# COMPOSITION OPERATORS ON THE FOCK SPACE OF VECTOR-VALUED ANALYTIC FUNCTIONS

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ABSTRACT. In this note, we study some properties of the composition operator $C_\varphi$ on the Fock space $\mathcal{F}_X^2$ of $X$-valued analytic functions in $\mathbb{C}$. We give a necessary and sufficient condition for a bounded operator on $\mathcal{F}_X^2$ to be a composition operator and for the adjoint operator of a composition operator to be also a composition operator on $\mathcal{F}_X^2$. We also give characterizations of normal, unitary and co-isometric composition operators on $\mathcal{F}_X^2$.

## 1. INTRODUCTION

Throughout this paper, let $X$ be a separable Hilbert space with an inner product $\langle\cdot,\cdot\rangle_X$ and $H_X(\mathbb{C})$ denote the space of all $X$-valued analytic functions defined in $\mathbb{C}$. For proerties of $X$-valued analytic functions, see [4]. The *$X$-valued Fock space* $\mathcal{F}_X^2$ over $\mathbb{C}$ is defined as follows:

$$\mathcal{F}_X^2 := \left\{ f \in H_X(\mathbb{C}) \,:\, \|f\|^2 := \frac{1}{2\pi}\int_{\mathbb{C}} \|f(z)\|_X^2 e^{-\frac{1}{2}|z|^2} dm(z) < \infty \right\},$$

where $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle_X}$ and $dm$ denotes the two-dimensional Lebesgue measure. $\mathcal{F}_X^2$ is a Hilbert space with inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{C}} \langle f(z), g(z)\rangle_X e^{-\frac{1}{2}|z|^2} dm(z).$$

Recently Carswell, MacCluer and Schuster [1] or Guo and Izuchi [2] studied the composition operator on the scalar-valued Fock space. Stević [6] and the author [7, 8, 9, 10] have considered the weighted composition operator on the Fock spaces and related spaces. In studies on these type of operators, the main subject is to describe operator theoretic properties of them in terms of function theoretic properties of their including functions. Some authors have studied composition operators on vector-valued analytic function spaces defined on the unit disk in $\mathbb{C}$ ([3, 5]). In [5], Sharma and Bhanu gave a necessary and sufficient condition for a bounded operator on

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the vector-valued Hardy space to be a composition operator. They also characterized the holomorphic self map of the unit disk that the adjoint operator of a composition operator is also a composition operator. The purpose of this note is to consider the problems of Sharma and Bhanu about a composition operator on $\mathcal{F}_X^2$.

## 2. PRELIMINARIES

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\{e_n : n \in \mathbb{Z}_+\}$ be an orthonormal basis for $X$. For $m, n \in \mathbb{Z}_+$, we define the vector-valued functions $e_{m,n} : \mathbb{C} \to X$ as follows

$$e_{m,n}(z) = \frac{z^m}{\sqrt{2^m\Gamma(m+1)}} e_n \qquad (z \in \mathbb{C}).$$

**Proposition 1.** *$\{e_{m,n} : m, n \in \mathbb{Z}_+\}$ is an orthonormal basis for $\mathcal{F}_X^2$.*

*Proof.* Since $\{e_n : n \in \mathbb{Z}_+\}$ is an orthonormal basis for $X$, we see that

$$\langle e_{m,n}, e_{k,l}\rangle = \begin{cases} \frac{1}{\sqrt{2^{m+k}\Gamma(m+1)\Gamma(k+1)}} \frac{1}{2\pi} \int_{\mathbb{C}} z^m \bar{z}^k e^{-\frac{1}{2}|z|^2} dm(z) & \text{if } n = l, \\ 0 & \text{if } n \neq l, \end{cases} \tag{1}$$

for $m, n, k, l \in \mathbb{Z}_+$. When $n = l$, a polar coordinate shows that

$$\frac{1}{\sqrt{2^{m+k}\Gamma(m+1)\Gamma(k+1)}} \frac{1}{2\pi} \int_{\mathbb{C}} z^m \bar{z}^k e^{-\frac{1}{2}|z|^2} dm(z) = \begin{cases} \frac{1}{2^m\Gamma(m+1)} \int_0^\infty r^{2m+1} e^{-\frac{1}{2}r^2} dr = 1 & \text{if } m = k, \\ 0 & \text{if } m \neq k. \end{cases} \tag{2}$$

By (1) and (2), we see that $\{e_{m,n} : m, n \in \mathbb{Z}_+\}$ is an orthonormal subset of $\mathcal{F}_X^2$. In order to prove that $\{e_{m,n} : m, n \in \mathbb{Z}_+\}$ is a basis for $\mathcal{F}_X^2$, we suppose that $f \in \mathcal{F}_X^2$ and $\langle f, e_{n,m}\rangle = 0$. Thus,

$$\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\bar{z}^m}{\sqrt{2^m\Gamma(m+1)}} \langle f(z), e_n\rangle_X e^{-\frac{1}{2}|z|^2} dm(z) = 0. \tag{3}$$

Since $\{z^m/\sqrt{2^m\Gamma(m+1)} : m \in \mathbb{Z}_+\}$ is an orthonormal basis for the (scalar-valued) Fock space $\mathcal{F}^2$ and $\langle f(\cdot), e_n\rangle_X$ is in $\mathcal{F}^2$, we have $\langle f(\cdot), e_n\rangle_X = 0$ in $\mathbb{C}$. Moreover $\{e_n : n \in \mathbb{Z}_+\}$ is an orthonormal basis for $X$, and so $f \equiv 0$. That is, $\{e_{m,n} : m, n \in \mathbb{Z}_+\}$ is a basis for $\mathcal{F}_X^2$. □

Fix $z \in \mathbb{C}$. Let $I_z(f)(w) = e^{-\frac{|z|^2}{4} + \frac{\langle w, z\rangle}{2}} f(w - z)$ for all $f \in \mathcal{F}_X^2$ and $w \in \mathbb{C}$. By using a change of variables, we see that $I_z$ is a unitary operator on $\mathcal{F}_X^2$ and $I_z{}^{-1} = I_{-z}$. These proerties and the subharmonicity of $\|f(\cdot)\|_X^2$ show the inequality $\|f(z)\|_X \leq e^{\frac{|z|^2}{4}} \|f\|$ (see e.g.[7]). This implies that the point evaluation is bounded on $\mathcal{F}^2_X$. For each $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$, we define $E^j_z(f) = \langle f(z), e_j\rangle_X$ for all $f \in \mathcal{F}^2_X$. The boundedness of the point evaluation shows that $E^j_z$ is a bounded linear functional on $\mathcal{F}^2_X$. By Riesz's representation theorem, there exists $K^j_z \in \mathcal{F}^2_X$ such that $E^j_z(f) = \langle f, K^j_z\rangle$ for all $f \in \mathcal{F}^2_X$. We designate $K^j_z$ as the reproducing kernel functions for $\mathcal{F}^2_X$. By Proposition 1, we can get the concrete formula for $K^j_z$.

**Proposition 2.** *For each $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$, $K^j_z$ is given by*

$$K^j_z(w) = e_j \cdot \exp\left(\frac{\overline{z}w}{2}\right) \quad (w \in \mathbb{C}) \quad \textit{and} \quad \|K^j_z\| = e^{\frac{|z|^2}{4}}.$$

## 3. Composition operators on $\mathcal{F}^2_X$

We begin this section by giving a necessary and sufficient condition for a bounded operator on $\mathcal{F}^2_X$ to be a composition operator.

**Theorem 1.** *Suppose that $T$ is a bounded operator on $\mathcal{F}^2_X$. Then $T$ is a composition operator if and only if for each $z \in \mathbb{C}$ there exists a unique $w \in \mathbb{C}$ such that $T^*K^j_z = K^j_w$ for all $j \in \mathbb{Z}_+$.*

*Proof.* If $T = C_\varphi$ for some holomorphic map $\varphi$ on $\mathbb{C}$, then for each $f \in \mathcal{F}^2_X$ and $j \in \mathbb{Z}_+$, we obtain that $\langle f, T^*K^j_z\rangle = \langle C_\varphi f, K^j_z\rangle = E^j_z(C_\varphi f) = \langle f(\varphi(z)), e_j\rangle_X = E^j_{\varphi(z)}(f) = \langle f, K^j_{\varphi(z)}\rangle$, and so $T^*K^j_z = K^j_{\varphi(z)}$.

Conversely, suppose that for each $z \in \mathbb{C}$ there exists a unique $w \in \mathbb{C}$ such that $T^*K^j_z = K^j_w$ for all $j \in \mathbb{Z}_+$. Then define the map $\varphi$ on $\mathbb{C}$ by $\varphi(z) = w$. Furthermore, $\frac{\varphi(z)}{\sqrt{2\Gamma(2)}} = \langle \frac{\varphi(z)}{\sqrt{2\Gamma(2)}} e_1, e_1\rangle_X = E^1_{\varphi(z)}(e_{1,1}) = \langle e_{1,1}, K^1_{\varphi(z)}\rangle = \langle e_{1,1}, T^*K^1_z\rangle = \langle Te_{1,1}, K^1_z\rangle = E^1_z(Te_{1,1}) = \langle Te_{1,1}(z), e_1\rangle_X$. This implies that $\varphi$ is a holomorphic map on $\mathbb{C}$. For each $f \in \mathcal{F}^2_X$, $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$, we obtain $\langle Tf(z), e_j\rangle_X = \langle Tf, K^j_z\rangle = \langle f, T^*K^j_z\rangle = \langle f, K^j_{\varphi(z)}\rangle = E^j_{\varphi(z)}(f) = \langle f(\varphi(z)), e_j\rangle_X$. Since $\{e_j : j \in \mathbb{Z}_+\}$ is an orthonomal basis for $X$, we see that $\langle Tf(z), x\rangle_X = \langle f(\varphi(z)), x\rangle_X$ for all $x \in X$, and so $T = C_\varphi$. □

From now on, till the end of this chapter, we suppose that $\varphi$ is a holomorphic map on $\mathbb{C}$ with $C_\varphi$ is a operator from $\mathcal{F}^2_X$ to $\mathcal{F}^2_X$.

**Theorem 2.** *$C^*_\varphi$ is a composition operator if and only if $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$.*

*Proof.* First we assume that $C^*_\varphi = C_\psi$ for some holomorphic map $\psi$ on $\mathbb{C}$. Let $\varphi(z) = \sum_{n=0}^\infty a_n z^n$ and $\psi(z) = \sum_{n=0}^\infty b_n z^n$. Since

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{it})e^{-int}dt$$

for all $n \in \mathbb{Z}_+$ and $0 < r < \infty$, we have

$$a_n r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{it})(re^{-it})^n dt = \frac{1}{2\pi}\int_0^{2\pi} \langle \varphi(re^{it})e_1, (re^{it})^n e_1\rangle_X dt.$$

Combining a polar coordinate with this, we have

$$\frac{1}{2\pi}\int_{\mathbf{C}}\langle\varphi(z)e_1, z^n e_1\rangle_X e^{-\frac{1}{2}|z|^2}dm(z) = a_n\int_0^\infty r^{2n+1}e^{-\frac{1}{2}r^2}dr, \tag{4}$$

for all $n \in \mathbb{Z}_+$.

On the other hand, it follows from $\frac{\varphi(z)}{\sqrt{2\Gamma(2)}}e_1 = C_\varphi e_{1,1}(z)$, $\frac{z^n}{2^n\Gamma(n+1)}e_1 = e_{n,1}(z)$ and $C_\varphi^* = C_\psi$ that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbf{C}}\langle\varphi(z)e_1, z^n e_1\rangle_X e^{-\frac{1}{2}|z|^2}dm(z)\\
&= \frac{1}{2\pi}\int_{\mathbf{C}}\langle C_\varphi e_{1,1}(z), e_{n,1}(z)\rangle_X e^{-\frac{1}{2}|z|^2}dm(z)\\
&= \langle C_\varphi e_{1,1}, e_{n,1}\rangle = \langle e_{1,1}, C_\varphi^* e_{n,1}\rangle = \langle e_{1,1}, C_\psi e_{n,1}\rangle\\
&= \frac{1}{2\pi}\int_{\mathbf{C}}\langle e_{1,1}(z), C_\psi e_{n,1}(z)\rangle_X e^{-\frac{1}{2}|z|^2}dm(z)\\
&= \frac{1}{2\pi}\int_{\mathbf{C}}\langle ze_1, \psi(z)^n e_1\rangle_X e^{-\frac{1}{2}|z|^2}dm(z) = \frac{1}{2\pi}\int_{\mathbf{C}} z\overline{\psi(z)^n}e^{-\frac{1}{2}|z|^2}dm(z)\\
&= \int_0^\infty r^2 e^{-\frac{1}{2}r^2}dr\frac{1}{2\pi}\int_0^{2\pi} e^{it}\overline{\psi(re^{it})^n}dt.
\end{aligned} \tag{5}$$

By (4) and (5), we have

$$a_n\int_0^\infty r^{2n+1}e^{-\frac{1}{2}r^2}dr = \int_0^\infty r^2 e^{-\frac{1}{2}r^2}dr\frac{1}{2\pi}\int_0^{2\pi} e^{it}\overline{\psi(re^{it})^n}dt, \tag{6}$$

for all $n \in \mathbb{Z}_+$.

Now, for each $j \in \mathbb{Z}_+$ and $f \in \mathcal{F}_X^2$, we see that $\langle f, C_\psi K_0^j\rangle = \langle C_\varphi f, K_0^j\rangle = E_0^j(C_\varphi f) = \langle f(\varphi(0)), e_j\rangle_X = \langle f, K_{\varphi(0)}^j\rangle$, and so $C_\psi K_0^j = K_{\varphi(0)}^j$ on $\mathbb{C}$. By Proposition 2, we also see that $C_\psi K_0^j(z) = K_0^j(\psi(z)) = e_j$. Thus we obtain $\|K_{\varphi(0)}^j\| = 1$. Since $\|K_{\varphi(0)}^j\| = \exp(\frac{|\varphi(0)|^2}{4})$ by Proposition 2, we have $\exp(\frac{|\varphi(0)|^2}{4}) = 1$, and so $a_0 = 0$. By the same argument, we also obtain $b_0 = 0$. Thus, for each $n \in \mathbb{Z}_+ \setminus \{0\}$, the first $n$ Taylor coefficients of $\varphi^n$ and $\psi^n$ are zero. If $\psi^n(z) = \sum_{k=n}^\infty B_k z^k$, then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-it}\psi(re^{it})^n dt = \begin{cases} B_1 r & \text{if } n = 1,\\ 0 & \text{if } n \geq 2,\end{cases}$$

for all $0 < r < \infty$ and $n \in \mathbb{Z}_+ \setminus \{0\}$. Since $B_1 = b_1$, we have for each $0 < r < \infty$ and $n \in \mathbb{Z}_+ \setminus \{0\}$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{it}\overline{\psi(re^{it})^n}dt = \begin{cases} \overline{b_1}r & \text{if } n = 1,\\ 0 & \text{if } n \geq 2.\end{cases} \tag{7}$$

By (6) and (7) we see that

$$a_n \int_0^\infty r^{2n+1} e^{-\frac{1}{2}r^2} dr = \begin{cases} \overline{b_1} \int_0^\infty r^3 e^{-\frac{1}{2}r^2} dr & \text{if } n = 1, \\ 0 & \text{if } n \geq 2, \end{cases}$$

for all $0 < r < \infty$ and $n \in \mathbb{Z}_+ \setminus \{0\}$. Since $\int_0^\infty r^3 e^{-\frac{1}{2}r^2} dr \neq 0$, we see that $a_1 = \overline{b_1} \in \mathbb{C}$ and $a_n = 0$ for all $n \geq 2$. That is, $\varphi(z) = a_1 z$.

Conversely, we assume that $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. Put $\psi(z) = \overline{\alpha} z$. It is clear that $\psi$ is a holomorphic map on $\mathbb{C}$. Take $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$. By Proposition 2, we see that $(C_\varphi^*)^* K_z^j(w) = K_z^j(\alpha w) = e_j \cdot \exp(\frac{\overline{z} \cdot \alpha w}{2}) = K_{\overline{\alpha} z}^j(w) = K_{\psi(z)}^j(w)$, for all $w \in \mathbb{C}$. Hence, it follows from Theorem 1 that $C_\varphi^*$ is a composition operator and $C_\varphi^* = C_\psi$. □

**Theorem 3.** *$C_\varphi$ is a normal operator if and only if $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$.*

*Proof.* Suppose that $C_\varphi$ is normal, we have that $\|C_\varphi^* K_0^j\| = \|C_\varphi K_0^j\|$ for all $j \in \mathbb{Z}_+$. Since $C_\varphi K_0^j(z) = K_0^j(\varphi(z)) = e_j$ and $\|K_{\varphi(0)}^j\| = \exp(\frac{|\varphi(0)|^2}{4})$ by Proposition 2, we have $\exp(\frac{|\varphi(0)|^2}{4}) = \|K_{\varphi(0)}^j\| = \|C_\varphi^* K_0^j\| = \|C_\varphi K_0^j\| = 1$. That is, $a_0 = 0$ where $\varphi(z) = \sum_{n=0}^\infty a_n z^n$. As in the proof of (7), we see that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{it} \overline{\varphi(re^{it})^m} dt = \begin{cases} \overline{a_1} r & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

Thus we have

$$\frac{1}{2\pi} \int_{\mathbb{C}} z \cdot \overline{\varphi(z)^m} e^{-\frac{1}{2}|z|^2} dm(z) = \begin{cases} 0 & \text{if } m \neq 1, \\ 2\Gamma(2)\overline{a_1} & \text{if } m = 1. \end{cases} \tag{8}$$

Parseval's identity and (8) show that

$$\|C_\varphi^* e_{1,1}\|^2 = \sum_{(m,n) \in \mathbb{Z}_+^2} |\langle C_\varphi^* e_{1,1}, e_{m,n} \rangle|^2 = \sum_{(m,n) \in \mathbb{Z}_+^2} |\langle e_{1,1}, C_\varphi e_{m,n} \rangle|^2$$

$$= \sum_{(m,n) \in \mathbb{Z}_+^2} \left| \frac{1}{2\pi} \int_{\mathbb{C}} \left\langle \frac{1}{\sqrt{2\Gamma(2)}} z e_1, \frac{1}{\sqrt{2^m \Gamma(m+1)}} \varphi(z)^m e_n \right\rangle_X e^{-\frac{1}{2}|z|^2} dm(z) \right|^2$$

$$= \sum_{m \in \mathbb{Z}_+} \left| \frac{1}{\sqrt{2^{m+1}\Gamma(2)\Gamma(m+1)}} \frac{1}{2\pi} \int_{\mathbb{C}} z \cdot \overline{\varphi(z)^m} e^{-\frac{1}{2}|z|^2} dm(z) \right|^2 = |a_1|^2. \tag{9}$$

On the other hand, Fubini's theorem gives

$$\|C_\varphi^* e_{1,1}\|^2 = \|C_\varphi e_{1,1}\|^2$$

$$= \frac{1}{2\pi} \int_{\mathbb{C}} \|e_{1,1}(\varphi(z))\|_X^2 e^{-\frac{1}{2}|z|^2} dm(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{|\varphi(z)|^2}{2\Gamma(2)} e^{-\frac{1}{2}|z|^2} dm(z)$$

$$= \frac{1}{2\Gamma(2)} \int_0^\infty re^{-\frac{1}{2}r^2} dr \frac{1}{2\pi} \int_0^{2\pi} \left\{ \sum_{j=0}^\infty a_j r^j e^{ijt} \right\} \overline{\left\{ \sum_{k=0}^\infty a_k r^k e^{ikt} \right\}} dt$$

$$= \frac{1}{2\Gamma(2)} \sum_{j=0}^\infty |a_j|^2 2^j \Gamma(2) = |a_1|^2 + \sum_{j\geq 2} 2^{j-1} |a_j|^2. \tag{10}$$

Equations (9) and (10) imply that $a_n = 0$ for all $n \geq 2$. Hence $\varphi(z) = a_1 z$.

Conversely, we suppose that $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. By Theorem 2, we obtain $C_\varphi^* = C_\psi$ where $\psi(z) = \overline{\alpha} z$. Hence we have $C_\varphi C_\varphi^* f(z) = f(\psi \circ \varphi(z)) = f(|\alpha|^2 z) = f(\varphi \circ \psi(z)) = C_\varphi^* C_\varphi f(z)$ for each $f \in \mathcal{F}_X^2$ and $z \in \mathbb{C}$. This implies that $C_\varphi$ is a normal operator on $\mathcal{F}_X^2$. □

**Corollary 1.** *$C_\varphi$ is self-adjoint if and only if $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{R}$.*

*Proof.* If $C_\varphi$ is self-adjoint, then $C_\varphi$ is normal. Hence, by Theorem 3, we have $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. Moreover, it follows from Theorem 2 that $C_\varphi^* = C_\psi$ where $\psi(z) = \overline{\alpha} z$. That is, $C_\varphi = C_\psi$ on $\mathcal{F}_X^2$. Since $e_{1,1} \in \mathcal{F}_X^2$, we see that $\frac{\varphi(z)}{\sqrt{2\Gamma(2)}} e_1 = C_\varphi e_{1,1}(z) = C_\psi e_{1,1}(z) = \frac{\psi(z)}{\sqrt{2\Gamma(2)}} e_1$, and so $\varphi = \psi$ in $\mathbb{C}$. This implies that $\alpha = \overline{\alpha}$, that is, $\alpha$ is real.

Conversely, we assume that $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{R}$. By Theorem 2 we have $C_\varphi^* = C_\psi$ where $\psi(z) = \overline{\alpha} z$. Since $\alpha \in \mathbb{R}$, we see that $\varphi(z) = \alpha z = \overline{\alpha} z = \psi(z)$ and so $C_\varphi = C_\psi$. Thus $C_\varphi^* = C_\varphi$, that is, $C_\varphi$ is self-adjoint. □

**Theorem 4.** *$C_\varphi$ is unitary if and only if $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.*

*Proof.* If $C_\varphi$ is unitary, then it is normal. Hence Theorem 3 and Theorem 2 show that there exists $\alpha \in \mathbb{C}$ such that $\varphi(z) = \alpha z$ and $C_\varphi^* = C_\psi$ where $\psi(z) = \overline{\alpha} z$. Take $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$. Since $C_\varphi C_\varphi^* = id_{\mathcal{F}_X^2}$ (the identity operator on $\mathcal{F}_X^2$), we obtain $K_z^j(|\alpha|^2 w) = K_z^j(\psi \circ \varphi(w)) = C_\varphi C_\psi K_z^j(w) = C_\varphi C_\varphi^* K_z^j(w) = K_z^j(w)$ for each $w \in \mathbb{C}$. By Proposition 2, we see that $e_j \cdot \exp(\frac{\overline{z}|\alpha|^2 w}{2}) = e_j \cdot \exp(\frac{\overline{z} w}{2})$, and $|\alpha| = 1$.

Conversely, we suppose that $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Theorem 3 and Theorem 2 imply that $C_\varphi$ is normal and $C_\varphi^* = C_\psi$ where $\psi(z) = \overline{\alpha} z$. Therefore, for each $f \in \mathcal{F}_X^2$ and $z \in \mathbb{C}$, $C_\varphi C_\varphi^* f(z) = f(\psi \circ \varphi)(z) = f(|\alpha|^2 z) = f(z)$, that is, $C_\varphi C_\varphi^* = id_{\mathcal{F}_X^2}$. The normality of $C_\varphi$ shows that $C_\varphi$ is unitary. □

A bounded operator $T$ on the Hilbert space $\mathcal{H}$ is said to be *co-isometry* if and only if the adjoint $T^*$ is isometry of $\mathcal{H}$. It is easily see that a unitary operator on $\mathcal{H}$ is co-isometry. The following theorem present the characterization of a co-isometric composition operator on $\mathcal{F}_X^2$.

**Theorem 5.** *$C_\varphi$ is co-isometry if and only if $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.*

*Proof.* First we assume that $C_\varphi$ is co-isometry. Since $C_\varphi^*$ is isometry, we have $\|C_\varphi^* f\| = \|f\|$ for all $f \in \mathcal{F}_X^2$. Proposition 2 shows that $\exp(\frac{|\varphi(z)|^2}{4}) = \|K_{\varphi(z)}^j\| = \|C_\varphi^* K_z^j\| = \|K_z^j\| = \exp(\frac{|z|^2}{4})$, and so $|\varphi(z)| = |z|$. Thus there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\varphi(z) = \alpha z$.

Conversely, we assume that $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. By Theorem 4, we see that $C_\varphi$ is a unitary operator on $\mathcal{F}_X^2$ and so $\|C_\varphi f\| = \|f\|$ for all $f \in \mathcal{F}_X^2$. Moreover, the unitarity of $C_\varphi$ implies that $C_\varphi$ is normal. So $\|C_\varphi f\| = \|C_\varphi^* f\|$ for all $f \in \mathcal{F}_X^2$. Hence $\|C_\varphi^* f\| = \|f\|$ for all $f \in \mathcal{F}_X^2$. This completes the proof. □

It follows from an elementary calculation that the entire function $\varphi(z) = \alpha z$ with $|\alpha| = 1$ induces a surjective composition operator on $\mathcal{F}_X^2$. Hence, the following result is an immediately consequence of Theorem 4 and Theorem 5.

**Corollary 2.** *The following conditions are equivalent :*

(a) $C_\varphi$ *is a unitary operator on* $\mathcal{F}_X^2$.

(b) $C_\varphi$ *is a co-isometry operator on* $\mathcal{F}_X^2$.

(c) $\varphi(z) = \alpha z$ *for some* $\alpha \in \mathbb{C}$ *with* $|\alpha| = 1$.

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