Edge-choosability of planar graphs without chordal 7-Cycles *

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Abstract

A graph G is edge-L-colorable, if for a given edge assignment $L = \{L(e) : e \in E(G)\}$, there exits a proper edge-coloring ϕ of G such that $\phi(e) \in L(e)$ for all $e \in E(G)$. If G is edge-L-colorable for every edge assignment L with $|L(e)| \geq k$ for $e \in E(G)$, then G is said to be edge-k-choosable. In this paper, we prove that if G is a planar graph without chordal 7-cycles, then G is edge-k-choosable, where $k = \max\{8, \Delta(G) + 1\}$.

Keywords: planar graph;edge-coloring; choosability; cycle; chord; combinatorial problem

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1 Introduction

Graphs considered in this paper are finite, simple and undirected. For a planar graph G, we denote its vertex set, edge set, face set, maximum degree, and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$, respectively.

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An *edge coloring* of a graph G is a mapping ϕ from $E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ for some positive integer k. An edge coloring is called *proper* if adjacent edges receive different colors. The *edge chromatic number* $\chi'(G)$ is the smallest integer k such that G has a proper edge-coloring into the set $\{1, 2, \ldots, k\}$. We say that L is an *edge assignment* for the graph G if it assigns a list $L(e)$ of possible colors to each edge e of G. If G has a proper edge-coloring ϕ such that $\phi(e) \in L(e)$ for each edge e of G, then we say that G is *edge-L-colorable* or ϕ is an *edge-L-coloring* of G. The graph G is *edge-k-choosable* if it is edge-L-colorable for every edge assignment L satisfying $|L(e)| \geq k$ for each edge $e \in E(G)$. The *edge choice number* $\chi'_l(G)$ of G is the smallest k such that G is edge-k-choosable.

The following conjecture was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris(see [7] and [12]), and this combinatorial problem is well known as the List Coloring Conjecture.

Conjecture 1.1. *If G is a multigraph, then $\chi'_l(G) = \chi'(G)$.*

The conjecture has been proved for a few special cases, such as bipartite multigraphs [5], complete graphs of odd order [6], multicircuits [21], graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic [2], and outerplanar graphs [20]. Vizing (see [14]) proposed a weaker conjecture as follows.

Conjecture 1.2. *Every graph G is edge-$(\Delta(G) + 1)$-choosable.*

Harris [8] shows that $\chi'_l(G) \leq 2\Delta(G) - 2$ if G is a graph with $\Delta(G) \geq 3$. This implies Conjecture 1.2 for the case $\Delta(G) = 3$. In 1999, Juvan, Mohar and Škrekovski [13] settled the case for $\Delta(G) = 4$. Some other special cases of Conjecture 1.2 have been confirmed such as complete graphs [6], graphs with girth at least $8\Delta(G)(\ln \Delta(G) + 1.1)$ [14], planar graphs with $\Delta(G) \geq 9$ [1], and planar graph with $\Delta(G) \neq 5$ and without two 3-cycles sharing a common vertex [18]. Suppose that G is a planar graph without k-cycles for some fixed integer $3 \leq k \leq 6$. Then it was shown that Conjecture 1.2 holds if G satisfies one of following conditions: (i) either $k = 3$ or $k = 4$ and $\Delta(G) \neq 5$ [22]; (ii) $k = 4$ [16]; (iii) $k = 5$ [19]; (iv) $k = 6$ and $\Delta(G) \neq 5$ [17], related known results on this topic we refer the readers to [3,9,10,11,15].

In this paper, we will consider planar graphs without chordal 7-cycles and get the following theorem.

Theorem 1.1. *Let G be a planar graph without chordal 7-cycles. Then G is edge-k-choosable, where $k = \max\{8, \Delta(G) + 1\}$.*

In Section 2, we will consider the structure of planar graphs without chordal 7-cycles. In Section 3, we will prove Theorem 1.1.

2 Structure lemma of some planar graphs

First, let us introduce some notation and definitions. Let $G = (V, E, F)$ be a planar graph. A vertex v is called a *d-vertex* or *d^+-vertex* if $d(v) = d$ or $d(v) \geq d$, respectively. For $f \in F$, we use $b(f)$ to denote the closed boundary walk of f and write $f = [u_1u_2...u_n]$ if $u_1, u_2, ..., u_n$ are the vertices on the boundary walk in clockwise order, with repeated occurrences of vertices allowed. The degree of a face f, denoted by $d(f)$, is the number of edge-steps in $b(f)$. Note that each cut-edge is counted twice. A *d-face* or *d^+-face* is a face of degree d or of degree at least d, respectively. Let $\delta(f)$ denote the minimum degree of vertices incident with f. When v is a d-vertex, we say that there are d faces incident with v. However, these faces are not required to be distinct, i.e., v may have repeated occurrences on the boundary walk of an incident face. Let $n_d(v)$ or $n_{d^+}(v)$ denote the number of d-faces or d^+-faces incident with vertex v with repeated occurrence of faces allowed, respectively.

We use the technique of "discharging" to prove the following Lemma which gives some information about the structure of a planar graph without chordal 7-cycles.

Lemma 2.1. *Let G be a planar graph without chordal 7-cycles. Then G contains one of the following configurations.*
(1) *An edge uv with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$.*
(2) *An even cycle c: $v_1v_2 \cdots v_{2n}v_1$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \cdots = d(v_{2n}) = \Delta(G)$.*

The proof is carried out by contradiction. Let G be a minimal counterexample to the lemma in terms of the number of vertices and edges. Then G is a connected planar graph with $\delta(G) \geq 3$ because there is no edge uv as in (1). Since G is a planar graph, by Euler's formula, we have

$$\sum_{v \in V}(3d(v) - 10) + \sum_{f \in F}(2d(f) - 10) = -10(|V| - |E| + |F|) = -20 < 0.$$

Now we define the initial weight function on $V(G) \cup F(G)$. Let $w(v) = 3d(v) - 10$ if $v \in V(G)$ and $w(f) = 2d(f) - 10$ if $f \in F(G)$. Thus the total sum of weights is the negative number -20. We are going to introduce discharging rules so that the total sum of weights is kept fixed while the discharging is in progress. However, once the discharging is finished, we can show that the resulting weight function w^* is nowhere negative. Thus, the following contradiction is arrived at and the existence of G is absurd.

$$0 \leq \sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) = -20.$$

By the choice of G, we have the following observations.

(P_1) Let uv be any edge of G. Then $d(u)+d(v) \geq \max\{10, \Delta(G)+3\}$. This implies that

$d(v)=\Delta(G) \geq 7$ if v neighbors a 3-vertex.

(P_2) Since G does not contain a 7-cycle with chords, it does not contain three adjacent triangles adjacent to a 4-face or a triangle adjacent to two 4-faces respectively.

If $d(v) > 5$, then v cannot be incident with five adjacent triangles.

(P_3) Let G_3 be the subgraph induced by the edges incident with the 3-vertices of G. Then G_3 contains a bipartite subgraph $G' = (V_1', V_2', E(G'))$, such that for any vertex $v \in V_1'$, $d_{G'}(v) = 2$; for any vertex $v \in V_2'$, $d_{G'}(v) = 1$. If $uv \in E(G')$ and $d_G(u) = 3$, then v is called a 3-*master* of u and u is called a *dependent* of v. Each 3-vertex has exactly two 3-masters and each vertex of degree Δ can be the 3-master of at most one 3-vertex.

Next we show that (P_3) is true. Clearly, G_3 does not contain odd cycles by (P_1). Thus G_3 is a bipartite graph with partite sets V_1, V_2, so that $V(G) = V_1 \cup V_2$ and for each vertex $v \in V_1$, $d_G(v) = 3$; for each vertex $v \in V_2$, $d_G(v) = \Delta$. By the choice of G, we know G_3 does not contain even cycles. Thus G_3 is a forest. For any component of G_3, we can select a vertex u with $d_G(u) = 3$ as the root of the tree. We define the distance between an edge and u to be the distance between one end of the edge having shorter distance to u and the root u. Thus, an edge incident with u is at distance 0 from the root. Then, we define edges of distance i from the root to be at level i, where $i = 0, 1, \cdots, m$ and m is the depth of the tree. We can select the edges which are not incident with u and at even level to form 3-paths v_1vv_2 such that $d_G(v) = 3$ and for the three edges incident with u, we select two of them to form a 3-path v_1uv_2. The selected edges form a bipartite subgraph G' with all the properties described in (P_3). This completes the proof of (P_3).

Note that each 3-vertex has exactly two 3-masters and each vertex of degree Δ can be the 3-master of at most one 3-vertex.

To prove the lemma, we are ready to consider a new weight w^* on G as follows:

R_1: From the 3-master to each 3-vertex transfer 2.

R_2: From each 3-vertex v to each incident 3-face f, transfer 1.

R_3: From each 4-vertex v to each incident face f, where $3 \leq d(f) \leq 4$, transfer $\frac{1}{2}$.

Let v be a 5-vertex and $\beta(v)$ be the weights transferred from v to its incident 4-face.

R_4: From each 5-vertex v to each incident face f, where $3 \leq d(f) \leq 4$, transfer

$\frac{5-\beta(v)}{n_3(v)}$, if $d(f) = 3$;

$\frac{1}{2}$, if $d(f) = 4$.

R_5: From each 6^+-vertex v to each incident face f, where $3 \le d(f) \le 4$, transfer

$\frac{7}{4}$, if $d(f) = 3$;

1, if $d(f) = 4$.

Let $\gamma(x \to y)$ denote the amount transferred out of an element x into another element y according to the above rules.

Next we will show that $w^*(x) \ge 0$ for all $x \in V \cup F$. Suppose that v is a d-vertex of G. If $d = 3$, then $w(v) = -1$. By (P_3), v has two 3-masters, and v is incident with at most three 3-faces; so by R_1 and R_2, $w^*(v) \ge w(v)+2\times2-1\times3 = 0$. If $d = 4$, then $w(v) = 2$. It follows from R_3 that $w^*(v) \ge w(v)-\frac{1}{2}\times4 = 0$. If $d = 5$ then $w(v) = 5$. From (P_2) we know that when $n_3(v) \ge 3$, $n_4(v) = 0$. So by R_4 $w^*(v) \ge w(v)-\frac{5}{n_3(v)}\times n_3(v) = 0$. When $n_3(v) \le 2$, it is obvious that $w^*(v) \ge 0$ by (P_2) and R_4. If $d = 6$, then $w(v) = 8$. From (P_2) we know that $n_3(v) \le 4$. When $n_3(v) = 4$ then $n_4(v) = 0$ by (P_2). So by R_5 $w^*(v) \ge w(v) - \frac{7}{4}\times 4 > 0$. When $n_3(v) = 3$ then $n_4(v) \le 1$ by (P_2). So by R_5 $w^*(v) \ge w(v) - \frac{7}{4}\times3-1 > 0$. By (P_2) and R_5 it is obvious that when $n_3(v) \le 2$, $w^*(v) \ge 0$. If $d = 7$, then $\omega(v) = 11$. From (P_2) we know that $n_3(v) \le 5$. When $n_3(v) = 5$, then $n_4(v) = 0$ by (P_2) and v can be the 3-master of a 3-vertex. So by R_1 and R_5 $w^*(v) \ge w(v) - \frac{7}{4}\times5-2 = \frac{1}{4} > 0$. When $n_3(v) = 4$, then $n_4(v) \le 1$ by (P_2) and v can be the 3-master of a 3-vertex. So by R_1 and R_5 $w^*(v) \ge w(v)-\frac{7}{4}\times4-1-2 > 0$. By (P_2), R_1 and R_5 it is obvious that when $n_3(v) \le 3$, $w^*(v) \ge 0$. If $d = 8$, then $\omega(v) = 14$. From (P_2) we know that $n_3(v) \le 6$. When $n_3(v) = 6$, then $n_4(v) = 0$ by (P_2) and v can be the 3-master of a 3-vertex. So by R_1 and R_5 $w^*(v) \ge w(v)-\frac{7}{4}\times6-2 = \frac{1}{2} > 0$. It is obvious that when $n_3(v) \le 5$, then $w^*(v) \ge 0$ by R_1 and R_5. If $d = 9$, we can use the same argument as above cases to verify that $w^*(v) \ge 0$. If $d \ge 10$, then $\omega(v) = 3d-10$. Let $n_3(v) = i$. Then $n_4(v) \le d-i$. So by R_1 and R_5 $w^*(v) \ge w(v)-\frac{7}{4}\times i-(d-i)-2 = 2d-12-\frac{3}{4}i \ge \frac{5}{4}d-12 \ge \frac{1}{2} > 0$.

Let f be any face of G, Clearly, $w^*(f) = w(f) \ge 0$ if $d(f) \ge 5$. We first consider the case that $f = v_1v_2v_3v_1$ is a 3-face with $d(v_1) \le d(v_2) \le d(v_3)$. Then $\omega(f) = -4$. If $d(v_1) = 3$, then $d(v_2) = d(v_3) = \Delta(G) \ge 7$ by (P_1). Thus by R_2 and R_5, $\gamma(v_1 \to f) = 1$, $\gamma(v_2 \to f) = \gamma(v_3 \to f) = \frac{7}{4}$, and $w^*(f) = w(f)+1+\frac{7}{4}\times2 > 0$. If $d(v_1) = 4$, then $d(v_2) \ge 6$, $d(v_3) \ge 6$ by (P_1). Thus according to R_3 and R_5, $\gamma(v_1 \to f) = \frac{1}{2}$, $\gamma(v_2 \to f) = \gamma(v_3 \to f) = \frac{7}{4}$, and $w^*(f) = w(f)+\frac{1}{2}+\frac{7}{4}\times2 = 0$. If $d(v_1) = 5$, then $d(v_2) \ge 5$, $d(v_3) \ge 5$. If $d(v_2) \ge 6$, by R_4 and R_5, it is obvious that $\omega^*(f) \ge 0$. If $d(v_2) = 5$ and $d(v_3) = 5$, we consider the following cases: when $n_3(v_1) = 5$, by (P_2) we know that $n_3(v_2)+n_4(v_2) \le 3$, and $n_3(v_3)+n_4(v_3) \le 3$. Then by R_4, $\gamma(v_1) \to f = 1$, $\gamma(v_2) \to f \ge \frac{5}{3}$ and $\gamma(v_3) \to f \ge \frac{5}{3}$. So $w^*(f) \ge w(f)+1+\frac{5}{3}\times2 = \frac{1}{3} > 0$. When $n_3(v_1) = 4$, by (P_2) we know that $n_3(v_2)+n_4(v_2) \le 3$, and $n_3(v_3)+n_4(v_3) \le 3$. Then by R_4, $\gamma(v_1) \to f \ge \frac{5}{4}$,

$\gamma(v_2) \to f \geq \frac{5}{3}$ and $\gamma(v_3) \to f \geq \frac{5}{3}$. So $w^*(f) \geq w(f) + \frac{5}{4} + \frac{5}{3} \times 2 = \frac{1}{3} > 0$. When $n_3(v_1) = 3$, then by (P_2) we know that $n_3(v_2) + n_4(v_2) \leq 4$, and $n_3(v_3) + n_4(v_3) \leq 3$ or $n_3(v_2) + n_4(v_2) \leq 3$, and $n_3(v_3) + n_4(v_3) \leq 4$. Then by R_4, $w^*(f) \geq w(f) + \frac{5}{4} \times 2 + \frac{5}{3} = \frac{1}{6} > 0$. when $n_3(v_1) \leq 2$, it is obvious that $\omega^*(f) \geq 0$. If $d(v_2) = 5$ and $d(v_3) \geq 6$. When $n_3(v_1) = 5$, then by (P_2) we know that $n_3(v_2) + n_4(v_2) \leq 3$. Then by R_4 and R_5, $\gamma(v_1) \to f = 1$, $\gamma(v_2) \to f \geq \frac{5}{3}$ and $\gamma(v_3) \to f \geq \frac{7}{4}$. So $w^*(f) \geq w(f) + 1 + \frac{5}{3} + \frac{7}{4} > 0$. If $d(v_1) \geq 6$, clearly $w^*(f) \geq 0$.

Next, we consider the case that $f = v_1v_2v_3v_4v_1$ is a 4-face, then $\omega(f) = -2$. If $\delta(f) \geq 4$, then $w^*(f) \geq w(f) + \frac{1}{2} \times 4 = 0$. Now assume that $\delta(f) = 3$. Without loss of generality, let $d(v_1) = 3$, then $d(v_2) = d(v_4) = \Delta \geq 7$, and $d(v_3) \geq 4$ since there is no even cycle as in (2). Thus by R_3 and R_5, $\gamma(v_2 \to f) = 1$, $\gamma(v_4 \to f) = 1$ and $\gamma(v_3 \to f) \geq \frac{1}{2}$. So $w^*(f) \geq w(f) + 1 \times 2 + \frac{1}{2} > 0$. This completes the proof of Lemma 2.1.

3 Proof of Theorem 1.1

The proof is carried out by contradiction. Let G be a minimal counterexample to the theorem. Then there is an edge assignment L with $|L(e)| \geq k$ for all $e \in E(G)$, where $k = \max\{8, \Delta(G) + 1\}$, such that G is not edge-L-colorable. By Lemma 2.1, we consider two cases as follows.

Case 1. G contains an edge uv with with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$. Consider the graph $G' = G - uv$. Then G' has an edge-L-coloring ϕ. Since there exist at most $\max\{7, \Delta(G)\}$ edges adjacent to uv and $|L(uv)| \geq \max\{8, \Delta(G)+1\}$, we can color uv with some color from $L(uv)$ that was not used by ϕ on the edges adjacent to uv. It is easy to see that the resulting coloring is an edge-L-coloring of G. This contradicts the choice of G.

Case 2. G contains an even cycle $C = v_1v_2 \cdots v_{2n}v_1$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$. Let G' be the subgraph of G obtained by deleting the edges of C. Then G' has an edge-L-coloring ϕ. Define an edge assignment L' of C such that $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in E(G')$ is adjacent to e in $G\}$ for each $e \in E(C)$. It is easy to see that $L'(e) \geq 2$ for each $e \in E(C)$. As has been proved independently by Vizing and by Erdős, Rubin, and Taylor (see [4]), a cycle of even length is 2-choosable. Thus C is edge-2-choosable and hence G is edge-L-colorable, which is a contradiction. This completes the proof.

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