Some remarks about the derivation operator and generalized Stirling numbers

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Abstract

Studying expressions of the form $(f(x)D)^p$, where $D = \frac{d}{dx}$ is the derivation operator, goes back to Scherk's Ph.D. thesis in 1823. We show that this can be extended as $\sum \gamma_{p;a}(f^{(0)})^{a(0)+1}(f^{(1)})^{a(1)}\dots(f^{(p-1)})^{a(p-1)}D^{p-\sum_i ia(i)}$, where the summation is taken over the p-tuples $(a_0, a_1, \dots, a_{p-1})$, satisfying $\sum_i a(i) = p-1$, $\sum_i ia(i) < p$, $f^{(i)} = D^i f$ and $\gamma_{p;a}$ is the number of increasing trees on the vertex set $[0,p]$ having $a(0)+1$ leaves and having $a(i)$ vertices with i children for $0 < i < p$. Thus, previously known results about increasing trees, lead us to some equalities containing coefficients $\gamma_{p;a}$. In the sequel, we consider the expansion of $(x^k D)^p$ and coefficients appearing there, which are called generalized Stirling numbers by physicists. Some results about these coefficients and their inverses are discussed through bijective methods. Particularly, we introduce and use the notion of (p,k)-forest in these arguments.

Keywords: formal power series; derivation operator; Stirling numbers;

1 Introduction

The derivation operator $\frac{d}{dx}$ (or briefly D) plays an important role in the theory of formal power series. There are some known results discussing

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about iteration of expressions of form $f(x)D$, of which, the followings are the most well-known. Here $\left\{ {p \atop m} \right\}$ (resp. $\left[{p \atop m} \right]$) stands for Stirling number of the second type (resp. signless Stirling number of the first type).

$$(xD)^p = \sum_{m=1}^{p} \left\{ {p \atop m} \right\} x^m D^m, \tag{1}$$

$$x^p D^p = \sum_{m=1}^{p} (-1)^{p-m} \left[{p \atop m} \right] (xD)^p. \tag{2}$$

These equations were obtained by H. Scherk in his ph.D. thesis defended in 1823 [9], in which, he has discussed expressions of the form $(f(x)D)^p$. The formula (1) is related to the case $f(x) = x$, but Scherk has also considered some other special cases such as $f(x) = e^{kx}$ and $f(x) = x^k$ as well as the general case. In the general case, he has obtained exact values for the first few coefficients as well as some infinite classes of particular coefficients. Recently these studies have been followed independently in quantum physics in studying creation and annihilation operators (See [3, 8, 7, 2] and the references therein). The relation between the coefficients appearing in these expansions and some combinatorial objects are also studied in these references.

In this paper we adopt a vector notation, which gives an easy description for the expansion of $(f(x)D)^p$ and a simple recurrence formula for the coefficients appearing in the expansion. The relation between these coefficients and increasing trees is already known [2]. By expanding this relation, we give a formula for these coefficients and we provide bijective proofs for some of identities satisfied by these coefficients; particularly, the appearance of Stirling and Eulerian numbers in these coefficients are justified.

We also discuss the expansion of $(x^k D)^p$ in a slightly different way from [7]. By studying the relation between this and the operator xD, not only we obtain the coefficients of the expansion as polynomials in terms of $k-1$ (with productions of Stirling numbers as their coefficients), but also we go towards a bijective proof for this expansion. This is done using the so called (p,k)-forests which are in fact some "increasing forests". Expansion of $x^{kp}D^p$ in terms with the general form $x^{(p-m)(k-1)}(x^k D)^m$, is also studied as in [7] as the inverse process.

2 Preliminaries and Notation

In this section, we introduce some definitions and notation which is useful in the rest of the paper. For a positive integer n, the rising factorial (resp.

falling factorial), denoted as $(x)_{\overline{n}}$ (resp. denoted as $(x)_n$), is defined as $(x)_{\overline{n}} = x(x+1)\dots(x+n-1)$ (resp. defined as $(x)_n = x(x-1)\dots(x-n+1)$). For a real number α, by $xD+\alpha$ we mean $xD+\alpha\mathbf{1}$ where $\mathbf{1}$ is the identity operator, thus the expression $(xD)_m$ is defined as $xD(xD-1)\cdots(xD-m+1)$.

Since the Stirling numbers and Eulerian numbers appear in these expansions, a brief introduction to these numbers and some related formulas appear in the sequel of this introduction, based on the notation of [6]. We recall that the *signless Stirling numbers of the first kind*, denoted by $\left[{n \atop k}\right]$, are the number of permutations of S_n having exactly k cycles. The *Stirling numbers of the first kind* are then defined as $(-1)^{n-k}\left[{n \atop k}\right]$. The *Stirling numbers of the second kind*, denoted by $\left\{{n \atop k}\right\}$, are the number of ways of partitioning an n-set into exactly k parts. The *Eulerian numbers*, denoted by $\left\langle{n \atop k}\right\rangle$, are the number of permutations $\pi \in S_n$ which have k descents (i.e. $|\{i : 1 \le i < n, \pi_i > \pi_{i+1}\}| = k$.) Both kinds of Striling numbers and the Eulerian numbers satisfy some binomial-type recurrence relations and nice identities (See [4, 6] for instance). Here, the following identity is useful.

$$(x)_{\overline{n}} = \sum_j \left[{n \atop j}\right] x^j. \tag{3}$$

It is concluded from (2) that the following equation holds

$$x^p D^p = (xD)_m. \tag{4}$$

The set of integers (resp. nonnegative integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{N}$). For integers m and n we denote the set $\{x \in \mathbb{Z} : m \le x \le n\}$ by $[m, n]$. We denote the set of infinite row vectors of nonnegative integers by $\mathbb{N}^\infty$, so each element $\mathbf{a} \in \mathbb{N}^\infty$ is represented as $\mathbf{a} = (\mathbf{a}(0), \mathbf{a}(1), \cdots, \mathbf{a}(p-1), \cdots)$. The vectors $\mathbf{j}, \mathbf{e_m}, \mathbf{n} \in \mathbb{N}^\infty$ are defined respectively by $\mathbf{j}(i) = 1, \mathbf{e_m}(i) = \delta_{m,i}$ and $\mathbf{n}(i) = i$ for any integer $i \ge 0$. The value of $f^{\mathbf{a}}$, for a vector $\mathbf{a} \in \mathbb{N}^\infty$ with finitely many nonzero components, is defined as

$$f^{\mathbf{a}} = \prod_i (f^{(i)})^{\mathbf{a}(i)},$$

where $f^{(0)} = f$ and for $j \ge 1$, we have $f^{(j)} = D^j f$. Also we define the set Λ_p by

$$\Lambda_p = \{\mathbf{a} \in \mathbb{N}_p^\infty : \mathbf{a}.\mathbf{j}^\top = p - 1,\ \mathbf{a}.\mathbf{n}^\top < p\}.$$

Let V be a finite ordered set with $v_0 = \min V$ (for instance, V can be considered as a finite set of integers). An increasing tree on V, is a tree T rooted at v_0 with $V(T) = V$, such that for any $v \in V$, the vertices in the unique $v_0 - v$ path P in T, appear increasingly. A *starlike increasing tree* is an increasing tree,in which, any vertex (except possibly the root) has at most one child. The increasing trees are widely studied in the literature (See Section 1.3 of [10]; for more information see [1]). For a vertex v of an increasing tree, we denote the number of its children by $d'(v)$.

In Proposition 8, a little about finite calculus and its notations is required: If $\{a_n\}_{n\geq 0}$ is a sequence of real (or complex) numbers, then the operator Δ on it is defined by $\Delta a_n = a_{n+1} - a_n$. It is easily proved that the following identity holds:

$$\Delta^p a_n = \sum_{i=0}^{p}(-1)^i \binom{p}{i} a_{n+p-i}$$

and in particular

$$\Delta^p a_n \Big]_{n=0} = \sum_{i=0}^{p}(-1)^i \binom{p}{i} a_{p-i}.$$

3 Expanding $(f(x)D)^p$

After testing some small cases, one can guess that $(fD)^p$ is expressed in the following form

$$\sum_{\sum_i a_i = p-1,\, \sum_i i a_i < p} \gamma_{p;a_0,a_1,\cdots,a_{p-1}} (f^{(0)})^{a_0+1} (f^{(1)})^{a_1} \cdots (f^{(p-1)})^{a_{p-1}} D^{p-\sum_i i a_i},$$

where the constants $\gamma_{p;a_0,a_1,\cdots,a_{p-1}}$ are nonnegative integers. This is formulated and proved shortly in the following theorem, by using the vector notation of Section 2.

Theorem 1. *Let p be a positive integer.*

(i) *We have*

$$(fD)^p = \sum_{\mathbf{a}} \gamma_{p;\mathbf{a}} f^{\mathbf{a}+\mathbf{e}_0} D^{p-\mathbf{a}.\mathbf{n}^T} \tag{5}$$

where the summation runs over the elements $\mathbf{a} \in \Lambda_p$. Equivalently, one can say that $\mathbf{a}$ runs over $\mathbb{N}^\infty$ but $\gamma_{p;\mathbf{a}} \neq 0$ only if $\mathbf{a} \in \Lambda_p$.

(ii) *For any $p \geq 2$ and for any $\mathbf{a} \in \mathbb{N}^\infty$, we have the following recurrence relation:*

$$\gamma_{p;\mathbf{a}} = \gamma_{p-1;\mathbf{a}-\mathbf{e}_0} + \sum_{i=0}^{p-2}(\mathbf{a}(i)+1)\gamma_{p-1;\mathbf{a}-\mathbf{e}_0+\mathbf{e}_i-\mathbf{e}_{i+1}}, \tag{6}$$

where by definition the terms $\gamma_{p-1;\mathbf{b}}$ in the right are 0 whenever $\mathbf{b} \notin \Lambda_{p-1}$.

(iii) *If* $\mathbf{a}(p-1) = 1$ *and* $\gamma_{p;\mathbf{a}} \neq 0$ *then* $\mathbf{a} = (p-2)\mathbf{e}_0 + \mathbf{e}_{p-1}$ *and* $\gamma_{p;(p-2)\mathbf{e}_0+\mathbf{e}_{p-1}} = 1$. *Furthermore, if* $\mathbf{a}(p-1) = 0$ *then the upper limit of the summation in (6) may be considered* $p-3$.

Proof. (i) The proof is by induction on p. The case $p = 1$ is trivial: $fD = (f^{(0)})^1 D$ so for $p = 1$ the only nonzero coefficient is $\gamma_{1;\mathbf{e}_0} = 1$. Now let the above equation holds for $p-1$. To prove it for p, we write

$$\begin{aligned}(fD)^p &= fD((fD)^{p-1})\\ &= fD\left(\sum_{\mathbf{a}} \gamma_{p-1;\mathbf{a}} f^{\mathbf{a}+\mathbf{e}_0} D^{p-1-\mathbf{a}.\mathbf{n}^\top}\right)\\ &= \sum_{\mathbf{a}} \gamma_{p-1;\mathbf{a}} fD\left(f^{\mathbf{a}+\mathbf{e}_0} D^{p-1-\mathbf{a}.\mathbf{n}^\top}\right)\\ &= \sum_{\mathbf{a}} \gamma_{p-1;\mathbf{a}} f^{\mathbf{a}+2\mathbf{e}_0} D^{p-\mathbf{a}.\mathbf{n}^\top} +\\ &\sum_{\mathbf{a}} \sum_{i=0}^{p-2} \gamma_{p-1;\mathbf{a}} \left(\mathbf{a}(i) + \mathbf{e}_0(i)\right) f^{\mathbf{a}+2\mathbf{e}_0-\mathbf{e}_i+\mathbf{e}_{i+1}} D^{p-1-\mathbf{a}.\mathbf{n}^\top}\end{aligned}$$

Now it is straightforward to show that all the terms in the right (ignoring the coefficients) are of the form $f^{\mathbf{b}+\mathbf{e}_0} D^{p-1-\mathbf{b}.\mathbf{n}^\top}$ for a vector $\mathbf{b}$ of length p satisfying $\mathbf{b}.\mathbf{j}^\top = p-1$ and $\mathbf{b}.\mathbf{n}^\top < p$.

(ii) From the proof of part (i) we have

$$\begin{aligned}\sum_{\mathbf{a}} \gamma_{p;\mathbf{a}} f^{\mathbf{a}+\mathbf{e}_0} D^{p-\mathbf{a}.\mathbf{n}^\top} &= \sum_{\mathbf{c}} \gamma_{p-1;\mathbf{c}} f^{\mathbf{c}+2\mathbf{e}_0} D^{p-\mathbf{c}.\mathbf{n}^\top} +\\ &\sum_{\mathbf{c}} \sum_{i=0}^{p-2} \gamma_{p-1;\mathbf{c}} \left(\mathbf{c}(i) + \mathbf{e}_0(i)\right) f^{\mathbf{c}+2\mathbf{e}_0-\mathbf{e}_i+\mathbf{e}_{i+1}} D^{p-1-\mathbf{c}.\mathbf{n}^\top}\end{aligned}$$

Calculating the coefficient of $f^{\mathbf{a}+\mathbf{e}_0} D^{p-\mathbf{a}.\mathbf{n}^\top}$ in both sides gives the following identity which concludes the result.

$$\gamma_{p;\mathbf{a}} = \gamma_{p-1;\mathbf{a}-\mathbf{e}_0} + \sum_{i=0}^{p-2} (\mathbf{a}(i) + 1)\gamma_{p-1;\mathbf{a}-\mathbf{e}_0+\mathbf{e}_i-\mathbf{e}_{i+1}}.$$

(iii) This is easily obtained by using previous parts.

□

A simple way to find a combinatorial interpretation for expression $(fD)^p$ is labeling the terms f and also D from right to left and adding an element f_0 in the right to obtain an expression of the form $f_p D_p f_{p-1} D_{p-1} \dots f_1 D_1 f_0$, and then expanding this expression and deleting the labels (except for f_0) at the end. The following theorem is then a simple observation.

Proposition 2. *Let p be a positive integer and $\mathbf{a} \in \Lambda_p$. Then the value of $\gamma_{p;\mathbf{a}}$ equals the number of distributions of p distinguished balls $D_1, \cdots, D_p$ into p distinguished urns $f_0, \cdots, f_{p-1}$ satisfying the following conditions:*
(i) The label of each ball is greater than the label of the urn containing it.
(ii) The number of the urns with positive label which contain exactly i balls is $\mathbf{a}(i)$, for $i = 0, \cdots, p-1$.

The vector $\mathbf{a}$ is called the *counting vector* of the distribution. As a simple application of the above Proposition, one can prove the recurrence relation (6) by the combinatorial argument obtained form it. In Section 5, we use slight modification of this approach to associate increasing trees to the coefficients γ.

4 The operators $(e^{kx}D)^p$ and $(x^kD)^p$

In this section we study the expansion of $(f(x)D)^p$ in two cases $f(x) = e^{kx}$ and $f(x) = x^k$. Both cases are considered in [9] and [7], but we study the case $f(x) = x^k$ slightly differently. It is remarkable that Stirling numbers of both kinds appear explicitly in our formulae.

The following proposition is mentioned in p.9-10 of [9] as well as [7]

Proposition 3. *Let p be a positive integer. Then we have*

$$(e^{kx}D)^p = e^{kpx} \sum_{m=1}^{p} \begin{bmatrix} p \\ m \end{bmatrix} k^{p-m} D^m, \tag{7}$$

$$e^{kpx} D^p = \sum_{m=1}^{p} \begin{Bmatrix} p \\ m \end{Bmatrix} (-k)^{p-m} e^{k(p-m)x} (e^{kx}D)^m. \tag{8}$$

The problem of extending $(x^kD)^p$ is a natural generalization of (1) which is studied both in [9] and [7]. First, we mention the following proposition which is easily proved by induction on p.

Proposition 4. *Let p be a positive integer.*

(i) [7, Lemma 1] *We have*

$$(x^kD)^p = x^{p(k-1)} \sum_{m=1}^{p} \alpha_{pm}(k) x^m D^m, \tag{9}$$

where $\alpha_{pm}(k)$ satisfies the following recurrence relation

$$\alpha_{pm}(k) = \alpha_{p-1,m-1}(k) + \big((p-1)(k-1) + m\big)\alpha_{p-1,m}(k), \tag{10}$$

$$\alpha_{p,0}(k) = 0, (p > 0), \quad \alpha_{p,p}(k) = 1, (p \geq 0), \quad \alpha_{p,q}(k) = 0, (0 \leq p < q). \tag{11}$$

(ii) [7, Lemma 14] *We have*

$$x^{kp}D^p = \sum_m \beta_{pm}(k)x^{(p-m)(k-1)}(x^kD)^m, \tag{12}$$

where $\beta_{pm}(k)$ *satisfies the following recurrence relation*

$$\beta_{pm}(k) = \beta_{p-1,m-1}(k) - (m(k-1)+p-1)\beta_{p-1,m}(k), \tag{13}$$
$$\beta_{p,0}(k) = 0, (p>0), \quad \beta_{p,p}(k) = 1, (p \geq 0), \quad \beta_{p,q}(k) = 0, (0 \leq p < q). \tag{14}$$

Note that from definitions of coefficients α and β, it is easily seen that they are inverses of each other, i.e. $\sum_m \alpha_{pm}\beta_{mq} = \sum_m \beta_{pm}\alpha_{mq} = \delta_{pq}$. Now we are going to calculate these coefficients, but we prefer to do this calculation by the auxiliary tool given in the next proposition.

Proposition 5. *Let p be a positive integer. The following identity holds between the operators $x^kD$and xD.*

$$(x^kD)^p = x^{p(k-1)}\prod_{i=0}^{p-1}(xD + i(k-1)).$$

Proof. This is proved easily by induction on p. □

In the next theorem, we calculate $\alpha_{pm}(k)$ and $\beta_{pm}(k)$ as polynomials of degree $p-m$ in terms of $k-1$, whose coefficients are given in terms of Stirling numbers as follows:

Theorem 6. *We have*

$$\alpha_{pm}(k) = \sum_{j=m}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{p-j} \tag{15}$$

$$\beta_{pm}(k) = (-1)^{p-m}\sum_{j=m}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{j-m} \tag{16}$$

Proof. To calculate $\alpha_{pm}(k)$, one can use corresponding recurrence relation (10). By the way, we give an alternative proof. By (9)and Proposition

5, we obtain

$$\begin{aligned}\sum_{m=1}^{p} \alpha_{pm}(k)x^m D^m &= \prod_{i=0}^{p-1} (xD + i(k-1)) \\ &= \sum_{j=1}^{p} \begin{bmatrix} p \\ j \end{bmatrix} (k-1)^{p-j}(xD)^j \\ &= \sum_{j=1}^{p} \begin{bmatrix} p \\ j \end{bmatrix} (k-1)^{p-j} \sum_{m=1}^{p} \begin{Bmatrix} j \\ m \end{Bmatrix} x^m D^m \\ &= \sum_{m=1}^{p} \sum_{j=1}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{p-j} x^m D^m.\end{aligned}$$

Hence, we obtain

$$\alpha_{pm}(k) = \sum_{j=1}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{p-j} = \sum_{j=m}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{p-j}.$$

Clearly (16) holds for $k = 1$, so we prove it in the case $k \neq 1$. From Theorem 5, by changing p to n, we obtain

$$s_n = \sum_{m} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} t_m,$$

where $s_n = x^{-n(k-1)}(1-k)^{-n}(x^k D)^n$ and $t_m = (1-k)^{-m}(xD)^m$. By inverting the above equation, we provide $t_n = \sum_m \left\{ {n \atop m} \right\} s_m$, which equals

$$(xD)^n = (1-k)^n \sum_{m} (1-k)^m x^{m(1-k)} \begin{Bmatrix} n \\ m \end{Bmatrix} (x^k D)^n.$$

Finally combining this with (2) yields equation (16). □

Corollary 7. *With previous notations, the following identity holds*

$$\beta_{pm}(k) = (1-k)^{p-m} \alpha_{pm}\left(\frac{k}{k-1}\right).$$

It is immediately seen from (15) that the case $k = 2$ gives famous Lah numbers which are defined as $L(p,m) = \sum_j \left[{p \atop j} \right] \left\{ {j \atop m} \right\}$, and satisfy $L(p,m) = \frac{p!}{m!}\binom{p-1}{m-1}$. (See for instance Exercise 2.13 of [5]). By using Corollary 7, $\beta_{pm}(2) = (-1)^{p-m}\alpha_{pm}(2)$. Hence

$$\begin{aligned}\alpha_{pm}(2) &= \frac{p!}{m!}\binom{p-1}{m-1} \\ \beta_{pm}(2) &= (-1)^{p-m}\frac{p!}{m!}\binom{p-1}{m-1}\end{aligned}$$

The following proposition may be considered as a generalization of these equations. Moreover it gives alternative formulas for computing $\alpha_{pm}(k)$ and $\beta_{pm}(k)$.

Proposition 8.

(i) *Let $p \geq m$. Then for any t we have*

$$\sum_j \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} t^j = \frac{p!}{m!} n_{pmt}.$$

where the value n_{pmt} is given by

$$n_{pmt} = \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} \binom{p + tm - t\ell - 1}{p}$$

and is equal to the number of $t \times m$ matrices with nonnegative integer entries and no zero column, and with p as the sum of their entries.

(ii) *Let $A(z) = \prod_{i=0}^{p-1} (z + i(k-1))$. We have*

$$\alpha_{pm}(k) = \frac{1}{m!} \Delta^m A(z)\Big]_{z=0}.$$

(iii) *Let $k \neq 1$ and $B(z) = \prod_{i=0}^{p-1} ((k-1)z + i)$. Then*

$$\beta_{pm}(k) = \frac{1}{m!(k-1)^m} \Delta^m B(z)\Big]_{z=0}.$$

Proof. We prove (i); parts (ii) and (iii) are easy conclusions of this. By replacing $\left\{ {j \atop m} \right\} = \frac{1}{m!} \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} (m-\ell)^j$, we obtain

$$\begin{aligned}
\sum_{j=1}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} t^j &= \sum_{j=1}^{p} \sum_{\ell=0}^{m} \frac{1}{m!} \begin{bmatrix} p \\ j \end{bmatrix} t^j (-1)^\ell \binom{m}{\ell} (m-\ell)^j \\
&= \sum_{\ell=0}^{m} \frac{(-1)^\ell}{m!} \binom{m}{\ell} \sum_{j=1}^{p} \begin{bmatrix} p \\ j \end{bmatrix} t^j (m-\ell)^j \\
&= \frac{1}{m!} \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} (t(m-\ell))_{\overline{p}} \\
&= \frac{p!}{m!} \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} \binom{t(m-\ell) + p - 1}{p} \\
&= \frac{p!}{m!} n_{pmt}
\end{aligned}$$

By using PIE, it is easily obtained that the value n_{pmt} equals the number of $t \times m$ matrices with nonnegative integer entries and no zero column, and with p as the sum of their entries. □

Remark 1. From Proposition 8, it is concluded that if k is a positive integer, then the value $\frac{m!}{p!}(k-1)^m|\beta_{pm}(k)|$ is an integer; in fact it equals the number of $(k-1) \times m$ matrices with nonnegative entries and with no zero column whose entries sum up to p.

5 Increasing trees and some bijections

In this section, first we mention some enumerative results about increasing trees in two propositions. The definition of an increasing tree related to a distribution with conditions of Proposition 2, is then a useful connector which concludes the rest of results of this section about coefficients $\gamma_{p;\mathbf{a}}$. The definitions and notations for increasing trees are as mentioned in Section 2. The proposition about increasing trees is well-known and is proved by a standard representation of permutations as in proposition 1.3.16 of [10].

Proposition 9. *Let $\mathcal{T}(p)$ be the set of increasing trees on the vertex set $V = [0,p]$. Then*

(i) *Let $\mathcal{S}'_m(p) \subseteq \mathcal{T}(p)$ be the set of starlike increasing trees on $[0,p]$ for which there exists a unique vertex $v \in [1,p]$ such that v has exactly m children and for any vertex $u \in [1,p] \setminus \{v\}$, u is a leaf. Then $|\mathcal{S}'_m(p)| = \binom{p}{m+1}$.*

(ii) *If $\mathcal{S}_m(p) \subseteq \mathcal{T}(p)$ is the set of starlike increasing trees with m leaves, then $|\mathcal{S}_m(p)| = \left\{ {p \atop m} \right\}$.*

(iii) *If $\tau_m(p) \subseteq \mathcal{T}(p)$ is the set of increasing trees with $m+1$ leaves, then $|\tau_m(p)| = \left\langle {p \atop m} \right\rangle$.*

(iv) *If $\mathcal{T}_m(p) \subseteq \mathcal{T}(p)$ is the set of increasing trees in which the root has m children, then $|\mathcal{T}_m(p)| = \left[{p \atop m} \right]$.*

Remark 2. Since the bijection used in the proof of the above proposition is required for future sections, we give a naive description of it (This is slightly modified with respect to the one used in [10]). Let T b an increasing tree on $[0,p]$ rooted at 0. The mentioned bijection is based on the sequence $\{c_i\}_{i=0}^p$ of vertices obtained from the depth first search of T, with the constraint that in each step, the greater child should be visited sooner. Conversely, having a sequence $\{c_i\}_{i=0}^p$ with $c_0 = 0$ and $\{c_1, \ldots, c_p\} = [1,p]$, one can recover the corresponding increasing tree on $[0,p]$ by constructing the array

$\{\text{PARENT}(v)\}_{v=1}^{p}$ which assigns to any vertex $v \neq 0$ in the tree its parent through the following algorithm:

1. $i = 0$.
2. $i = i + 1$.
3. If $(i = 1)$ then PARENT$(c_i) = 0$, go to step 2.
4. $v = c_{i-1}$.
5. while$(c_i < v)$ let $v =$PARENT(v).
6. PARENT$(c_i) = v$.
7. If $(i < p)$ go to step 2.
8. Stop.

The problem of the enumeration of increasing trees on $V = [0, p]$ which satisfy $d'(i) = \ell_i$ for $i = 1, \cdots, p-1$ where $\{\ell_i\}_{i=1}^{p-1}$ is a given sequence of nonnegative integers , is considered in the next proposition. (Note that since $d'(p) = 0$ and $d'(0) = p - \sum_{i=1}^{p-1} d'(i)$, these values are excluded from the sequence ℓ_i.)

Proposition 10. *Let $\ell_1, \ell_2, \cdots, \ell_{p-1}$ be a sequence of nonnegative integers and let $V = [0, p]$. Then*

(i) *There exists an increasing tree T on $V = [0, p]$ with $d'_T(v) = \ell_v$ for $v = 1, \cdots, p-1$ if and only if $\sum_{i=j}^{p-1} \ell_i \leq p - j$ for $j = 1, \cdots, p-1$.*

(ii) *The number of increasing trees mentioned in part (i) is obtained as $\frac{g(\ell_1, \cdots, \ell_{p-1})}{\ell_1! \cdots \ell_{p-1}!}$ where*

$$g(\ell_1, \cdots, \ell_{p-1}) = (2 - \ell_{p-1})_*(3 - \ell_{p-1} - \ell_{p-2})_* \cdots (p - 1 - \sum_{i=2}^{p-1} \ell_i)_*$$

and for a real number x, the value of $(x)_*$ is defined to be x if $x > 0$ and 0 otherwise.

Proof. (i) We denote the set of children of the vertex i by L_i for $i = 1, \cdots, p-1$. Obviously theses are disjoint sets and for any j , $1 \leq j \leq p-1$, we have $\bigcup_{i=j}^{p-1} L_i \subseteq [j+1, p]$. Therefore, from the existence of an increasing tree with the given out-degree sequence, the required inequalities hold. Conversely, if the inequalities $\sum_{i=j}^{p-1} \ell_i \leq p-j$ holds for $j = 1, \cdots, p-1$, one can construct L_i's as follows: The set L_{p-1} is a subset of size ℓ_{p-1} of $\{p\}$ (Thus either $\ell_{p-1} = 0, L_{p-1} = \emptyset$ or $\ell_{p-1} = 1, L_{p-1} = \{p\}$.) Now suppose that L_i is constructed for $i = p-1, p-2, \cdots, i'+1$. Then $L_{i'} \subseteq [i'+1, p] \setminus \bigcup_{i=i'+1}^{p-1} L_i$, but the set in the right contains exactly $h_{i'} = p - i' - \sum_{i=i'+1}^{p-1} \ell_i$ elements. On the other hand, from the given inequalities we obtain $h_{i'} \geq \ell_{i'}$ thus it is possible to construct $L_{i'}$.

(ii) It is concluded from the construction of Part (i).

□

Definition 1. Consider a distribution $\mathcal{D}$ of p distinguishable balls $D_1, D_2, \ldots, D_p$ into p distinguishable urns $f_0, f_1, \ldots, f_{p-1}$ satisfying conditions (i) and (ii) of Proposition 2. We associate a graph $T(\mathcal{D})$ with the vertex set $V = [0, p]$ to the distribution $\mathcal{D}$, as as follows: If the ball D_i is put into the urn f_j, then $\{i, j\}$ is an edge of $T(\mathcal{D})$. It is clear that $T(\mathcal{D})$ is an increasing tree rooted at 0.

The following theorem is concluded from the above definition and Proposition 2.

Proposition 11. *With the conditions of Proposition 2, the value of* $\gamma_{p;\mathbf{a}}$ *equals the number of increasing trees on* $[0, p]$ *in which*
(i) The number of the leaves is $\mathbf{a}(0) + 1$.
(ii) The number of the nodes which have exactly i *children is* $\mathbf{a}(i)$ *for* $i = 1, \cdots, p$.

Corollary 12. *The following identities hold.*

(i) *Suppose that* $1 \leq m \leq p-1$. *Then* $\gamma_{p;(p-2)\mathbf{e}_0+\mathbf{e}_m} = \binom{p}{m+1}$.

(ii) *Let* $1 \leq m \leq p$. *Then* $\gamma_{p;(m-1)\mathbf{e}_0+(p-m)\mathbf{e}_1} = \left\{ {p \atop m} \right\}$.

(iii) $$\sum_{\mathbf{a}.\mathbf{n}^{\mathsf{T}}=p-m} \gamma_{p;\mathbf{a}} = \begin{bmatrix} p \\ m \end{bmatrix}$$

(iv) $$\sum_{\mathbf{a}.\mathbf{e}_0{}^{\mathsf{T}}=m} \gamma_{p;\mathbf{a}} = \left\langle {p \atop m} \right\rangle.$$

Proposition 13. *All parts are concluded from Proposition 9. We also mention that parts (i) and (ii) can be also proved using recurrence relation (6). Moreover, parts (ii) and (iii) may also be concluded from expansions of* $(xD)^p$ *and* $(e^x D)^p$.

The following theorem gives a nonrecursive formula to compute the coefficient $\gamma_{p;\mathbf{a}}$.

Theorem 14. *The coefficient $\gamma_{p;\mathbf{a}}$ can be computed as follows*

$$\gamma_{p;\mathbf{a}} = \frac{1}{(0!)^{\mathbf{a}(0)}(1!)^{\mathbf{a}(1)}\dots((p-1)!)^{\mathbf{a}(p-1)}} \sum g(\ell_1, \ell_2, \dots, \ell_{p-1})$$

Where the summation runs over all $(p-1)$-tuple $(\ell_1, \ell_2, \dots, \ell_{p-1})$ of integers satisfying $\{\ell_1, \ell_2, \dots, \ell_{p-1}\} = \{\mathbf{a}(0).0, \mathbf{a}(1).1, \dots, \mathbf{a}(p-1).(p-1)\}$ (which means that the number of i's appearing in the sequence $\{\ell_i\}_{i=1}^{p-1}$ is $\mathbf{a}(i)$ for $i = 0, \cdots, p-1$).

Proof. The proof is straightforward by using Proposition 10 and the definition of $\gamma_{p;\mathbf{a}}$. (Note that the summation given above, contains $\frac{(p-1)!}{\mathbf{a}(0)!\cdots\mathbf{a}(p-1)!}$ summands, some of which may equal 0.) □

6 Expansion of $(x^k D)^p$

The expansion of $(x^k D)^p$ is given in [7]. Here we would like to obtain this result through a pure combinatorial discussion. An immediate usage of the results of the previous section, suggests to bound the capacity of each urn f_i, $(i > 0)$ by k, but here we propose a more useful model: The problem of expanding $(x^k D)^p$ is related to the expansion of the following expression

$$x_{pk}x_{pk-1}\cdots x_{pk-k+1}D_p\cdots x_{ik}x_{ik-1}\cdots x_{ik-k+1}D_i\cdots x_k x_{k-1}\cdots x_1 D_1,$$

which itself is related to a (p, k)-distribution defined as follows (Note that the rightmost position is reserved for an urn called x_0).

Definition 4. For positive integers p and k, a *(p, k)-distribution* is a distribution of balls D_i, $(1 \le i \le p)$ into the urns x_j, $(0 \le j \le pk)$ such that each urn (except x_0 whose capacity is not bounded) contains at most one ball, and a ball D_i can be put into an urn x_j only if $\lceil \frac{j}{k} \rceil < i$.

Proposition 15. *We have $(x^k D)^p = x^{p(k-1)} \sum_{m=1}^{p} \alpha_{pm}(k) x^m D^m$, where $\alpha_{pm}(k)$ is the number of (p, k)-distributions in which the urn x_0 contains exactly m balls.*

To relate a (p, k)-distribution to a graph, we add a new definition. Before this, we mention that a rooted tree T with root r is sometimes denoted by (T, r) to emphasize on the root.

Definition 5. A *(p, k)-forest* (related to a (p, k)-distribution,) is defined as graph F on the vertex set $V(F) = \{0, 1, \cdots, (p-1)k + 1\} \cup$

$\{r_1, \cdots, r_{p-1}\}$, which is a disjoint union of starlike increasing trees $(S_0, 0)$, $(S_1, r_1), \cdots, (S_{p-1}, r_{p-1})$, where the symbols r_i are conventionally considered smaller than any positive integer, and the following conditions hold.

(1) The children of r_i are vertices $(i-1)k+2, (i-1)k+3, \cdots, ik$, for $i = 1, \cdots, p-1$.

(2) If the ball D_m is put into urn x_0, then $(m-1)k+1$ is a child of 0 in S_0.

(3) If the ball D_m is put into urn x_n, then $(m-1)k+1$ is a child of n.

We mention that $(p-1)(k-1)$ edges of the forest are determined by condition (1), independent of the related (p,k)-distribution. We note that the existence of vertices r_i and these edges in the (p,k)-forest, gives it a more symmetric structure. The other edges of the forest are obtained from the related (p,k)-distribution.

Also the vertex 1 is always a child of 0. If $1 \leq u \leq (p-1)k$ and $u \not\equiv 1(\mod k)$, then u is a child of some r_i. Consequently, the vertices of the form $u \equiv 1(\mod k)$, which are not in $V(S_0)$, are at distance at least 2 from some r_i in S_i. Furthermore, for any such vertex u, there exists a unique ancestor of the form $(i-1)k + r'$ with $(2 \leq r' \leq k)$.

Definition 6. The mapping $\Omega_{pk} : F \to ((T,0), C'_k)$ maps a (p,k)-forest F into an increasing tree $(T,0)$ and an associated coloring of its edges with the color set $C = \{0, 1, \cdots, k-1\}$ constructed as follows:

(1) For any $v > 0$ such that $(v-1)k+1 \in V(S_0)$, add an edge $\{0, v\}$ into T and color it with color 0.

(2) Let $i > 0$. For any vertex of the form $(v-1)k+1$ in $V(S_i)$ add an edge $\{i, v\}$ in T. Moreover, if the vertex $(v-1)k+1$ in S_i has an ancestor of the form $(i-1)k+r+1$, with $1 \leq r \leq k-1$, (note that this ancestor is unique,) then color the edge $\{i, v\}$ with color r.

Example 1. The forest shown in Figure 1 is the corresponding forest (according to the Definition 5) to the term $x_{12}x_{11}x_{10}x_9^{\{4\}}x_8x_7x_6x_5x_4^{\{3\}}x_3x_2^{\{2\}}x_1D^{\{1\}}$ of the expansion of $(x^3D)^4$

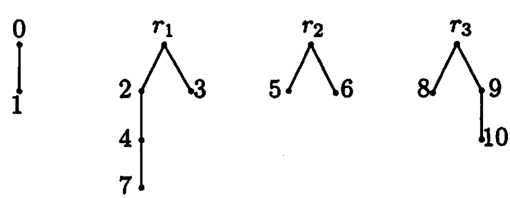

Figure 1

And the colored increasing tree shown in Figure 2, is the corresponding colored increasing tree (according to the Definition 6) to the forest shown in Figure 1.

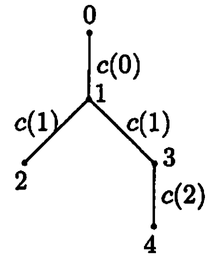

Figure 2

Theorem 16. *Using the previous notations we have*

(i) *There exists a bijection* $\Xi : F \to ((T,0), C'_k, (S_0,0))$.

(ii) *The mapping in part (i) gives a bijective proof for the following identity*

$$\alpha_{pm}(k) = \sum_{j=m}^{p} \begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{p-j}.$$

Proof. (i) Having the (p,k)-forest F, we obtain $(S_0, 0)$ as its component which contain the vertex 0. The other components of the triple in the right (which are $(T,0)$ and C'_k,) are constructed uniquely from F by the mapping Ω_{pk} as mentioned before.
Now suppose that $(S_0,0)$, $(T,0)$, and C'_k are given. To construct F, it is enough to construct the components (S_i, r_i) for $i = 1, \cdots, p-1$. For this, the following process should be followed for $i = 1, \cdots p-1$: Fix i and consider all of the edges $\{i,v\}$ in T. For any child v of i, let $(v-1)k+1 \in V(S_i)$ and consider the vertex $(v-1)k+1$ in S_i as a descendent of $(i-1)k+r+1$, where r is the color of the edge $\{i,v\}$.

(ii) To calculate $\alpha_{pm}(k)$, by Proposition 15, we should calculate the number of (p,k)-distributions in which x_0 contains m balls. By part (i), this equals the number of triples $((T,0), C'_k, (S_0,0))$ in which the vertex 0 has m children in S_0. Suppose that the vertex 0 has j children in T; thus $|V(S_0) \setminus \{0\}| = j$. There are then $\begin{bmatrix} p \\ j \end{bmatrix}$ choices for T, and for any one of them, by Proposition 9 (ii), there are $\begin{Bmatrix} j \\ m \end{Bmatrix}$ choices for S_0. It remains to count the number of colorings C'_k of T: There are j edges of the form $\{0,v\} \in A(T)$ with color 0; For any of the remaining $p-j$ edges we have $k-1$ choices. Thus there are $(k-1)^{p-j}$ such colorings. We conclude that there are $\begin{bmatrix} p \\ j \end{bmatrix} \begin{Bmatrix} j \\ m \end{Bmatrix} (k-1)^{p-j}$ possibilities for such a triple. Finally, summing up over j, gives the desired value. □

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