_{a,b}), d(u,z|K_{a,b})\} =$$
$$a((1 \times b) + 2(ab - b)) + b((1 \times a) + 2(ab - a)) =$$
$$a(b + 2ab - 2b) + b(a + 2ab - 2a) = ab(2a + 2b - 2) = 2ab(a + b - 1).$$

4. Edge Wiener indices of the sum of some familiar graphs

In the previous section, we found $Min(G)$ and $Max(G)$ the familiar graphs P_n, C_n, S_n, K_n and $K_{a,b}$. The Wiener index and edge Wiener indices of these graphs have been computed previously [13, 10]. So, we have the following tables.

Graph (G)	P_n	C_n n is odd	C_n n is even
$\|V(G)\|$	n	n	n
$\|E(G)\|$	$n-1$	n	n
$W(G)$	$\binom{n+1}{3}$	$\frac{n}{8}(n^2-1)$	$\frac{n^3}{8}$
$W_{e_0}(G)$	$\binom{n}{3}$	$\frac{n}{8}(n^2-1)$	$\frac{n^3}{8}$
$W_{e_4}(G)$	$\binom{n-1}{2}\frac{n+3}{3}$	$\frac{n}{8}(n^2+4n-13)$	$\frac{n}{8}(n^2+4n-8)$
$Min(G)$	$2\binom{n}{3}$	$\frac{n}{4}(n-1)^2$	$\frac{n^2}{4}(n-2)$
$Max(G)$	$2\binom{n+1}{3}$	$\frac{n}{4}(n-1)(n+3)$	$\frac{n^2}{4}(n+2)$

Table 1. Some topological indices of P_n and C_n

Graph (G)	S_n	K_n	$K_{a,b}$
$\lvert V(G)\rvert$	n	n	$a+b$
$\lvert E(G)\rvert$	$n-1$	$\binom{n}{2}$	ab
$W(G)$	$(n-1)^2$	$\binom{n}{2}$	$(a+b)^2-ab$ $-a-b$
$W_{e_0}(G)$	$\binom{n-1}{2}$	$\binom{n}{2}\binom{n-1}{2}$	$\frac{ab}{2}(2ab-a-b)$
$W_{e_4}(G)$	$2\binom{n-1}{2}$	$3\binom{n+1}{4}$	$ab(ab-1)$
$Min(G)$	$2\binom{n-1}{2}$	$3\binom{n}{3}$	$ab(a+b-2)$
$Max(G)$	$2(n-1)^2$	$n\binom{n}{2}$	$2ab(a+b-1)$

Table 2. Some topological indices of S_n, K_n and $K_{a,b}$

According to the tables, Theorem 2.11 and Corollary 2.12, we can easily obtain the edge Wiener indices of the sum of each pair of the above-mentioned graphs. Specially, we can obtain the edge Wiener indices of graphene, C_4-nanotubes and C_4- nanotori as P_n+P_m, P_n+C_m and C_n+C_m, respectively [11, 2].

Example 4.1. Suppose that $G=P_n+P_m$, where n and m are not necessarily equal, then G is graphene. The edge Wiener indices of G, are as follows:

$$W_{e_0}(P_n+P_m)=\frac{m^3}{6}(2n-1)^2+\frac{m^2}{6}(4n^3-12n^2+8n-3)-$$
$$\frac{m}{3}(2n^3-4n^2+2n-1)+\frac{n}{6}(n^2-3n+2),$$

$$W_{e4}(P_n+P_m)=\frac{m^3}{6}(2n-1)^2+\frac{m^2}{6}(4n^3-7n+3)-\frac{m}{6}(4n^3+7n^2-2n-2)+$$
$$\frac{n}{6}(n^2+3n+2)$$

Example 4.2. Suppose that $G = P_n + C_m$, where n and m are not necessarily equal, then $G = TUC_4[m,n]$ is a C_4-nanotube [11]. The edge Wiener indices of G, are as follows:

Case 1. If m is odd, then

$$W_{e0}(P_n + C_m) = \frac{m^3}{8}(2n-1)^2 + \frac{m^2}{6}(4n^3 - 6n^2 + 5n - 3) + \frac{m}{8}(4n^2 - 8n + 3)$$

$$W_{e4}(P_n + C_m) = \frac{m^3}{8}(2n-1)^2 + \frac{m^2}{6}(4n^3 + 6n^2 - 10n + 3) - \frac{m}{8}(16n^2 - 3)$$

Case 2. If m is even, then

$$W_{e0}(P_n + C_m) = \frac{m^3}{8}(2n-1)^2 + \frac{m^2}{6}(4n^3 - 6n^2 + 5n - 3) + \frac{m}{2}(n-1)^2$$

$$W_{e4}(P_n + C_m) = \frac{m^3}{8}(2n-1)^2 + \frac{m^2}{6}(4n^3 + 6n^2 - 10n + 3) - \frac{m}{2}(n^2 + 2n - 1)$$

Example 4.3. Suppose that $G = C_n + C_m$, where n and m are not necessarily equal, then $G = TC_4[m,n]$ is a C_4- nanotorus [2]. The first edge Wiener index of G is equal to: $W_{e0}(C_n + C_m) = \frac{m^3}{2}n^2 + \frac{m^2}{2}n(n^2+1) + \frac{m}{2}n(n-2)$

The second edge Wiener index of G is as follows:

Case 1. If m, n are odd, then

$$W_{e4}(C_n + C_m) = \frac{m^3}{2}n^2 + \frac{m^2}{2}n(n^2 + 4n - 4) - mn(2n+1)$$

Case 2. If m, n are even, then

$$W_{e4}(C_n + C_m) = \frac{m^3}{2}n^2 + \frac{m^2}{2}n(n^2 + 4n - 1) - \frac{m}{2}n(n+2)$$

Case 3. If n is odd and m is even, then

$$W_{e4}(C_n + C_m) = \frac{m^3}{2}n^2 + \frac{m^2}{2}n(n^2 + 4n - 4) - \frac{m}{2}n(n+2)$$

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