Signed and minus total domination on subclasses of bipartite graphs

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Abstract

In this paper we study the signed and minus total domination problems for two subclasses of bipartite graphs: biconvex bipartite graphs and planar bipartite graphs. We present a unified method to solve the signed and minus total domination problems for biconvex bipartite graphs in $O(n+m)$ time. We also prove that the decision problem corresponding to the signed (respectively, minus) total domination problem is NP-complete for planar bipartite graphs of maximum degree 3 (respectively, maximum degree 4).

Keywords: Graph algorithms; Minus total dominating functions; Signed total dominating functions; Biconvex bipartite graphs; Planar bipartite graphs

1 Introduction

Total Domination is a fundamental concept in graph theory. It plays an important role as an often studied NP-complete problem in the literature and has been surveyed in [8, 9, 11]. Recently two variations of total domination, *signed total domination* and *minus total domination*, have been studied in [7, 10, 12, 14, 21, 22, 23, 24, 25]. However, few papers studied the algorithmic complexity of these two problems. From the algorithmic point of view, the signed and minus total domination problems are polynomial-time solvable for trees [7] and chordal bipartite graphs [14], while the decision

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problems corresponding to these two problems are NP-complete for bipartite graphs and doubly chordal graphs [7, 14]. In [14], Lee introduced the concept of *R-total domination* as follows.

Definition 1. Suppose that $G = (V, E)$ is a finite, simple, undirected graph. Let $\mathcal{P}$ be a subset of real numbers. Let $f : V \to \mathcal{P}$ be a function which assigns to each $v \in V$ a value in $\mathcal{P}$. The set $\mathcal{P}$ is called the *weight set* of f. Let $f(S) = \sum_{u \in S} f(u)$ for any subset S of V. Then $f(V)$ is called the *weight* of f.

Definition 2. Let ℓ, d, I_1 be fixed integers and $\ell, d > 0$. Let $\mathcal{P}$ be the weight set $\{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$. Suppose that $G = (V, E)$ is a graph and R is a labeling function which assigns an integer $R(v)$ to each $v \in V$. An *R-total dominating function* of $G = (V, E)$ is a function $f : V \to \mathcal{P}$ such that $f(N_G(v)) \geq R(v)$ for all vertices $v \in V$. The *R-total domination number* $\gamma_{t,R}(G)$ is the minimum weight of an R-total dominating function of G. The *R-total domination problem* is to find an R-total dominating function of G of minimum weight.

The concept of R-total domination is similar to that of *labeled domination* introduced by Lee and Chang [13]. It includes the total domination, signed total domination, and minus total domination problems as special cases. Any polynomial-time algorithm for the R-total domination problem gives a unified approach to the signed and minus total domination problems. Lee showed that the R-total domination problem for chordal bipartite graphs and trees can be solved in $O(n^2)$ and $O(n + m)$ time, respectively. Note that biconvex bipartite graphs are a subclass of chordal bipartite graphs [2]. The R-total domination problem for biconvex bipartite graphs can also be solved in $O(n^2)$ time.

In this paper we study the signed and minus total domination problems on two classes of bipartite graphs: biconvex bipartite graphs and planar bipartite graphs. We present a linear-time algorithm for the R-total domination problem on biconvex bipartite graphs. The algorithm improves the complexity of the R-total domination problem for biconvex bipartite graphs. This paper also shows that the decision problem corresponding to the signed (respectively, minus) total domination problem is NP-complete for planar bipartite graphs of maximum degree 3 (respectively, maximum degree 4).

The rest of this paper is organized as follows. Section 2 reviews the definitions and properties of some classes of graphs that will be studied in this paper. Section 3 deals with the R-total domination problem for biconvex bipartite graphs. Section 4 shows that the decision problem corresponding to the signed (respectively, minus) total domination problem is NP-complete for planar bipartite graphs of maximum degree 3 (respectively, maximum degree 4).

2 Preliminaries

Let $G = (V, E)$ be a finite, simple, undirected graph with *vertex set* V and *edge set* E. Unless stated otherwise, it is understood that $|V| = n$ and $|E| = m$. We also use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. We denote by $G[W]$ the subgraph of G induced by the vertex set $W \subseteq V$. For any vertex $v \in V$, the *neighborhood* of v in G is $N_G(v) = \{u \in V | (u, v) \in E\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $deg_G(v) = |N_G(v)|$. The number, $\max\{deg_G(v) \mid v \in V\}$, is called the *maximum degree* of G. A *clique* is a subset of pairwise adjacent vertices of V. A clique is *maximum* if there is no clique of G of larger cardinality.

A *vertex cover* of a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that for each edge $(u, v) \in E$, at least one of u and v belongs to V'. The *vertex cover number* of G, denoted by $\tau(G)$, is the minimum cardinality of a vertex cover of G. The vertex cover problem is to find a vertex cover of G of minimum cardinality.

A total dominating set D of a graph G is a subset $S \subseteq V(G)$ such that $|D \cap N_G(v)| \geq 1$ for every vertex $v \in V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. The *total domination problem* is to find a total dominating set of G of minimum cardinality.

Suppose that $G = (V, E)$ is a graph. A function $f : V \rightarrow \{0, 1\}$ is a *total dominating function* of G if $f(N_G(v)) \geq 1$ for every vertex $v \in V$. A total dominating set can be viewed as a total dominating function f and $\gamma_t(G) = \min\{f(V) \mid f$ is a total dominating function of $G\}$. A function $f : V \rightarrow \mathcal{P}$ is a *signed* (respectively, *minus*) *total dominating function* of G if $\mathcal{P}$ is $\{-1, 1\}$ (respectively, $\{-1, 0, 1\}$). The *signed* (respectively, *minus*) *total domination number* of G, denoted by $\gamma_t^s(G)$ (respectively, $\gamma_t^-(G)$), is the minimum weight of a signed (respectively, minus) total dominating function of G. The signed (respectively, minus) total domination problem is to find a signed (respectively, minus) total dominating function of G of minimum weight.

Given a graph $G = (V, E)$, a vertex v is *simplicial* if all vertices of $N_G[v]$ form a clique. The ordering $v_1, v_2, \ldots, v_n$ of the vertices of V is a *perfect elimination ordering* of G if for all $i \in \{1, \ldots, n\}$, v_i is a simplicial vertex of the subgraph G_i of G induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. A *chord* of a cycle is an edge between two vertices of the cycle that is not an edge of the cycle. A graph G is called a *chordal* graph if each cycle in G of length at least 4 has at least one chord. Rose [17] showed the characterization that a graph is *chordal* if and only if it has a perfect elimination ordering. Let $N_i[v]$ denote the closed neighborhood of v in G_i. A perfect elimination ordering is called a *strong elimination ordering* if it has the following property:

For $i < j < k$ if v_j and v_k belong to $N_i[v_i]$ in G_i, then $N_i[v_j] \subseteq N_i[v_k]$. Farber [4] showed that a graph is *strongly chordal* if and only if it admits a strong elimination ordering. Currently, the fastest algorithms for recognizing a strongly chordal graph and giving a strong elimination ordering run in $O(m \log n)$ [16] or $O(n^2)$ time [19].

A graph $G = (V, E)$ is a *bipartite graph* if V can be partitioned into two sets A and B such that every edge has its ends in different sets. We call the sets, A and B, the bipartition of V and use $G = (A, B, E)$ to denote a bipartite graph. A bipartite graph is a *chordal* bipartite graph if every cycle of length at least 6 has a chord. A bipartite graph $G = (A, B, E)$ is a *biconvex* bipartite graph if both A and B can be ordered so that for every vertex v in $A \cup B$, vertices in $N_G(v)$ occur consecutively in the ordering. Then the ordering of the vertices in $V = A \cup B$ is called a *biconvex ordering* of G.

Given a bipartite graph $G = (A, B, E)$, an ordering of the vertices of A has the *adjacency property* if for each vertex $b \in B$, $N_G(b)$ consists of vertices which are consecutive in the ordering of A. An ordering of the vertices of A has the *enclosure property* if for every pair of $b, b' \in B$ with $N_G(b) \subset N_G(b')$, vertices in $N_G(b') - N_G(b)$ occur consecutively in the ordering of A. A *strong ordering* of the vertices of $G = (A, B, E)$ consists of an ordering of A and an ordering of B such that for all $(a, b'), (a', b) \in E$, where $a, a' \in A$ and $b, b' \in B$, $a < a'$ and $b < b'$ imply $(a, b), (a', b') \in E$. A *bipartite permutation graph* is a bipartite graph $G = (A, B, E)$ with a strong ordering of $A \cup B$ [20]. Bradstädt et al. [3] showed that given a strong ordering of $A \cup B$, both A and B have the adjacency and the enclosure properties if all isolated vertices of G appear at the beginning of the orderings of A and B.

Biconvex (respectively, chordal) bipartite graphs are a superclass of bipartite permutation (respectively, biconvex bipartite) graphs [2]. Lipski et al. [15] (respectively, Spinrad et al. [20]) gave a linear-time algorithm to recognize whether a given graph is a biconvex bipartite (respectively, bipartite permutation) graph and producing a biconvex (respectively, strong) ordering of the vertices if so.

Definition 3. Suppose that $G = (A, B, E)$ is a bipartite graph. Let G_A (respectively, G_B) be the graph obtained by adding all possible edges between vertices of A (respectively, B) such that the set A (respectively, B) is a clique of G_A (respectively, G_B).

Lemma 1 shows a connection between chordal bipartite graphs and strongly chordal graphs.

Lemma 1 ([1, 4]). *The graphs G_A and G_B obtained from a chordal bipartite graph $G = (A, B, E)$ are strongly chordal graphs.*

3 R-total domination on biconvex bipartite graphs

In this section, we develop a linear-time algorithm for the R-total domination problem on biconvex bipartite graphs. Suppose that $G = (A, B, E)$ is a biconvex bipartite graphs with $|A \cup B| = n$ and $|E| = m$. Section 3.1 gives an algorithm to compute a strong elimination ordering of G_A (respectively, G_B) from a biconvex ordering of G in $O(n + m)$ time. Using strong elimination orderings of G_A and G_B, Section 3.2 gives a linear-time algorithm to solve the R-total domination problem for a biconvex bipartite graph G.

It is clear that an R-total dominating function of a graph does not exist if the graph contains an isolated vertex. Throughout this section, we assume that all graphs considered here do not contain isolated vertices.

3.1 From a biconvex ordering to a strong elimination ordering

A vertex v is *simple* of a graph G if for any two vertices $x, y \in N_G(v)$ either $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$. An ordering $v_1, v_2, \ldots, v_n$ of the vertices in G is called a *simple elimination ordering* if for each $1 \leq i \leq n$, the vertex v_i is a simple vertex of $G[\{v_i, v_{i+1}, \ldots, v_n\}]$. From this definition, we know that strong elimination orderings are simple elimination orderings, but the converse is not necessarily true.

Sawada and Spinrad [18] presented a linear-time algorithm for transforming a simple elimination ordering of a strongly chordal graph into a strong elimination ordering. Based upon their algorithm, we develop an algorithm in this section for a given biconvex bipartite graph $G = (A, B, E)$ to transform a biconvex ordering of G into a strong elimination ordering of G_A (respectively, G_B) in $O(n + m)$ time.

Definition 4. Suppose that $G = (A, B, E)$ is a biconvex bipartite graph. We use $\langle A, B \rangle$ (respectively, $\langle B, A \rangle$) to denote a biconvex ordering $v_1, v_2, \ldots, v_n$, where $A = \{v_1, \ldots, v_{|A|}\}$ and $B = \{v_{|A|+1}, \ldots, v_n\}$ (respectively, $B = \{v_1, \ldots, v_{|B|}\}$ and $A = \{v_{|B|+1}, \ldots, v_n\}$).

Lemma 2 shows that there is a biconvex ordering of a biconvex bipartite $G = (A, B, E)$, which is also a simple elimination ordering of G_A (respectively, G_B).

Lemma 2. *Suppose that $G = (A, B, E)$ is a biconvex bipartite graph with a biconvex ordering $\langle A, B \rangle$ (respectively, $\langle B, A \rangle$). The following statements are true.*

(1) *The biconvex ordering* $\langle A, B\rangle = v_1, v_2, \ldots, v_n$ *is a simple elimination ordering of* G_B.
(2) *The biconvex ordering* $\langle B, A\rangle = v_1, v_2, \ldots, v_n$ *is a simple elimination ordering of* G_A.

Proof. In the following, we just show the correctness of statement (1) since statement (2) can be proved in the similar way.

By Lemma 1, the graphs G_A and G_B obtained from G are strongly chordal graphs. Let G_i be the subgraph of G_B induced by $\{v_i, v_{i+1}, \ldots, v_n\}$, where $1 \le i \le n$. Let $N_i[v]$ denote the closed neighborhood of v in G_i. It can be easily verified that $N_i[v_i]$ is a clique of G_i for $1 \le i \le n$. Therefore, the ordering $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering of G_B. Suppose that there exist three positive integers i, j, and k such that $1 \le i < j < k \le n$ and $v_j, v_k \in N_i(v_i)$. We prove the biconvex ordering $v_1, v_2, \ldots, v_n$ is a simple elimination ordering of G_B by showing that either $N_i[v_j] \subseteq N_i[v_k]$ or $N_i[v_k] \subseteq N_i[v_j]$. We consider the following cases:

Case 1: $|A| + 1 \le i \le n$. Then $V(G_i) \subseteq B$. Note that B is a clique of G_B. Hence, $N_i[v_j] = N_i[v_k]$.

Case 2: $1 \le i \le |A|$. Then $v_i \in A$, $B \subset V(G_i)$, and $v_j, v_k \in B$. Clearly $(N_i[v_j] \cap B) = (N_i[v_k] \cap B)$. If $N_i[v_j] \cap A = \{v_i\}$, then we have $N_i[v_j] \subseteq N_i[v_k]$. Assume that $N_i[v_j] \cap A$ contains at least two vertices. By definition of the biconvex ordering, the vertices in $N_i[v_j] \cap A$ (respectively, $N_i[v_k] \cap A$) are consecutive in the ordering. This implies that either $(N_i[v_j] \cap A) \subseteq (N_i[v_k] \cap A)$ or $(N_i[v_k] \cap A) \subseteq (N_i[v_j] \cap A)$. Hence, either $N_i[v_j] \subseteq N_i[v_k]$ or $N_i[v_k] \subseteq N_i[v_j]$.

Following the discussion above, the biconvex ordering $v_1, v_2, \ldots, v_n$ is a simple elimination ordering of G_B. □

Theorem 1. *Let* $G = (A, B, E)$ *be a biconvex bipartite graph with* $|A \cup B| = n$ *and* $|E| = m$. *A simple elimination ordering of* G_A *(respectively,* G_B*) can be computed from* G *in* $O(n + m)$ *time.*

Proof. It follows from Lemma 2 and the result that a biconvex ordering of a biconvex bipartite graph can be computed in $O(n + m)$ time [15]. □

Following the arguments similar to those for proving Lemma 2, we can prove that there is a strong ordering of a bipartite permutation graph $G = (A, B, E)$, which is also a simple elimination ordering of G_A (respectively, G_B). Furthermore, we show in Lemma 3 that there is a strong ordering of a bipartite permutation graph $G = (A, B, E)$, which is also a *strong* elimination ordering of G_A (respectively, G_B).

Lemma 3. *Suppose that* $G = (A, B, E)$ *is a bipartite permutation graph with a strong ordering* $v_1, v_2, \ldots, v_n$ *(respectively,* $u_1, u_2, \ldots, u_n$*), where*

$A = \{v_1, \ldots, v_p\}$ and $B = \{v_{p+1}, \ldots, v_n\}$ (respectively, $B = \{u_1, u_2, \ldots, u_q\}$ and $A = \{u_{q+1}, u_{q+2}, \ldots, u_n\}$. The following statements are true.

(1) *The strong ordering $v_1, \ldots, v_n$ is a strong elimination ordering of G_B.*
(2) *The strong ordering $u_1, \ldots, u_n$ is a strong elimination ordering of G_A.*

Proof. In the following we just show the correctness of statement (1) since the statement (2) can be proved in the similar way.

By Lemma 1, the graphs G_A and G_B obtained from G are strongly chordal graphs. Let G_i be the subgraph of G_B induced by $\{v_i, v_{i+1}, \ldots, v_n\}$, where $1 \leq i \leq n$. Let $N_i[v]$ denote the closed neighborhood of v in G_i. It can be easily verified that $N_i[v_i]$ is a clique of G_i for $1 \leq i \leq n$. Therefore, the ordering $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering. Suppose that i, j, and k are positive integers. Let $1 \leq i < j < k \leq n$ and $v_j, v_k \in N_i[v_i]$. We prove the ordering $v_1, v_2, \ldots, v_n$ is a strong elimination ordering by showing that $N_i[v_j] \subseteq N_i[v_k]$. We consider the following cases:

Case 1: $p+1 \leq i \leq n$. Then $V(G_i) \subseteq B$. Note that B is a clique of G_B. Hence, $N_i[v_j] = N_i[v_k]$.

Case 2: $1 \leq i \leq p$. Then $v_i \in A$, $B \subset V(G_i)$, and $v_j, v_k \in B$. Clearly $(N_i[v_j] \cap B) = (N_i[v_k] \cap B)$. If $N_i[v_j] \cap A = \{v_i\}$, then we have $N_i[v_j] \subseteq N_i[v_k]$. If there is a vertex $v_\ell \in (N_i[v_j] \cap A)$ and $v_\ell \neq v_i$, then $i < \ell < j < k$. In this case, $(v_i, v_k), (v_\ell, v_j) \in E(G_i)$. By definition of the strong ordering, $(v_\ell, v_k) \in E(G_i)$. We have $v_\ell \in N_i[v_k]$. Hence, $N_i[v_j] \subseteq N_i[v_k]$. Following the discussion above, the ordering $v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_B. □

Theorem 2. *Suppose that $G = (A, B, E)$ is a bipartite permutation graph with $|A \cup B| = n$ and $|E| = m$. The graphs G_A and G_B obtained from G are strongly chordal graphs, and strong elimination orderings of G_A and G_B can be computed in $O(n+m)$ time, respectively.*

Proof. It follows from Lemmas 1 and 3, and the result that a strong ordering of a bipartite permutation graph can be computed in $O(n+m)$ time [20]. □

In the rest of the subsection, we give the function SimpleToStrong(G, $\langle X, Y \rangle$) for transforming a biconvex ordering $\langle X, Y \rangle$ of a biconvex bipartite graph $G = (X, Y, E)$ into a strong elimination ordering of G_Y. The function SimpleToStrong(G, $\langle X, Y \rangle$) includes the function MakeSets(G, $\langle X, Y \rangle$) for partitioning the vertices in $X \cup Y$ into a list $\mathcal{L}$ of disjoint sets. Let $H = (X, Y, E)$ be the biconvex bipartite graph with $X = \{a, b, c\}$ and $Y = \{d, e, f, g\}$ as shown in Figure 1. The function MakeSets(H, X, Y) returns the list of sets $\mathcal{L} = \{a\}$, $\{b\}$, $\{d, g\}$, $\{c\}$, $\{e, f\}$. We can visit each set in order, output the vertices within each set, and then obtain a new

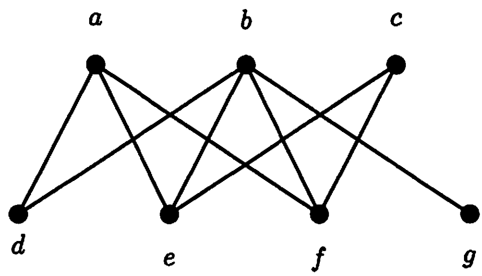

Figure 1: A biconvex bipartite graph.

ordering a, b, d, g, c, e, f. We can also obtain the orderings a, b, g, d, c, e, f and a, b, g, d, c, f, e in this fashion. To simplify the discussion, we use $\langle \mathcal{L} \rangle$ to denote an arbitrary ordering obtained from $\mathcal{L}$ as we did above.

For any biconvex bipartite graph $G = (A, B, E)$, Lemma 4 shows that an ordering $\langle \mathcal{L} \rangle$ obtained from the function MakeSets($G, \langle A, B \rangle$) (respectively, MakeSets($G, \langle B, A \rangle$)) is also a simple elimination ordering of G_B (respectively, G_A)

Lemma 4. *Suppose that $G = (A, B, E)$ is a biconvex bipartite graph. Let $\mathcal{L} = S_1, S_2, \ldots, S_r$ (respectively, $\hat{\mathcal{L}} = \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_r$) be returned from the function* MakeSets($G, \langle A, B \rangle$) *(respectively,* MakeSets($G, \langle B, A \rangle$)*). The following statements are true.*

(1) *Let $\langle \mathcal{L} \rangle = w_1, \ldots, w_n$. Then the ordering $\langle \mathcal{L} \rangle$ is a simple elimination ordering of G_B. If there exists $i < j < k$ such that (w_i, w_j), (w_i, w_k), and (w_j, w_k) are edges of G_B and there exists a positive integer ℓ such that $i < \ell$ and (w_j, w_ℓ) is an edge of G_B, but (w_k, w_ℓ) is not, then w_j and w_k must belong to the same set in $\mathcal{L}$.*

(2) *Let $\langle \hat{\mathcal{L}} \rangle = w_1, \ldots, w_n$. Then the ordering $\langle \hat{\mathcal{L}} \rangle$ is a simple elimination ordering of G_A. If there exists $i < j < k$ such that (w_i, w_j), (w_i, w_k), and (w_j, w_k) are edges of G_A and there exists a positive integer ℓ such that $i < \ell$ and (w_j, w_ℓ) is an edge of G_A, but (w_k, w_ℓ) is not, then w_j and w_k must belong to the same set in $\hat{\mathcal{L}}$.*

Proof. In the following, we just show the correctness of statement (1) since statement (2) can be proved in the similar way.

Note that G_B is obtained from G by adding all possible edges between the vertices of B such that the set B is a clique of G_B. By Lemma 2, the biconvex ordering $\langle A, B \rangle = v_1, \ldots, v_n$ is a simple elimination ordering of G_B. If $|A| = 1$, then the ordering $\langle \mathcal{L} \rangle$ is a simple elimination ordering of

Function MakeSets($G, \langle X, Y\rangle$)

$\mathcal{L} \leftarrow$ empty list of sets;
$v_1, v_2, \ldots, v_n \leftarrow \langle X, Y\rangle$;
for $i \leftarrow 1$ **to** $|X|$ **do**
    $S \leftarrow \{v_i\}$; $S' \leftarrow \{u \mid u \in N_G(v_i)$ and $\deg_G(v_i) = \deg_G(u) + |Y| - 1\}$;
    append S to $\mathcal{L}$;
    if $S' \neq \emptyset$ **then**
        append S' to $\mathcal{L}$;
    end if
    remove S from X; remove S' from Y; update the neighborhoods and degrees;
end for
if $Y \neq \emptyset$ **then**
    append Y to $\mathcal{L}$;
end if;
return $\mathcal{L}$;

G_B. We therefore assume that $|A| \geq 2$. Let S be a set in $\mathcal{L}$. Following the function MakeSets($G, \langle A, B\rangle$), either $S \subseteq B$ or S consists of precisely one vertex in A. Let $V = A \cup B$ and let h be a positive integer such that $1 < h \leq r$ and $S_h = \{v_{|A|}\}$. Then $(S_{h+1} \cup S_{h+2} \cup \cdots \cup S_r) \subseteq B$ and each vertex in S_ℓ is simple in the graph $G_B[V - S_1 - S_2 - \cdots - S_{\ell-1}]$ for $h \leq \ell \leq r$.

Clearly $S_1 = \{v_1\}$ and v_1 is a simple vertex in G_B. In the following, we show that each vertex in S_i is simple in the graph $G_B[V - S_1 - S_2 - \cdots - S_{i-1}]$ for $2 \leq i \leq h-1$. Now consider the set S_2.

Case 1: S_2 consists of precisely one vertex in A. Then $S_2 = \{v_2\}$. It can be easily verified that v_2 is a simple vertex in the graph $G_B[V - S_1]$.

Case 2: $S_2 \subseteq B$. By the function MakeSets($G, \langle A, B\rangle$), any vertex $x \in S_2$ must be a neighbor of v_1 and $deg_G(v_1) = deg_G(x) + |B| - 1 = deg_{G_B}(x)$. By the construction of G_B, $N_G[v_1] = N_{G_B}[v_1]$. We have $|N_{G_B}[x]| = |N_{G_B}[v_1]|$. If $N_{G_B}[x] \neq N_{G_B}[v_1]$, then there is a vertex y adjacent to v_1 but not adjacent to x. This contradicts the fact that v_1 is a simple vertex in G_B. Thus $N_{G_B}[v_1] = N_{G_B}[x]$ which implies that v_1 and each vertex in S_2 are simple in G_B. Therefore, each vertex in S_2 is simple in the graph $G_B[V - S_1]$.

Using the arguments similar to those for proving Cases 1 and 2, it follows that each vertex in S_ℓ are simple in the graphs $G_B[V - S_1 - S_2 - \cdots - S_{\ell-1}]$ for $3 \leq \ell \leq h-1$. Following the discussion above, the ordering $\langle \mathcal{L}\rangle$ is a simple elimination ordering of G_B.

Now suppose that there exists $i < j < k$ such that (w_i, w_j), (w_i, w_k),

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Function SimpleToStrong(G, ⟨X, Y⟩)
  L' ← MakeSets(G, ⟨X, Y⟩); L ← L';
  v_1, ..., v_n ← ⟨L⟩;
  for t ← n down to 1 do
     if v_t ∈ X then
        for each set S ∈ L containing a vertex in N_G[v_t] do
           if S − N_G[v_t] ≠ ∅ then
              replace S in the list L with the two sets S − N_G[v_t],
              S ∩ N_G[v_t];
           end if
        end for
     end if
  end for
  return ⟨L⟩
```

and (w_j, w_k) are edges of G_B and there exists a positive integer ℓ such that $i < \ell$ and (w_j, w_ℓ) is an edge of G_B, but (w_k, w_ℓ) is not. Note that w_i, w_k, and w_ℓ are adjacent to w_j. If w_j is in A, then $w_i, w_k, w_\ell \in B$. Since B is a clique of G_B, w_ℓ is adjacent to w_k. This contradicts the assumption that (w_k, w_ℓ) is not an edge of G_B. Therefore $w_j \in B$.

Let $G_j = G_B[\{w_j, \ldots, w_n\}]$. If $j < \ell$, then since w_j is simple in G_j, either $N_{G_j}[w_\ell] \subseteq N_{G_j}[w_k]$ or $N_{G_j}[w_k] \subseteq N_{G_j}[w_\ell]$. This implies w_ℓ is adjacent to w_k. This contradicts the fact that (w_k, w_ℓ) is not an edge of G_B. Therefore, we have $\ell < j$.

Assume for contrary that w_j and w_k belong to different sets S_x and S_y in the list $\mathcal{L}$, respectively. Since $j < k$, $x < y$. Note that $w_j \in B$ and $S_x \subseteq B$. By the function MakeSets(G, $\langle A, B\rangle$), the set S_{x-1} consists of precisely one vertex in A. Let w_t be the vertex in S_{x-1}. Then $\ell \leq t < j$. Let $G_t = G_B[\{v_t, \ldots, v_n\}]$. It can be proved by contradiction that $N_{G_t}[w_t] = N_{G_t}[w_j]$. Therefore, w_k is adjacent to w_t and $w_k \in B$. This implies that there exists a vertex $z \in A$ such that z is adjacent to w_k, but not adjacent to w_j in G_t. Otherwise w_k would be in the same set as w_j. However, this contradicts the fact that w_t is a simple vertex in G_t. Thus w_j and w_k must belong to the same set in $\mathcal{L}$. □

Theorem 3. *Suppose that $G = (A, B, E)$ is a biconvex bipartite graph. The following statements are true.*

(1) *The function* SimpleToStrong(G, $\langle A, B\rangle$) *outputs a strong elimination ordering of G_B.*
(2) *The function* SimpleToStrong(G, $\langle B, A\rangle$) *outputs a strong elimination ordering of G_A.*

Proof. In the following, we just show the correctness of statement (1) since statement (2) can be proved in the similar way.

The function SimpleToStrong($G, \langle A, B\rangle$) starts by obtaining a list of sets $\mathcal{L}'$ returned by MakeSets (G, $\langle A$, $B\rangle$). The list $\mathcal{L}$ is a copy of the list $\mathcal{L}'$. In Step 2, the ordering $v_1, v_2, \ldots, v_n$ is updated to one that can be obtained from $\mathcal{L}$. By Lemma 4, the ordering $v_1, v_2, \ldots, v_n$ is a simple elimination ordering of G_B. In Steps 3–11, the function processes vertices in the ordering $v_n, v_{n-1}, \ldots, v_1$. As a visited vertex v_t is in A, the function replaces each set S in $\mathcal{L}$ that contains a neighbor of v_t with two sets $S - N_G[v_t]$ and $S \cap N_G[v_t]$ if $S - N_G[v_t]$ is not empty. Note that the set $S - N_G[v_t]$ is placed before $S \cap N_G[v_t]$.

The list $\mathcal{L}$ is finalized at the end of Step 11. Clearly each set $S \in \mathcal{L}'$ is either the same as a set in $\mathcal{L}$ or partitioned into at least two consecutive sets in $\mathcal{L}$. Thus the ordering $\langle \mathcal{L} \rangle$, returned from Step 12, is an arbitrary ordering that can also be obtained from $\mathcal{L}'$ by visiting each set of $\mathcal{L}'$ in order and outputting the vertices within each set. Hence it is a simple elimination ordering of G_B, too.

Let $w_1, w_2, \ldots, w_n$ be the ordering returned from Step 12 of the function SimpleToStrong(G, $\langle A$, $B\rangle$). As we mentioned above, the ordering is a simple elimination ordering that can also be obtained from the list $\mathcal{L}'$. Now suppose that there exists $i < j < k$ such that (w_i, w_j) and (w_i, w_k) are edges of G_B. Let $G_i = G_B[\{w_i, w_{i+1}, \ldots, w_n\}]$. Since the ordering $w_1, \ldots, w_n$ is a simple elimination ordering of G_B, the vertex w_i is simple in G_i and either $N_{G_i}[v_j] \subseteq N_{G_i}[v_k]$ or $N_{G_i}[v_k] \subseteq N_{G_i}[v_j]$. Therefore (w_j, w_k) is an edge of G_B.

Assume for contrary that there exists a positive integer ℓ such that $i < \ell$ and (w_j, w_ℓ) is an edge of G_B, but (w_k, w_ℓ) is not. Since w_i is simple in G_i, we have $N_{G_i}[w_k] \subset N_{G_i}[w_j]$. By Lemma 4, w_j and w_k must belong to the same set in $\mathcal{L}'$. For each set S in $\mathcal{L}'$, either $S \subseteq B$ or S consists of precisely one vertex in A. Thus $w_j, w_k \in B$ and $w_\ell \in A$. After the iteration of Steps 3–11 when $t = \ell$, the vertex w_k will be in a set that precedes the set containing w_j. This contradicts $j < k$. Hence $w_1, \ldots, w_n$ is a strong elimination ordering of G_B. □

Theorem 4. *The functions* SimpleToStrong(G, $\langle X$, $Y\rangle$) *and* MakeSets(G, $\langle X$, $Y\rangle$) *can be implemented in* $O(m)$ *time.*

Proof. The analysis and implementation of the functions SimpleToStrong(G, $\langle X$, $Y\rangle$) and MakeSets(G, $\langle X$, $Y\rangle$) are similar to those of the algorithm for transforming a simple elimination ordering of a strongly chordal graphs into a strong elimination ordering. We refer to [18] for the details. Hence these two functions can be implemented in $O(m)$ time. □

Corollary 1. *Let $G = (A, B, E)$ be a biconvex bipartite graph with $|A \cup B| = n$ and $|E| = m$. A strong elimination ordering of G_A (respectively, G_B) can be computed from G in $O(n + m)$ time.*

Proof. It follows from Theorems 1, 3, and 4. □

3.2 A linear-time algorithm

Let ℓ, d, I_1 be fixed integers and $\ell, d > 0$. Let $\mathcal{P}$ be the weight set $\{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$. Suppose that $G = (A, B, E)$ is a bipartite graph with a labeling function R which assigns an integer $R(v)$ to each vertex $v \in V(G)$. Let R_A (respectively, R_B) be a labeling function of G which assigns an integer $R_A(v)$ (respectively, $R_B(v)$) to each vertex in G such that $R_A(v) = I_1 \cdot deg_G(v)$ (respectively, $R_B(v) = I_1 \cdot deg_G(v)$) for every $v \in A$ (respectively, $v \in B$), and $R_A(v) = R(v)$ (respectively, $R_B(v) = R(v)$) for every $v \in B$ (respectively, $v \in A$).

Definition 5. An R_A-total dominating function f of a bipartite graph $G = (A, B, E)$ is called an *R_A^*-total dominating function* of G if $f(v) = I_1 + (\ell - 1) \cdot d$ for every $v \in B$. An R_B-total dominating function g of G is called an *R_B^*-total dominating function* of G if $g(v) = I_1 + (\ell - 1) \cdot d$ for every $v \in A$.

Lemma 5 shows that a minimum R-total dominating function of a chordal bipartite graph G can be obtained from a minimum R_A^*-total dominating function and a minimum R_B^*-total dominating function of G.

Lemma 5 ([14]). *Suppose that $G = (A, B, E)$ is a bipartite graph with a labeling function R as mentioned above. Let f_A (respectively, f_B) be a minimum R_A^*-total (respectively, R_B^*-total) dominating function of G. Let f be a function of G defined by $f(v) = f_A(v)$ for every $v \in A$ and $f(v) = f_B(v)$ for every $v \in B$. Then f is a minimum R-total dominating function of G.*

We give the function MRTD(G,$\langle X, Y\rangle, R, I_1, \ell, d$) for computing a minimum R_Y^*-dominating function of a biconvex bipartite graph G. The MRTD(G, $\langle X, Y\rangle$, R, I_1, ℓ, d) takes G, $\langle X, Y\rangle$, R, I_1, ℓ, and d as inputs. Input G represents a biconvex bipartite graph, and X and Y are the bipartition of G. Input $\langle X, Y\rangle$ is a biconvex ordering of G. Input R is a labeling function assigning an integer $R(v)$ to each vertex $v \in X \cup Y$. Inputs ℓ, d, I_1 are integers and $\ell, d > 0$. The weight set $\mathcal{P}$ is assumed to be the set $\{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$. The function MRTD($G, \langle X, Y\rangle, R, I_1, \ell, d$) finds a minimum R_Y^*-dominating function of a biconvex bipartite graph G.

Lemma 6. *If the function f initialized by the function* MRTD(G, $\langle X, Y\rangle$, *R, I_1, ℓ, d) in Steps 6–8 is not an R_Y^*-total dominating function of G, then G has no R-total dominating functions.*

Function MRTD($G, \langle X, Y\rangle, R, I_1, \ell, d$)

1: **for** every vertex $v \in X \cup Y$ **do**
2: **if** $v \in Y$ **then** $R_Y(v) = I_1 \cdot deg_G(v)$;
3: **else** $R_Y(v) = R(v)$;
4: **end for**
5: $v_1, \dots, v_n \leftarrow$ SimpleToStrong($G, \langle X, Y\rangle$);
6: **for** $i \leftarrow 1$ **to** n **do**
7: $f(v_i) \leftarrow I_1 + (\ell - 1) \cdot d$;
8: **end for**
9: **for** $i \leftarrow 1$ **to** n **do**
10: **if** $R_Y(v_i) > f(N_G(v_i))$
11: **then** stop and return the infeasibility of the problem;
13: **end for**
14: **for** $i \leftarrow 1$ **to** n **do**
15: **if** $v_i \in Y$ **then**
16: $M \leftarrow \min\{f(N_G(v)) - R_Y(v) | v \in N_G(v_i)\}$;
17: $f(v_i) \leftarrow \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}$;
18: **end for**
19: return the function f;

Proof. Note that the maximum value in $\mathcal{P}$ is $I_1 + (\ell - 1) \cdot d$. We may assume that $\ell > 1$. The function MRTD($G, \langle X, Y\rangle, R, I_1, \ell, d$) in Steps 6–8 assigns the maximum value in $\mathcal{P}$ to all vertices in G. The function f has the largest weight among all R_Y^*-total (respectively, R-total) dominating functions if f is an R_Y^*-total (respectively, R-total) dominating function of G. Since the minimum value in $\mathcal{P}$ is I_1, $f(N_G(v)) > R_Y(v)$ for every vertex $v \in Y$. If there exists a vertex $v \in X \cup Y$ such that $R_Y(v) > f(N_G(v))$, then $v \in X$ and $R(v) = R_Y(v) > f(N_G(v))$. This implies that the function f is neither an R_Y^*-total dominating function nor an R-total dominating function of G, and thus G has no R-total dominating functions. □

Lemma 7. *The function f returned from Step 19 of the function* MRTD(G, $\langle X, Y\rangle$, R, I_1, ℓ, d) *is an R_Y^*-dominating function of G.*

Proof. Following Theorem 3, the ordering $v_1, \dots, v_n$ obtained by Step 5 is a strong elimination ordering of G_Y. By Lemma 6, the function f initialized in Steps 6–8 is an R_Y^*-total dominating function of G if the function does not stop in Step 11. In Steps 14–18, the function MRTD($G, \langle X, Y\rangle, R, I_1, \ell, d$) processes vertices in the ordering $v_1, v_2, \dots, v_n$ to decrease the weight of the function f. In the following, we show that at the end of each iteration of Steps 14–18, the new function f obtained by changing the value of $f(v_i)$ in Step 17 is still an R_Y^*-total dominating function of G.

The function f at the beginning of the first iteration of Steps 14–18 is an R^*_Y-total dominating function initialized in Steps 6–8. We assume that at the beginning of the i-th iteration of Steps 14–18, the function f is an R^*_Y-total dominating function of G. Suppose that $v_i \in X$. Then the value of $f(v_i)$ will not be changed and thus $f(v_i) = I_1 + (\ell - 1) \cdot d$. The function f at the end of the i-th iteration is the same as the function f at the beginning of the i-th iteration. Suppose that $v_i \in Y$. Let $x = \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}$. Then $x \geq I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d$. We have

$$x \geq I_1 + (\ell - 1 - \frac{M}{d}) \cdot d \Rightarrow M \geq (I_1 + (\ell - 1) \cdot d) - x.$$

Since $M = \min\{f(N_G(v)) - R_Y(v) | v \in N_G(v_i)\}$, $f(N_G(v)) - R_Y(v) \geq (I_1 + (\ell - 1) \cdot d) - x$ for every vertex $v \in N_G(v_i)$. We have $f((N_G(v) - \{v_i\})) + f(v_i) - (I_1 + (\ell - 1) \cdot d) + x \geq R_Y(v)$ for every vertex $v \in N_G(v_i)$. Note that $f(v_i) = I_1 + (\ell - 1) \cdot d$ before the execution of Step 17. Therefore, the new function f obtained by changing the value of $f(v_i)$ in Step 17 is still an R^*_Y-total dominating function of G. Following the discussion above, we know that the function f returned from Step 19 of the function MRTD$(G, \langle X, Y \rangle, R, I_1, \ell, d)$ is an R^*_Y-total dominating function of G. □

Lemma 8. *The function f found by the function* MRTD$(G, \langle X, Y \rangle, R, I_1, \ell, d)$ *is a minimum R^*_Y-total dominating function of G.*

Proof. By Lemmas 6 and 7, the function f found by the function MRTD$(G, \langle X, Y \rangle, R, I_1, \ell, d)$ is an R^*_Y-dominating function of G. In the following, we let $V = X \cup Y$ and show that f is a minimum R^*_Y-total dominating function of G. Among all minimum R^*_Y-total dominating functions of G, we let h be a minimum R^*_Y-total dominating function of G such that the cardinality of $\{v | v \in V, f(v) = h(v)\}$ is maximum. We claim that $f(v) = h(v)$ for every vertex $v \in V$. Assume for contrary that W is a nonempty set of all vertices w with $f(w) \neq h(w)$. Suppose that t is the smallest index such that $v_t \in W$. Obviously, $v_t \in Y$ and $W \subseteq Y$. We consider the following cases.

Case 1: $h(v_t) < f(v_t)$. By the function MRTD$(G, \langle X, Y \rangle, R, I_1, \ell, d)$, $f(v_t) = \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}$ at the end of the t-th iteration (where an *iteration* here is understood as one iteration of Steps 14–18). We consider the following two cases:

Case 1.1: $f(v_t) = I_1$. Then $h(v_t) < f(v_t) = I_1$ which contradicts the assumption that $h(v_t) \in \mathcal{P}$ since I_1 is the smallest number in $\mathcal{P}$.

Case 1.2: $f(v_t) = I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d$. Then $h(v_t) \leq f(v_t) - d = I_1 + (\lceil \ell - \frac{M}{d} \rceil - 2) \cdot d$. Let v_α be a vertex in $N_G(v_t)$ such that $M = f(N_G(v_\alpha)) - R_Y(v_\alpha)$. Note that at the beginning of the t-th iteration, $f(v_t) = I_1 + (\ell - 1) \cdot d$. Therefore, $f(N_G(v_\alpha) - \{v_t\}) = M + R_Y(v_\alpha) - (I_1 + (\ell - 1) \cdot d)$ before the execution of Step 17 at the t-iteration. Since only the value of $f(v_t)$ was changed at the t-th iteration, $f(N_G(v_\alpha) - \{v_t\})$ is still equal to $M + R_Y(v_\alpha) - (I_1 + (\ell-1)\cdot d)$ at the end of t-th iteration.

Note that $h(v_x) = f(v_x)$ for every index $x < t$. At the end of the t-th iteration, $h(v_x) \le f(v_x) = I_1 + (\ell-1)\cdot d$ for every index $x > t$. Then,

$$\begin{aligned} h(N_G(v_\alpha)) &\le f(N_G(v_\alpha) - \{v_t\}) + h(v_t) \\ &\le f(N_G(v_\alpha) - \{v_t\}) + I_1 + (\lceil \ell - \tfrac{M}{d} \rceil - 2)\cdot d \\ &= M + R_Y(v_\alpha) - (I_1 + (\ell-1)\cdot d) + I_1 + (\lceil \ell - \tfrac{M}{d} \rceil - 2)\cdot d \\ &\le M + R_Y(v_\alpha) - (I_1 + (\ell-1)\cdot d) + (I_1 + ((\ell - \tfrac{M}{d}) - 1)\cdot d) \\ &= M + R_Y(v_\alpha)) - (I_1 + (\ell-1)\cdot d) + (I_1 + (\ell-1)\cdot d - M) \\ &= R_Y(v_\alpha) \end{aligned}$$

Hence $h(N_G(v_\alpha) < R_Y(v_\alpha)$ which contradicts the assumption that h is an R_Y^*-total dominating function.

Case 2: $f(v_t) < h(v_t)$. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_\ell\}$ where $p_1 = I_1$, $p_2 = I_1 + d, \ldots, p_\ell = I_1 + (\ell-1)\cdot d$. We let $f(v_t) = p_i$ and $h(v_t) = p_j$ for $1 \le i < j \le \ell$. Let $X' = \{v | v \in N_G(v_t), h(N_G(v)) - p_j + p_i < R_Y(v)\}$. We have $X' \neq \emptyset$. Otherwise, $h(N_G(v)) - p_j + p_i \ge R_Y(v)$ for every $v \in N_G(v_t)$ and there is an R_Y^*-total dominating function g with $g(V) < h(V)$ by setting $g(v_t) = h(v_t) - p_j + p_i = p_i$ and $g(v) = h(v)$ for every vertex $v \in V - \{v_t\}$. It leads to a contradiction to the assumption that h is a minimum R_Y^*-total dominating function.

Note that $h(v_x) = f(v_x)$ for every index $x < t$. Since $h(N_G(v)) - a_j + a_i < R_Y(v)$ and $f(N_G(v)) \ge R_Y(v)$ for every vertex $v \in X'$, $N_G(v) \cap \{v_x | v_x \in W, t < x, \text{ and } h(v_x) < f(v_x)\} \neq \emptyset$. Let $Y'(v) = N_G(v) \cap \{v_x | v_x \in W, t < x, \text{ and } h(v_x) < f(v_x)\}$ for every vertex $v \in X'$. Clearly $Y'(v) \subseteq Y$ for every vertex $v \in X'$.

Let s be the smallest index of vertices in X'. Let b be the smallest index of $Y'(v_s)$. Note that $X' \subseteq N_G(v_t) \subseteq X$. Since $h(v_t)$, $f(v_t)$, $h(v_b)$, and $f(v_b)$ are in $\mathcal{P}$, there exist two positive integers α_1 and α_2 such that $h(v_t) = f(v_t) + \alpha_1 \cdot d$ and $f(v_b) = h(v_b) + \alpha_2 \cdot d$. We define a function h' as follows.

(1) If $\alpha_1 \le \alpha_2$, $h'(v_t) = h(v_t) - \alpha_1 \cdot d = f(v_t)$, $h'(v_b) = h(v_b) + \alpha_1 \cdot d$ and $h'(v) = h(v)$ for every vertex $v \in V - \{v_t, v_b\}$.
(2) If $\alpha_1 > \alpha_2$, $h'(v_t) = h(v_t) - \alpha_2 \cdot d$, $h'(v_b) = h(v_b) + \alpha_2 \cdot d = f(v_b)$, and $h'(v) = h(v)$ for every vertex $v \in V - \{v_t, v_b\}$.

Clearly, $h(V) = h'(V)$ and $|\{v | v \in V, f(v) = h'(v)\}| \ge |\{v | v \in V, f(v) = h(v)\}| + 1$. We prove $h'(N_G(v)) \ge R_Y(v)$ for every vertex $v \in V$ by showing that $X' \subseteq N_G(v_b)$. Since $v_s \in X'$ and $v_b \in Y'(v_s)$, $v_s \in X$ and $v_t, v_b \in Y$. Apparently $s \neq t$. For $1 \le i \le n$, we let $N_i[v]$ (respectively $N_i^Y[v]$) denote the closed neighborhood of a vertex $v \in V$ in the subgraph of G (respectively, G_Y) induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. For every vertex $v \in V$, we let $N_i(v) = N_i[v] - \{v\}$ and $N_i^Y(v) = N_i^Y[v] - \{v\}$. We consider the following two cases:

Case 2.1: $s < t$. Then $s < t < b$. Following Theorem 3, the ordering $v_1, \dots, v_n$ obtained by Step 5 of the function $\mathsf{MRTD}(G, \langle X, Y\rangle, R, I_1, \ell, d)$ is a strong elimination ordering of G_Y. By definition of the strong elimination ordering, $N_s^Y[v_t] \subseteq N_s^Y[v_b]$. Then $N_s(v_t) = (N_s^Y[v_t] \cap X) \subseteq (N_s^Y[v_b] \cap X) = N_s(v_b)$. Since $X' \subseteq N_s(v_t)$, we have $X' \subseteq N_s(v_b) \subseteq N_G(v_b)$.

Case 2.2: $s > t$. By definition of the strong elimination ordering, $N_t^Y[v_s] \subseteq N_t^Y[v]$ for every vertex $v \in X'$. Since $v_b \in N_t^Y[v_s]$, $v_b \in N_t^Y[v]$ for every vertex $v \in X'$. Since $v_b \in Y$, $v_b \notin X$. Therefore, $X' \subseteq N_t^Y[v_b] \cap X = N_t(v_b) \subseteq N_G(v_b)$.

Hence, h' is a minimum R_Y^*-total dominating function such that the cardinality of $\{v | v \in V, f(v) = h'(v)\}$ is larger than that of $\{v | v \in V, f(v) = h(v)\}$ which contradicts the assumption that the cardinality of $\{v | v \in V, f(v) = h(v)\}$ is maximum.

Following the discussion above, W does not exist. Hence, f is a minimum R_Y^*-total dominating function of G. □

Lemma 9. *The function* $\mathsf{MRTD}(G, \langle X, Y\rangle, R, I_1, \ell, d)$ *finds a minimum* R_Y^**-total dominating function of a biconvex bipartite graph* $G = (X, Y, E)$ *in* $O(n + m)$ *time.*

Proof. Steps 1–4 can be done in $O(\sum_{v \in X \cup Y}(deg_G(v) + 1)) = O(n + m)$ time. By Theorem 4, Step 5 can be done in $O(m)$ time. The initialization of a function f in Steps 6–8 can be done in $O(n)$ time.

For each vertex $v_i \in X \cup Y$, we can use $d(v_i)$ to keep track of $f(N_G(v_i))$ and use $m(v_i)$ to keep track of $d(v_i) - R_Y(v_i)$. Following the initialization of a function f in Steps 6–8, we initialize $d(v_i) = (I_1 + (\ell - 1) \cdot d) \cdot deg_G(v_i)$ and $m(v_i) = d(v_i) - R_Y(v_i)$. The initialization of $d(v_i)$ and $m(v_i)$ can be done in $O(deg_G(v_i) + 1)$ time.

While $f(v_i)$ is replaced by a number $x \in \mathcal{P}$, $d(v)$ and $m(v)$ are respectively decreased by $(I_1 + (\ell - 1) \cdot d) - x$ for every vertex $v \in N_G(v_i)$. This can be done in $O(deg_G(v_i) + 1)$ time. At i-th iteration, $1 \le i \le n$, M can be computed in $O(deg_G(v_i))$ time by verifying $m(v)$ for every vertex $v \in N_G(v_i)$. Following the discussion above, the running time of the function $\mathsf{MRTD}(G, \langle X, Y\rangle, R, I_1, \ell, d)$ is $O(\sum_{v_i \in X \cup Y}(deg_G(v_i) + 1)) = O(n + m)$. □

Theorem 5. *Given a biconvex bipartite graph* $G = (A, B, E)$ *with* $|A \cup B| = n$ *and* $|E| = m$*, the R-total domination problem can be solved in* $O(n + m)$ *time.*

Proof. By Lemma 8, a function f_A (respectively, f_B) obtained by $\mathsf{MRTD}(G, \langle B, A\rangle, R, I_1, \ell, d)$ (respectively, $\mathsf{MRTD}(G, \langle A, B\rangle, R, I_1, \ell, d)$) is a minimum R_A^*-total (respectively, R_B^*-total) dominating function of G. Following Lemmas 5 and 9, the R-total domination problem is linear-time solvable for a biconvex bipartite graph G. □

4 NP-completeness results

In this section, we show that the signed (respectively, minus) total domination problem is NP-complete on planar bipartite graphs of maximum degree 3 (respectively, maximum degree 4). Before presenting the NP-completeness results, we restate the vertex cover, total domination, signed total domination, and minus total domination problems as decision problems.

(1) **The vertex cover problem:**
Instance: A graph $G = (V, E)$ and a positive integer K.
Question: Is $\tau(G) \leq K$?

(2) **The total domination problem:**
Instance: A graph $G = (V, E)$ and an integer K.
Question: Is $\gamma_t(G) \leq K$?

(3) **The signed total domination problem:**
Instance: A graph $G = (V, E)$ and an integer K.
Question: Is $\gamma_t^s(G) \leq K$?

(4) **The minus total domination problem:**
Instance: A graph $G = (V, E)$ and an integer K.
Question: Is $\gamma_t^-(G) \leq K$?

Theorem 6. *The total domination problem is NP-complete on planar bipartite graphs of maximum degree 3.*

Proof. The total domination problem on planar bipartite graphs of maximum degree 3 is clearly in NP. It is known that vertex cover problem is NP-complete on planar graphs of maximum degree 3 [5, 6]. In the following, we show the NP-completeness of the total domination problem on planar bipartite graphs of maximum degree 3 by reducing the vertex cover problem on planar graphs of maximum degree 3 to it in polynomial time.

Let $G = (V, E)$ be a planar graph of maximum degree 3. Let $E = \{e_1, e_2, \ldots, e_m\}$. Assume that $e_i = (u_i, v_i)$ for $1 \leq i \leq m$. We construct the graph H using the following steps:

(1) Let $V(H) = V \cup W$, where $W = \{w_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq 2\}$.

(2) Let $E_1 = \{(u_i, w_{i,1}), (w_{i,1}, v_i) \mid 1 \leq i \leq m\}$. In other words, we replace each edge e_i by two edges $(u_i, w_{i,1})$ and $(w_{i,1}, v_i)$ for $1 \leq i \leq m$.

(3) Let $E(H) = E_1 \cup E_2$, where $E_2 = \{(w_{i,1}, w_{i,2}) \mid 1 \le i \le m\}$.

It is clear that the graph H can be constructed from G in polynomial time and that H is a planar bipartite graph of maximum degree 3.

Claim 1. $\gamma_t(H) = \tau(G) + m$.

Proof. Let $W_1 = \{w_{i,1} \mid 1 \le i \le m\}$ and $W_2 = \{w_{i,2} \mid 1 \le i \le m\}$. Suppose that S is a vertex cover of G of $\tau(G)$ vertices. Let $D = S \cup W_1$. It can be easily verified that D is a total dominating set of H. We have $\gamma_t(H) \le \tau(G) + m$.

Conversely, let D be a minimum total dominating set of H. Necessarily, D contains all vertices in W_1. Suppose that there is a vertex $w_{i,2} \in W_2$ such that $w_{i,2} \in D$. Then, D contains at most one vertex of u_i and v_i. Otherwise, D would contain them both. The set $D' = (D - \{w_{i,2}\})$ would be a total dominating set of H with $|D'| < |D|$. However, this contradicts the assumption that D is a *minimum* total dominating set of H. We may assume that $u_i \notin D$. The set $D' = (D - \{w_{i,2}\}) \cup \{u_i\}$ is still a minimum total dominating set of H. Hence, there exists a minimum total dominating set $\hat{D}$ of H such that $\hat{D} \cap W_2 = \emptyset$ and W_1 is a subset of $\hat{D}$. Let w be a vertex in W_1. Let $u, v \in N_H(w)$ and $(u, v) \in E$. Since $\hat{D} \cap W_2 = \emptyset$, at least one vertex of u and v is in $\hat{D}$. Then $\hat{D} - W_1$ is a vertex cover of G. We have $\tau(G) \le \gamma_t(H) - m$. Following the discussion above, $\gamma_t(H) = \tau(G) + m$. □

The above claim implies that for a positive integer K, $\tau(G) \le K$ if and only if $\gamma_t(H) \le K + m$. □

Theorem 7. *The minus total domination problem is NP-complete on planar bipartite graphs of maximum degree 4.*

Proof. The minus total domination problem on planar bipartite graphs of maximum degree 3 is clearly in NP. We have shown in Theorem 6 that the total domination problem is NP-complete on planar bipartite graphs of maximum degree 3. In the following we show the NP-completeness of the minus total domination problem on planar bipartite graphs of maximum degree 4 by reducing the total domination problem on planar bipartite graphs of maximum degree 3 to it in polynomial time.

Given a planar bipartite graph G of maximum degree 3, we construct the graph H by adding a path of length 4, say $v - v_1 - v_2 - v_3 - v_4$, to each vertex $v \in V(G)$. That is, $V(H) = V(G) \cup (\bigcup_{v \in V}\{v_1, v_2, v_3, v_4\})$ and $E(H) = E(G) \cup (\bigcup_{v \in V}\{(v, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_4)\})$. It can be easily verified that H is a planar bipartite graph of maximum degree 4. Let $|V(G)| = n$. By the arguments similar to those for proving the NP-completeness of the minus total domination problem on bipartite graphs in [7], we have the following claim.

Claim 2. $\gamma_t^-(H) = \gamma_t(H) = \gamma_t(G) + 2n$

The claim implies that for a positive integer K, $\gamma_t(G) \le K$ if and only if $\gamma_t^-(H) \le K + 2n$. □

Theorem 8. *The signed total domination problem is NP-complete on planar bipartite graphs of maximum degree 3.*

Proof. The signed total domination problem on planar bipartite graphs of maximum degree 3 is clearly in NP. It is known that vertex cover problem is NP-complete on planar graphs of maximum degree 3 [5, 6]. In the following, we show the NP-completeness of the signed total domination problem on planar bipartite graphs of maximum degree 3 by reducing the vertex cover problem on planar graphs of maximum degree 3 to it in polynomial time.

Let $G = (V, E)$ be a planar graph of maximum degree 3. Let $E = \{e_1, e_2, \ldots, e_m\}$ and let $e_i = (u_i, v_i)$ for $1 \le i \le m$. We construct the graph H using the following steps:

(1) Let $V(H) = V \cup W$, where $W = \{w_{i,j} \mid 1 \le i \le m \text{ and } 1 \le j \le 3\}$.

(2) Let $E_1 = \{(u_i, w_{i,1}), (w_{i,1}, v_i) \mid 1 \le i \le m\}$. In other words, we replace each edge e_i by two edges $(u_i, w_{i,1})$ and $(w_{i,1}, v_i)$ for $1 \le i \le m$.

(3) Let $E(H) = E_1 \cup E_2$, where $E_2 = \{(w_{i,1}, w_{i,2}), (w_{i,2}, w_{i,3}) \mid 1 \le i \le m\}$.

It is clear that H can be constructed from G in polynomial time and that H is a planar bipartite graph of maximum degree 3. Let $|V| = n$.

Claim 3. $\gamma_t^s(H) = 3m - n + 2\tau(G)$.

Proof. Let S be a vertex cover of G of $\tau(G)$ vertices. Let $h : V(H) \to \{-1, 1\}$ be a function of H such that $h(v) = 1$ if $v \in S \cup W$ and $h(v) = -1$ if $v \in V(H) - (S \cup W)$. It can be easily verified that h is a signed total dominating function of H. We have $\gamma_t^s(H) \le (\tau(G) - (n - \tau(G))) + 3m = 3m - n + 2\tau(G)$.

Conversely, let h be a minimum signed total dominating function of H. For $1 \le i \le m$ and $1 \le j \le 3$, we now consider the vertices u_i, v_i, and $w_{i,j}$. Necessarily, $h(w_{i,2}) = 1$. Note that $N_H(w_{i,2}) = \{w_{i,1}, w_{i,3}\}$. Since $N_H(w_{i,2}) \ge 1$, the function h cannot assign the value -1 to $w_{i,1}$ and $w_{i,3}$. Therefore, $h(w_{i,1}) = h(w_{i,2}) = h(w_{i,3}) = 1$. Note that $N_H(w_{i,1}) = \{w_{i,2}, u_i, v_i\}$. Since $N_H(w_{i,1}) \ge 1$ and $h(w_{i,2}) = 1$, the function h assigns the value -1 to at most one vertex of u_i and v_i. In other words, h assigns the value 1 to at least one vertex of u_i and v_i. Hence, the set $\{v \in V \mid h(v) = 1\}$ is a vertex cover of G. Let $K' = |\{v \in V \mid h(v) = 1\}|$. The weight of h

is $\gamma_t^s(H) = 3m + K' + (n - K') = 3m - n + 2K'$. We have $\tau(G) \le K' = \frac{\gamma_t^s(H)-(3m-n)}{2}$. Following the discussion above, $\gamma_t^s(H) = 3m-n+2\tau(G)$. □

The above claim implies that for a positive integer K, $\tau(G) \le K$ if and only if $\gamma_t^s(H) \le 3m - n + 2K$. □

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