On the hypercompetition numbers of hypergraphs

BORAM PARK *

Department of Mathematics Education
Seoul National University, Seoul 151-742, Korea
kawa22@snu.ac.kr

YOSHIO SANO †

Pohang Mathematics Institute
POSTECH, Pohang 790-784, Korea
ysano@postech.ac.kr

Abstract

The competition hypergraph $C\mathcal{H}(D)$ of a digraph D is the hypergraph such that the vertex set is the same as D and $e \subseteq V(D)$ is a hyperedge if and only if e contains at least 2 vertices and e coincides with the in-neighborhood of some vertex v in the digraph D. Any hypergraph with sufficiently many isolated vertices is the competition hypergraph of an acyclic digraph. The hypercompetition number $hk(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is defined to be the smallest number of such isolated vertices.

In this paper, we study the hypercompetition numbers of hypergraphs. First, we give two lower bounds for the hypercompetition numbers which hold for any hypergraphs. And then, by using these results, we give the exact hypercompetition numbers for some family of uniform hypergraphs. In particular, we give the exact value of the hypercompetition number of a connected graph.

Keywords: competition graph; competition number; competition hypergraph; hypercompetition number

* The author was supported by Seoul Fellowship.

† This work was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (MEST) (No. 2010-0029638).

1 Introduction

All hypergraphs considered in this paper may have isolated vertices but have no loops, where a vertex v in a hypergraph is called *isolated* if v is not contained in any hyperedge in the hypergraph, and a hyperedge e in a hypergraph is called a *loop* if e consists of exactly one vertex. So all the hyperedges of hypergraphs, in this paper, have at least two vertices. If (x, y) is an arc of a digraph D, then x is called an *in-neighbor* of y in D and y is called an *out-neighbor* of x in D. The *in-neighborhood* $N_D^-(v)$ of a vertex v in a digraph D is the set of in-neighbors of v in D.

The notion of a competition graph was introduced by Cohen [2] in 1968 and has arisen from ecology. The *competition graph* $C(D)$ of a digraph D is the graph which has the same vertex set as D and has an edge between vertices u and v if and only if there exists a common out-neighbor of u and v in D. Any graph together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Roberts [6] defined the *competition number* $k(G)$ of a graph G to be the minimum number k such that G together with k isolated vertices is the competition graph of an acyclic digraph. Since Cohen introduced the notion of a competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [3] and Lundgren [4]).

The notion of a competition hypergraph was introduced by Sonntag and Teichert [7] as a variant of a competition graph. The *competition hypergraph* $C\mathcal{H}(D)$ of a digraph D is the hypergraph such that the vertex set is the same as D and $e \subseteq V(D)$ is a hyperedge if and only if e contains at least 2 vertices and e coincides with the in-neighborhood of some vertex v in the digraph D (see [7, 8, 9, 10] for studies on competition hypergraphs of digraphs). Any hypergraph with sufficiently many isolated vertices is the competition hypergraph of an acyclic digraph. The *hypercompetition number* $hk(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is defined to be the smallest number of such isolated vertices. Though Sonntag and Teichert called it just the *competition number* of $\mathcal{H}$ and denoted it by $k(\mathcal{H})$, we use the terminology "hypercompetition number" and the notation $hk(\mathcal{H})$ to avoid confusion in the case where we regard graphs as hypergraphs. A hypergraph $\mathcal{H}$ is called a *graph* if $|e| = 2$ for any hyperedge $e \in E(\mathcal{H})$. The following example shows the difference between the (ordinary) competition number of a graph and the hypercompetition number of a graph.

Example. Let $\mathcal{G}$ be a triangle, i.e.,

$$V(\mathcal{G}) = \{v_1, v_2, v_3\}, \quad E(\mathcal{G}) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}.$$

Since the digraph D defined by $V(D) = V(\mathcal{G}) \cup \{z\}$ and $A(D) = \{(v_1, z), (v_2, z), (v_3, z)\}$ is acyclic and its competition graph $C(D)$ is the graph $\mathcal{G}$ with one isolated vertex z, the competition number $k(\mathcal{G})$ of $\mathcal{G}$ is at most one. But the hypercompetition number $hk(\mathcal{G})$ of $\mathcal{G}$ is equal to 2, which follows from Theorem 2.6.

Opsut [5] showed that the computation of the competition number of an arbitrary graph is an NP-hard problem. On the other hand, we can show that the hypercompetition number of a connected graph is computed easily.

In this paper, we study the hypercompetition numbers of hypergraphs. First we give two lower bounds for the hypercompetition numbers which hold for any hypergraph, and then we give several formulas for the hypercompetition numbers for some families of uniform hypergraphs.

2 Main results

We introduce notation and terminologies used in this section. For a (hyper)graph $\mathcal{H}$ and a finite set I, we denote by $\mathcal{H} \cup I$ the (hyper)graph such that $V(\mathcal{H} \cup I) = V(\mathcal{H}) \cup I$ and $E(\mathcal{H} \cup I) = E(\mathcal{H})$. The *degree* $\deg_{\mathcal{H}}(v)$ of a vertex v in a hypergraph $\mathcal{H}$ is defined to be the number of hyperedges containing the vertex v. We say two vertices u and v are *adjacent* in $\mathcal{H}$ if there is a hyperedge e in $\mathcal{H}$ such that $\{u, v\} \subset e$.

A hypergraph $\mathcal{H}$ is called *r-uniform* if each hyperedge of the hypergraph $\mathcal{H}$ has the same size r, where $2 \leq r \leq |V(\mathcal{H})|$. Note that 2-uniform hypergraphs are graphs.

A sequence $v_0 v_1 \cdots v_k$ of distinct vertices of a hypergraph $\mathcal{H}$ is called a *path* if there exist k distinct hyperedges $e_1, e_2, \ldots, e_k$ such that e_i contains $\{v_{i-1}, v_i\}$ for each $1 \leq i \leq k$. A sequence $v_0 v_1 \cdots v_k$ of vertices of a hypergraph $\mathcal{H}$ with $v_0 = v_k$ is called a *cycle* if there exist k distinct hyperedges $e_1, e_2, \ldots, e_k$ such that e_i contains $\{v_{i-1}, v_i\}$ for each $1 \leq i \leq k$. We say that $\mathcal{H}$ is *connected* if there exists a path between any two vertices of $\mathcal{H}$. A *connected component* of $\mathcal{H}$ is a maximal connected subhypergraph of $\mathcal{H}$.

For a digraph D, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D. For simplicity, we denote a set $\{(x, v) \mid x \in S\}$ by $S \to v$.

2.1 Two lower bounds for the hypercompetition number of a hypergraph

In this section, we give two lower bounds for the hypercompetition numbers of hypergraphs. Opsut [5] showed the following two lower bounds for competition numbers:

(LB1) For any graph G, $k(G) \geq \theta_e(G) - |V(G)| + 2$;

(LB2) For any graph G, $k(G) \geq \min_{v \in V(G)} \theta(N_G(v))$,

where $\theta_e(G)$ is the smallest number of cliques in G that cover the edges of G, $\theta(H)$ is the smallest number of cliques in a graph H that cover the vertices of H and $N_G(v) := \{u \in V(G) \mid uv \in E(G)\}$ is the open neighborhood of a vertex v in the graph G.

Our first lower bound for hypercompetition numbers, which corresponds to (LB1), is as follows:

Theorem 2.1. *Let $\mathcal{H}$ be a hypergraph. Then*

$$hk(\mathcal{H}) \geq |E(\mathcal{H})| - |V(\mathcal{H})| + \min_{e \in E(\mathcal{H})} |e|.$$

Proof. Let n and k be the number of vertices in a hypergraph $\mathcal{H}$ and the hypercompetition number $hk(\mathcal{H})$ of the hypergraph $\mathcal{H}$, respectively. Then there exists an acyclic digraph D such that $C\mathcal{H}(D) = \mathcal{H} \cup \{z_1, \ldots, z_k\}$. Furthermore, D can be chosen such that $v_1, v_2, \ldots, v_n, z_1, \ldots, z_k$ is an acyclic ordering of D. Let l be the smallest index such that $\{v_1, v_2, \ldots, v_l\}$ contains a hyperedge of $\mathcal{H}$. If there is a vertex v_j in the set $\{v_1, v_2, \ldots, v_l\}$ such that v_j has at least two in-neighbors in D, then $N_D^-(v_j)$ is a hyperedge of $\mathcal{H}$ and so $\{v_1, \ldots, v_{j-1}\}$ contains a hyperedge of $\mathcal{H}$, which contradicts the choice of l. Therefore, $|N_D^-(v)| \leq 1$ for any $v \in \{v_1, v_2, \ldots, v_l\}$. Then all in-neighborhoods of size at least 2 are in-neighborhoods of vertices in $V(D) \setminus \{v_1, v_2, \ldots, v_l\}$. So $n + k - l \geq |E(\mathcal{H})|$, i.e., $k \geq |E(\mathcal{H})| - n + l$. Since $\min_{e \in E(\mathcal{H})} |e| \leq l$, we have $|E(\mathcal{H})| - n + \min_{e \in E(\mathcal{H})} |e| \leq |E(\mathcal{H})| - n + l \leq k$. □

We present our second lower bound for hypercompetition numbers, which corresponds to (LB2).

Theorem 2.2. *Let $\mathcal{H}$ be a hypergraph. Then*

$$hk(\mathcal{H}) \geq \min_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v).$$

Proof. Let n and k be the number of vertices in a hypergraph $\mathcal{H}$ and the hypercompetition number $hk(\mathcal{H})$ of the hypergraph $\mathcal{H}$, respectively. Let $m := \min_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v)$. Then there exists an acyclic digraph D such that $C\mathcal{H}(D) = \mathcal{H} \cup \{z_1, \ldots, z_k\}$, and so there is an acyclic ordering $v_1, v_2, \ldots, v_n, z_1, \ldots, z_k$ of D. Since v_n is contained in at least m hyperedges, v_n has at least m out-neighbors in D. Thus $m \leq k$. □

For a hypergraph $\mathcal{H}$ with no isolated vertices, $\min_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v) \geq 1$ and so the following corollary holds (this is also justified by the fact that any acyclic digraph has a vertex which has no out-neighbors).

Corollary 2.3. *For a hypergraph $\mathcal{H}$ with no isolated vertex, $hk(\mathcal{H}) \geq 1$.*

2.2 The hypercompetition numbers of uniform hypergraphs

In this subsection, we give the exact values of the hypercompetition numbers of several kinds of uniform hypergraphs by using results in Subsection 2.1. Roberts [6] showed the following results for the (ordinary) competition numbers of graphs:

(R1) For a triangle free connected graph G, $k(G) = |E(G)| - |V(G)| + 2$;

(R2) For a chordal graph G, $k(G) \leq 1$, and the equality holds if and only if G has no isolated vertex.

The first result (R1) gives a graph family which satisfies the equality of (LB1) and the second result (R2) gives a graph family which satisfies the equality of (LB2). In this section, we found two hypergraph families that correspond to (R1) and (R2), respectively.

An ordering $v_1, v_2, \ldots, v_n$ of the vertices of an r-uniform hypergraph $\mathcal{H}$ is called an *elimination ordering* of $\mathcal{H}$ if, for each $r \leq i \leq n$, the vertex v_i has degree one in the subhypergraph of $\mathcal{H}$ induced by $\{v_1, \ldots, v_i\}$. Note that if an r-uniform hypergraph $\mathcal{H}$ has an elimination ordering then $|E(\mathcal{H})| = n - r + 1$.

Lemma 2.4. *Let n and r be positive integers with $r < n$ and $\mathcal{H}$ be a connected r-uniform hypergraph with n vertices which has an elimination ordering. Then $hk(\mathcal{H}) = 1$.*

Proof. Let n and t be the numbers of vertices and hyperedges in a hypergraph $\mathcal{H}$, respectively. Let $v_1, v_2, \ldots, v_n$ be an elimination ordering of $\mathcal{H}$. For each $r \leq i \leq n$, let e_i be the unique hyperedge containing v_i in the subhypergraph of $\mathcal{H}$ induced by $\{v_1, \ldots, v_i\}$. We define a digraph D by

$$\begin{aligned} V(D) &:= V(\mathcal{H}) \cup \{z\}, \\ A(D) &:= \left(\bigcup_{i=r}^{n-1} (e_i \to v_{i+1}) \right) \cup (e_n \to z). \end{aligned}$$

Then we can check that $C\mathcal{H}(D) = \mathcal{H} \cup \{z\}$ and that D is acyclic. Thus $hk(\mathcal{H}) \leq 1$. By Corollary 2.3, we have $hk(\mathcal{H}) \geq 1$. Hence the theorem holds. □

The following theorem gives a family of hypergraphs whose hypercompetition numbers satisfy the equality of Theorem 2.1, that corresponds to (R1).

Theorem 2.5. *Let n and r be positive integers such that $r < n$, and $\mathcal{H}$ be a connected r-uniform hypergraph with n vertices. Suppose that $\mathcal{H}$ has a spanning subhypergraph $\mathcal{H}_0$ which has an elimination ordering. Then*

$$hk(\mathcal{H}) = |E(\mathcal{H})| - |V(\mathcal{H})| + r.$$

Proof. Let n and t be the number of vertices and hyperedges in a hypergraph $\mathcal{H}$, respectively. By Lemma 2.4, there exists an acyclic digraph D_0 such that $C\mathcal{H}(D_0) = \mathcal{H}_0 \cup \{z_1\}$. If $E(\mathcal{H}) \setminus E(\mathcal{H}_0) = \emptyset$, then $|E(\mathcal{H})| = |E(\mathcal{H}_0)| = n-r+1$, i.e., $hk(\mathcal{H}) = 1$ and the theorem holds. Suppose that $E(\mathcal{H}) \setminus E(\mathcal{H}_0) \neq \emptyset$. Let $E(\mathcal{H}) \setminus E(\mathcal{H}_0) := \{e_1, e_2, \ldots, e_{t-n+r-1}\}$. We define a digraph D by

$$\begin{aligned} V(D) &:= V(\mathcal{H}) \cup \{z_1, z_2, \ldots, z_{t-n+r}\}, \\ A(D) &:= A(D_0) \cup \left(\bigcup_{i=1}^{t-n+r-1} (e_i \to z_{i+1}) \right). \end{aligned}$$

Then we can check that $C\mathcal{H}(D) = \mathcal{H} \cup \{z_1, z_2, \ldots, z_{t-n+r}\}$ and that D is acyclic. Thus $hk(\mathcal{H}) \leq t-n+r$. By Theorem 2.1, we have $hk(\mathcal{H}) \geq t-n+r$. Hence the theorem holds. □

Let us consider 2-uniform hypergraphs, i.e., graphs. Opsut [5] showed that the computation of the competition number of an arbitrary graph is an NP-hard problem. On the other hand, we can show that the hypercompetition number of a graph is computed easily from Theorem 2.5.

Corollary 2.6. *For a connected graph $\mathcal{G}$,*

$$hk(\mathcal{G}) = |E(\mathcal{G})| - |V(\mathcal{G})| + 2.$$

Proof. Let $|V(\mathcal{G})| = n$, and T be a spanning tree of $\mathcal{G}$. Let v_n be a pendent vertex of T, and then take a pendent vertex v_{n-1} of $T - v_n$. For each $2 \leq i \leq n-1$, we take a pendent vertex v_i of $T - \{v_{i+1}, \ldots, v_n\}$. Then the ordering $v_1, v_2, \ldots, v_n$ of the vertices of $\mathcal{G}$ is an elimination ordering of $\mathcal{H}$. By Theorem 2.5, $hk(\mathcal{G}) = |E(\mathcal{G})| - |V(\mathcal{G})| + 2$. □

A *complete r-uniform hypergraph* $\mathcal{K}(n,r)$ is the hypergraph with $|V(\mathcal{H})| = n$ and $E(\mathcal{H}) = \binom{V(\mathcal{H})}{r}$, where $\binom{V(\mathcal{H})}{r}$ denotes the family of all r-subsets of $V(\mathcal{H})$. We can obtain the hypercompetition numbers of complete uniform hypergraphs as a corollary of Theorem 2.5.

Corollary 2.7. *For $2 \leq r \leq n$, it holds that*

$$hk(\mathcal{K}(n,r)) = \binom{n}{r} - n + r.$$

Proof. Let $2 \leq r \leq n$. By Theorem 2.1, we have $hk(\mathcal{K}(n,r)) \geq \binom{n}{r} - n + r$. If $r = n$, since $\mathcal{K}(n,n)$ is the only n-uniform hypergraph with n vertices, then it trivially holds that $hk(\mathcal{K}(n,n)) = 1 = \binom{n}{n} - n + n$. Suppose that $r < n$. Let $V(\mathcal{K}(n,r)) = \{v_1, v_2, \ldots, v_n\}$. Then the spanning subgraph $\mathcal{H}_0$ with hyperedge set $\{\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} \in E(\mathcal{K}(n,r)) \mid 1 \leq i \leq n-r+1\}$ has an elimination ordering $v_1, v_2, \ldots, v_n$. By Theorem 2.5, we have $hk(\mathcal{K}(n,r)) = \binom{n}{r} - n + r$. Hence, the corollary holds. □

Next, we present a family of hypergraphs whose hypercompetition numbers satisfy the equality of the inequality in Theorem 2.2. From (R2), it is well known that since a forest with no isolated vertex is a chordal graph, its (ordinary) competition number is exactly one. We generalize this result to the case for hypergraphs by showing that for an r-uniform hypergraph $\mathcal{H}$ with no isolated vertices and no cycles, $hk(\mathcal{H}) = 1$. We need the following lemma.

Lemma 2.8. *Let $\mathcal{H}$ be a hypergraph. If the number of vertices of degree one in $\mathcal{H}$ is at least $|E(\mathcal{H})| - 1$, then $hk(\mathcal{H}) \leq 1$ and the equality holds if and only if $\mathcal{H}$ has no isolated vertex.*

Proof. Let n and t be the number of vertices and hyperedges in a hypergraph $\mathcal{H}$, respectively. Let Q be the set of vertices of degree one in $\mathcal{H}$ and let $q := |Q|$. Label the hyperedges of $\mathcal{H}$ as $\{e_1, e_2, \ldots, e_t\}$ so that $\{e_1, \ldots, e_l\}$ is the set of distinct hyperedges containing a vertex of Q. Since each vertex in Q is contained in a unique hyperedge of $\mathcal{H}$, we have $l \leq q$. Label the vertices of $\mathcal{H}$ as $\{v_1, v_2, \ldots, v_n\}$, where $\{v_1, \ldots, v_q\}$ are the vertices of Q and $\{v_1, \ldots, v_l\}$ were chosen from Q such that $v_i \in e_i$ for $1 \leq i \leq l$. Now we define a digraph D by

$$V(D) := V(\mathcal{H}) \cup \{v_0\} \quad \text{and} \quad A(D) := \bigcup_{i=1}^{t} (e_i \to v_{i-1}).$$

By definition, $E(C\mathcal{H}(D)) = \{e_1, \ldots, e_t\}$, so $C\mathcal{H}(D) = \mathcal{H} \cup \{v_0\}$. It remains to show that D is acyclic. We prove it by showing that $v_n, \ldots, v_1, v_0$ is an acyclic ordering of D. Consider an arc (x, v_{i-1}) where $x \in e_i$. If $1 \leq i \leq l$ then $e_i \subseteq \{v_i, v_{l+1}, \ldots, v_n\}$, and so $x = v_j$ with $j > i - 1$. On the other hand, if $i > l$, then $e_i \subseteq \{v_{q+1}, \ldots, v_n\}$, and again $x = v_j$ with $j > i - 1$. So all the arcs in D are of the form (v_j, v_{i-1}) with $j > i - 1$, and this shows that $v_n, \ldots, v_1, v_0$ is an acyclic ordering of D and therefore D is acyclic.

For determining when $hk(\mathcal{H}) = 1$, suppose that $\mathcal{H}$ has an isolated vertex. Then the hypergraph $\mathcal{H}_0$ obtained from $\mathcal{H}$ by deleting the set I of isolated vertices of $\mathcal{H}$ also has q vertices of degree one and t hyperedges. The above argument shows that $hk(\mathcal{H}_0) \leq 1$. Thus $hk(\mathcal{H}) = hk(\mathcal{H}_0 \cup I) = 0$. On the other hand, if $\mathcal{H}$ has no isolated vertices, then $hk(\mathcal{H}) = 1$ by the above argument and Corollary 2.3. This proves the lemma. □

The following lemma is well-known.

Lemma 2.9 ([1, p.392]). *Let $\mathcal{H}$ be a hypergraph and p be the number of connected components of $\mathcal{H}$. Then $\mathcal{H}$ has no cycle if and only if*

$$\sum_{e \in E(\mathcal{H})} (|e| - 1) = |V(\mathcal{H})| - p.$$

Now we will show that for an r-uniform hypergraph $\mathcal{H}$ with no isolated vertices and no cycles, $hk(\mathcal{H}) = 1$.

Theorem 2.10. *Let r be a positive integer with $r \geq 3$, and $\mathcal{H}$ be an r-uniform hypergraph with no isolated vertex. If $\mathcal{H}$ has no cycle, then $hk(\mathcal{H}) = 1$.*

Proof. We prove by induction on the number of connected components of $\mathcal{H}$. Suppose that $\mathcal{H}$ is a connected hypergraph. Let n and t be the numbers of vertices and hyperedges in $\mathcal{H}$, respectively. Since $|e| = r$ for any $e \in E(\mathcal{H})$ and $\mathcal{H}$ has no cycle, we obtain by Lemma 2.9 that

$$n = (r-1)t + 1. \tag{1}$$

Also we have

$$\sum_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v) = \sum_{e \in E(\mathcal{H})} |e| = rt. \tag{2}$$

Since $\mathcal{H}$ is connected, $\deg_{\mathcal{H}}(v) \geq 1$ for any $v \in V(\mathcal{H})$. Let q be the number of vertices of degree one in $\mathcal{H}$. Then,

$$\sum_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v) \geq 2(n-q) + q = 2n - q. \tag{3}$$

By (1), (2), and (3), we have

$$q \geq 2n - rt = 2(rt - t + 1) - rt = (r-2)t + 2 \geq t + 2.$$

By Lemma 2.8, it holds that $hk(\mathcal{H}) = 1$.

Suppose that the statement holds for hypergraphs with p connected components where $p \geq 1$. Now suppose that $\mathcal{H}$ has $p+1$ connected components. Take a connected component $\mathcal{H}_1$ of $\mathcal{H}$. Let $\mathcal{H}_2$ be the union of the connected components of $\mathcal{H}$ other than $\mathcal{H}_1$. Then $\mathcal{H}_2$ has p components and it has no isolated vertex and no cycle. By induction hypothesis, we have $hk(\mathcal{H}_2) = 1$. Then, there exists an acyclic digraph D_1 (resp. D_2) such that $C\mathcal{H}(D_1) = \mathcal{H}_1 \cup \{z_1\}$ (resp. $C\mathcal{H}(D_2) = \mathcal{H}_2 \cup \{z_2\}$), where z_1 (resp. z_2) is a new isolated vertex. Without loss of generality, we may assume that $N^+_{D_2}(z_2) = \emptyset$. Since D_1 is acyclic, there exists a vertex v in D_1 which has no in-neighbor in D_1. We define a digraph D by

$$\begin{aligned} V(D) &:= V(\mathcal{H}) \cup \{z_1\}, \\ A(D) &:= A(D_1) \cup \big(A(D_2) \setminus (N^-_{D_2}(z_2) \to z_2)\big) \cup (N^-_{D_2}(z_2) \to v). \end{aligned}$$

Then D is acyclic and $C\mathcal{H}(D) = \mathcal{H} \cup \{z_1\}$ and so $hk(\mathcal{H}) \leq 1$. Since $\mathcal{H}$ has no isolated vertex, we have $hk(\mathcal{H}) \geq 1$ by Corollary 2.3. Hence $hk(\mathcal{H}) = 1$. □

References

[1] C. Berge: *Graphs and hypergraphs*, Translated from the French by Edward Minieka, North-Holland Mathematical Library, Vol. 6, (North-Holland Publishing Co., Amsterdam, 1973).

[2] J. E. Cohen: Interval graphs and food webs: a finding and a problem, *Document 17696-PR*, RAND Corporation, Santa Monica, CA (1968).

[3] S. -R. Kim: The competition number and its variants, in J. Gimbel, J.W. Kennedy, and L.V. Quintas (eds.), *Quo Vadis Graph Theory?*, *Annals of Discrete Mathematics*, Vol. **55** (1993) 313–325.

[4] J. R. Lundgren: Food webs, competition graphs, competition-common enemy graphs, and niche graphs, in Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, *IMH Volumes in Mathematics and Its Application* **17** Springer-Verlag, New York, (1989) 221–243.

[5] R. J. Opsut: On the computation of the competition number of a graph, *SIAM Journal on Algebraic and Discrete Methods* **3** (1982) 420–428.

[6] F. S. Roberts: Food webs, competition graphs, and the boxicity of ecological phase space, *Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976)* (1978) 477–490.

[7] M. Sonntag and H. -M. Teichert: Competition hypergraphs, *Discrete Appled Mathematics* **143** (2004) 324–329.

[8] M. Sonntag and H. -M. Teichert: Competition hypergraphs of digraphs with certain properties. I. Strong connectedness, *Discussiones Mathematicae Graph Theory* **28** (2008) 5–21.

[9] M. Sonntag and H. -M. Teichert: Competition hypergraphs of digraphs with certain properties. II. Hamiltonicity, *Discussiones Mathematicae Graph Theory* **28** (2008) 23–34.

[10] M. Sonntag and H. -M. Teichert: Competition hypergraphs of products of digraphs, *Graphs and Combinatorics* **25** (2009) 611–624.