ON KONHAUSER MATRIX POLNOMIALS

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ABSTRACT. The main purpose of this paper is to define pair of Konhauser matrix polynomials and obtain some properties such as recurrence relations and matrix differential equation for Konhauser matrix polynomials.

1. INTRODUCTION

In [10], the pair of Konhauser polynomials are defined as

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{r=0}^{n}\frac{(x)^r}{r!}\left(\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)+\alpha)\right)_n\right)$$

and

$$Z_n^\alpha(x;k) = \frac{\Gamma(\alpha+kn+1)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{x^{kr}}{\Gamma(\alpha+kr+1)}$$

where $\alpha > -1$ and $k \in \mathbb{Z}^+$. These polynomials are biorthogonal with respect to weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$. Actually, it is hold that

$$\int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k)Y_m^\alpha(x;k)dx = \begin{cases} 0 & ; \quad n \neq m \\ \neq 0 & ; \quad n = m \end{cases}. \tag{1.1}$$

Furthermore, $Y_n^\alpha(x;k)$ is a polynomial with respect to x and $Z_n^\alpha(x;k)$ is a polynomial with respect to x^k. As equivalent to (1.1) biorthogonal condition, the following orthogonality properties can be

$$\int_0^\infty x^\alpha e^{-x} Z_n^\alpha(x;k)x^i dx = \begin{cases} 0 & ; \quad i = 0,1,2,...,n-1 \\ \neq 0 & ; \quad i = n \end{cases} \tag{1.2}$$

$$\int_0^\infty x^\alpha e^{-x} Y_n^\alpha(x;k)x^{ki} dx = \begin{cases} 0 & ; \quad i = 0,1,2,...,n-1 \\ \neq 0 & ; \quad i = n \end{cases}. \tag{1.3}$$

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For $k = 1$, conditions (1.1),(1.2) and (1.3) reduces to the orthogonality requirement satisfied by the generalized Laguerre polynomials.

In the recent papers, matrix polynomials have significant emergent. Some results in the theory of classical orthogonal polynomials and special functions have been extended to orthogonal matrix polynomials and special matrix functions, see [2, 3, 5, 6, 8]. In [7], these matrix polynomials are orthogonal as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. Laguerre, Hermite and Gegenbauer matrix polynomials have been introduced and studied in [3, 8, 9] for matrices in $\mathbb{C}^{N\times N}$. Our main aim in this paper is to define and to prove new properties for the pair of Konhauser matrix polynomials.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus in [4], it follows that:

$$f(A)g(A) = g(A)f(A). \tag{1.4}$$

Hence, if $B \in \mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$. Furthermore the identity matrix and the zero matrix of $\mathbb{C}^{N\times N}$ will be denoted by I and $\mathbf{0}$, respectively.

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying $(-k) \notin \sigma(A)$ for $k \in \mathbb{Z}^+$ and λ be a complex number whose real part is positive. Then the Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ are defined by [8]:

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!}(A+I)_n\left[(A+I)_k\right]^{-1}(\lambda x)^k \quad , \quad n \geq 0. \tag{1.5}$$

Such matrix polynomials have the following generating matrix function:

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n = (1-t)^{-A-I}e^{-\frac{\lambda x t}{1-t}} \; ; \; x \in \mathbb{C}\;, t \in \mathbb{C},\; |t| < 1. \tag{1.6}$$

2. The definition of couple Konhauser matrix polynomials

In this section, we define the pair of Konhauser matrix polynomials with the help of generating matrix functions. Let us suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying condition

$$Re(\mu) > -1 \text{ for every } \mu \in \sigma(A) \tag{2.1}$$

and λ is complex number with $\text{Re}(\lambda) > 0$. Let us consider the function

$$F(x,w) = (1-w)^{-\frac{1}{k}(A+I)} \exp\left[-\lambda x\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right] \tag{2.2}$$

defined for complex values of x and w with $|w|<1$. Note that $F(x,w)$ be holomorphic matrix function in $|w|<1$. Therefore, we can write

$$F(x,w) = \sum_{n=0}^{\infty} Y_n^{(A,\lambda)}(x;k) w^n \tag{2.3}$$

at $|w|<1$. By Taylor expansion at $|w|<1$ and binomial expansion, we get

$$\begin{aligned} F(x,w) &= (1-w)^{-\frac{1}{k}(A+I)} \sum_{r=0}^{\infty} \frac{\left(\lambda x\left\{1-(1-w)^{-\frac{1}{k}}\right\}\right)^r}{r!} \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(\lambda x)^r}{r!}(-1)^s \binom{r}{s} (1-w)^{-\frac{1}{k}((s+1)I+A)} \\ &= \sum_{r=0}^{\infty} \frac{(\lambda x)^r}{r!} \sum_{s=0}^{r}(-1)^s\binom{r}{s} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{1}{k}((s+1)I+A)\right)_n}{n!} w^n\right\} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n} \frac{(\lambda x)^r}{r!}\left(\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_n\right)\right\}\frac{w^n}{n!}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) and by identification of the coefficients of w^n, we obtain explicit expression for $Y_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials which is defined

$$Y_n^{(A,\lambda)}(x;k) = \frac{1}{n!}\sum_{r=0}^{n}\frac{(\lambda x)^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_n. \tag{2.5}$$

Let us suppose that A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying conditions

$$\left.\begin{array}{c} Re(\mu) > -1 \text{ for every } \mu \in \sigma(A), \\ AB = BA, \end{array}\right\} \tag{2.6}$$

and λ is complex number with $\text{Re}(\lambda) > 0$. Let us consider the matrix function

$$F(x,t) = (1-t)^{-B}\,{}_1F_k\left[\begin{array}{c} B \\ \\ \frac{1}{k}(A+I)\ ;\ldots;\ \frac{1}{k}(A+kI) \end{array};\frac{-t(\lambda x)^k}{(1-t)k^k}\right] \tag{2.7}$$

where ${}_1F_k$ is defined as

$$ {}_1F_k\left[\begin{array}{ccc} B & & \\ & & ;x \\ A_1 & ;\ldots; & A_k \end{array}\right] = \sum_{n=0}^{\infty}\frac{(B)_n}{n!}\left[(A_k)_n\right]^{-1}\ldots\left[(A_1)_n\right]^{-1}x^n $$

for $A_1,\ldots,A_k$ and B are matrices in $\mathbb{C}^{N\times N}$ satisfying condition A_i+sI is invertible for $s\geq 0$, $i=1,\ldots,k$ and $|t|<1$, $\left|\frac{-t(\lambda x)^k}{(1-t)k^k}\right|<1$. Note that $F(x,t)$ be holomorphic function in domain $|t|<1$ and $\left|\frac{-t(\lambda x)^k}{(1-t)k^k}\right|<1$. Therefore, we can write

$$ F(x,t)=\sum_{n=0}^{\infty}(B)_n Z_n^{(A,\lambda)}(x;k)\left[(A+I)_{kn}\right]^{-1}t^n \tag{2.8} $$

at the domain. By (2.7), we get

$$ F(x,t)= $$

$$ (1-t)^{-B}\sum_{r=0}^{\infty}\frac{(B)_r}{r!}\left(\frac{1}{k}(A+kI)\right)_r^{-1}\ldots\left(\frac{1}{k}(A+I)\right)_r^{-1}\left[\frac{-t(\lambda x)^k}{(1-t)k^k}\right]^r. \tag{2.9} $$

From Pochhammer symbol, we have

$$ k^{-kr}\left(\frac{1}{k}(A+I)\right)_r^{-1}\ldots\left(\frac{1}{k}(A+kI)\right)_r^{-1}=\Gamma(A+I)\Gamma^{-1}(A+(kr+1)I). \tag{2.10} $$

Substituting (2.10) in (2.9) and using Taylor series of $(1-t)^{-B-rI}$, it follows that

$$ \begin{aligned} F(x,t) &= (1-t)^{-B}\sum_{r=0}^{\infty}\frac{(B)_r}{r!}\Gamma(A+I)\Gamma^{-1}(A+(kr+1)I)\left[\frac{-t(\lambda x)^k}{(1-t)k^k}\right]^r \\ &= \Gamma(A+I)\sum_{r=0}^{\infty}\Big\{\frac{(B)_r}{r!}\Gamma^{-1}(A+(kr+1)I) \\ &\quad\times(-1)^r t^r(\lambda x)^{kr}(1-t)^{-B-rI}\Big\} \\ &= \Gamma(A+I)\sum_{r=0}^{\infty}\Big\{\frac{(B)_r}{r!}\Gamma^{-1}(A+(kr+1)I)(-1)^r t^r(\lambda x)^{kr} \\ &\quad\times\sum_{n=0}^{\infty}\frac{(B+rI)_n}{n!}t^n\Big\}. \end{aligned} \tag{2.11} $$

Using following equation

$$ (B)_r(B+rI)_{n-r}=(B)_n\ , $$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(B)_n Z_n^{(A,\lambda)}(x;k)\left[(A+I)_{kn}\right]^{-1}t^n \\
&= \Gamma(A+I)\sum_{n=0}^{\infty}\left\{\frac{(B)_n}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\Gamma^{-1}(A+(kr+1)I)(\lambda x)^{kr}\right\}t^n.
\end{aligned}$$

By identification of the coefficients of t^n, we obtain explicit expression for $Z_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials which is defined

$$Z_n^{(A,\lambda)}(x;k)=\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\Gamma^{-1}(A+(kr+1)I)(\lambda x)^{kr}. \tag{2.12}$$

3. Biorthogonal property of the pair of Konhauser matrix polynomials

For the pair of Konhauser matrix polynomials, we will discuss the following integral

$$J_{m,n}=\int_0^{\infty}x^A e^{-\lambda x}Z_n^{(A,\lambda)}(x;k)Y_m^{(A,\lambda)}(x;k)dx. \tag{3.1}$$

Now, we try to evaluate the integral in (3.1). Substituting (2.5) and (2.12) in (3.1), we have

$$\begin{aligned}
J_{m,n} &= \int_0^{\infty}\left\{x^A e^{-\lambda x}\frac{\Gamma(A+(kn+1)I)}{n!}\right. \\
&\quad\times\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj} \\
&\quad\left.\times\sum_{r=0}^{m}\frac{(\lambda x)^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_m\right\}dx \\
&= \frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I) \\
&\quad\times\frac{1}{m!}\sum_{r=0}^{m}\frac{1}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_m \\
&\quad\times(\lambda)^{(kj+r)I}\int_0^{\infty}x^{A+(kj+r)I}e^{-\lambda x}dx.
\end{aligned}$$

By using Gamma matrix function from [6], we can write

$$J_{m,n} = (\lambda)^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!\,m!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I)$$

$$\times\sum_{r=0}^{m}\frac{\Gamma(A+(kj+r+1)I)}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_m. \quad (3.2)$$

Let $f(t)$ be a polynomial of degree m. By using the method in [1], we have

$$f(t) = \sum_{r=0}^{m}\binom{t}{r}\Delta^r f(0),\ \Delta^r f(0) = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}f(s)$$

or

$$f(t) = \sum_{r=0}^{m}\frac{(-t)_r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}f(s). \quad (3.3)$$

Taking the following matrix polynomials

$$f(t) = \left(\frac{1}{k}((t+1)I+A)\right)_m$$

in (3.3), we can write

$$\left(\frac{1}{k}((t+1)I+A)\right)_m = \sum_{r=0}^{m}\frac{(-t)_r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_m.$$

Also for $t = -(kj+1)I - A$, we get

$$(-jI)_m = \sum_{r=0}^{m}\frac{((kj+1)I+A)_r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_m.$$

By using Pochhammer symbol, it follows that

$$((kj+1)I+A)_r = \Gamma((kj+r+1)I+A)\Gamma^{-1}((kj+1)I+A).$$

By latest equation and (1.4), we have

$$\begin{aligned}(-jI)_m &= \sum_{r=0}^{m}\frac{\Gamma((kj+r+1)I+A)\Gamma^{-1}((kj+1)I+A)}{r!}\\ &\quad\times\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}((s+1)I+A)\right)_m. \quad (3.4)\end{aligned}$$

Substituting (3.4) in (3.2), we can write

$$\begin{aligned} J_{m,n} &= (\lambda)^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(-jI)_m}{m!} \\ &= (-1)^m(\lambda)^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j}{m} \\ &= (\lambda)^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\binom{n}{m}(1-1)^{n-m} \\ &= (\lambda)^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\delta_{n,m}. \end{aligned}$$

Thus, this equation show that $Z_n^{(A,\lambda)}(x;k)$ and $Y_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials are biorthogonal with respect to weight matrix function $x^A e^{-\lambda x}$. For $k=1$, we meet orthogonality properties of $L_n^{(A,\lambda)}(x)$ Laguerre matrix polynomials (see [8]).

4. $Z_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials

Now, we show that $Z_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials are orthogonal with respect to (λx) basic polynomial of $Y_n^{(A,\lambda)}(x;k)$. It is hold that

$$\int_0^\infty x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)(\lambda x)^i dx = \begin{cases} 0 & ; \quad i=0,1,2,...,n-1 \\ \neq 0 & ; \quad i=n \end{cases}. \tag{4.1}$$

Substituting (2.12) in left-hand side of (4.1), we get

$$\begin{aligned} &\int_0^\infty x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)(\lambda x)^i dx \\ &= \frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I)(\lambda)^{(kj+i)I} \\ &\quad \times\int_0^\infty x^{A+(kj+1)I}e^{-\lambda x}dx \\ &= \lambda^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!} \\ &\quad \times\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I)\Gamma(A+(kj+i+1)I). \end{aligned}$$

Furthermore, we can write

$$\frac{d^i}{dx^i}\left[x^{(kj+i)I+A}\right]\Big|_{x=1} = \Gamma(A+(kj+i+1)I)\Gamma^{-1}(A+(kj+1)I). \quad (4.2)$$

By using (4.2), we obtain

$$\begin{aligned}
&\int_0^\infty x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)(\lambda x)^i dx \\
&= \lambda^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left\{\frac{d^i}{dx^i}\left[x^{(kj+i)I+A}\right]\Big|_{x=1}\right\} \\
&= \lambda^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\frac{d^i}{dx^i}\left\{x^{A+iI}\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^{kj}\right\}\Bigg|_{x=1} \\
&= \lambda^{-A-I}\frac{\Gamma(A+(kn+1)I)}{n!}\frac{d^i}{dx^i}\left\{x^{A+iI}(1-x^k)^n\right\}\big|_{x=1} \qquad (4.3)
\end{aligned}$$

which is zero matrix for $i = 0, 1, ..., (n-1)$, but is different from zero matrix for $i = n$. Therefore, the matrix polynomials (2.12) satisfy orthogonality condition (4.1).

Next we obtain recurrence relations and matrix differential equation of $(k+1)$ order satisfied by $Z_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials. Considering the difference

$$nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k)$$

from (2.12), we have

$$\begin{aligned}
&nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k) \\
&= nk\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj} \\
&\quad -k((kn-k+1)I+A)_k\frac{\Gamma(A+(kn-k+1)I)}{(n-1)!} \\
&\quad \times\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj}. \qquad (4.4)
\end{aligned}$$

By Pochhammer symbol, we get

$$((kn-k+1)I+A)_k = \Gamma^{-1}(A+(kn-k+1)I)\Gamma(A+(kn+1)I). \quad (4.5)$$

Combining (4.4) and (4.5), we can write

$$
\begin{aligned}
& nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k) \\
= \;& k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\sum_{j=0}^{n}\left\{(-1)^j\left(\binom{n}{j}-\binom{n-1}{j}\right)\right. \\
& \left.\times\,\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj}\right\} \\
= \;& x\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma^{-1}(A+(kj+1)I)\,(\lambda kj)\,(\lambda x)^{kj-1} \\
= \;& x\frac{d}{dx}Z_n^{(A,\lambda)}(x;k).
\end{aligned}
$$

Thus, we have following matrix recurrence relation:

$$
nkZ_n^{(A,\lambda)}(x;k)-k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k) = x\frac{d}{dx}Z_n^{(A,\lambda)}(x;k). \quad (4.6)
$$

Also, we get

$$
\begin{aligned}
& nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k) \\
= \;& k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\sum_{j=0}^{n}\left\{(-1)^j\left(\binom{n}{j}-\binom{n-1}{j}\right)\right. \\
& \left.\times\,\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj}\right\}. \qquad (4.7)
\end{aligned}
$$

Using the relation

$$
\binom{n}{j}-\binom{n-1}{j}=\binom{n-1}{j-1},
$$

we have

$$
\begin{aligned}
& nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k) \\
= \;& k(\lambda x)^k\frac{\Gamma(A+kI+(k(n-1)+1)I)}{(n-1)!} \\
& \times\sum_{j=1}^{n}(-1)^j\tbinom{n-1}{j-1}\Gamma^{-1}(A+kI+(k(j-1)+1)I)(\lambda x)^{k(j-1)}. \quad (4.8)
\end{aligned}
$$

Taking $(j+1)$ instead of j, (4.8) can be written

$$
\begin{aligned}
& nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k) \\
= \;& -k(\lambda x)^k\frac{\Gamma(A+kI+(k(n-1)+1)I)}{(n-1)!} \\
& \times\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\Gamma^{-1}(A+kI+(kj+1)I)(\lambda x)^{kj} \\
= \;& -k(\lambda x)^k Z_{n-1}^{(A+kI,\lambda)}(x;k). \qquad (4.9)
\end{aligned}
$$

From (4.6) and (4.9), we have

$$\frac{d}{dx} Z_n^{(A,\lambda)}(x;k) = -k\lambda^k x^{k-1} Z_{n-1}^{(A+kI,\lambda)}(x;k). \tag{4.10}$$

For $k = 1$, (4.6),(4.9) and (4.10) reduces to recurrence relations for the Laguerre matrix polynomials.

With the help of (4.7), we try to find matrix differential equation satisfied by $Z_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials. By using the relation

$$nkZ_n^{(A,\lambda)}(x;k) - k((kn-k+1)I+A)_k Z_{n-1}^{(A,\lambda)}(x;k)$$

$$= k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\sum_{j=0}^{n}(-1)^j\left(\binom{n}{j}-\binom{n-1}{j}\right)\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj}$$

and (4.6), we get

$$x\frac{d}{dx}Z_n^{(A,\lambda)}(x;k) =$$

$$k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1}\Gamma^{-1}(A+(kj+1)I)(\lambda x)^{kj}. \tag{4.11}$$

Multiplying (4.11) by x^A and taking the k-th derivative with respect to x, we have

$$\begin{aligned}
&\frac{d^k}{dx^k}\left[x^{A+I}\frac{d}{dx}Z_n^{(A,\lambda)}(x;k)\right]\\
&= \lambda^{-A}k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\\
&\quad\times\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1}\Gamma^{-1}(A+(kj+1)I)\frac{d^k}{dx^k}(\lambda x)^{kjI+A}\\
&= \lambda^{-A+kI}k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\\
&\quad\times\sum_{j=1}^{n}\left\{(-1)^j\binom{n-1}{j-1}\Gamma^{-1}(A+(kj+1)I)(kjI+A)\right.\\
&\quad\left.\times((kj-1)I+A)...((kj-k+1)I+A)(\lambda x)^{k(j-1)I+A}\right\}.
\end{aligned} \tag{4.12}$$

From Pochhammer symbol, we get

$$\begin{aligned}
&\Gamma^{-1}(A+(kj+1)I)(kjI+A)((kj-1)I+A)...((kj-k+1)I+A)\\
=\ &\Gamma^{-1}(A+(k(j-1)+1)I).
\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we can write

$$\begin{aligned}&\frac{d^k}{dx^k}\left[x^{A+I}\frac{d}{dx}Z_n^{(A,\lambda)}(x;k)\right]\\ =\ & \lambda^{-A+kI}k\frac{\Gamma(A+(kn+1)I)}{(n-1)!}\\ &\times\sum_{j=1}^{n}(-1)^j\binom{n-1}{j-1}\Gamma^{-1}(A+(k(j-1)+1)I)(\lambda x)^{k(j-1)I+A}.\end{aligned}$$

Using the relation

$$\Gamma((kn+1)I+A)=((kn-k+1)I+A)_k\,\Gamma((kn-k+1)I+A),$$

we have

$$\begin{aligned}&\frac{d^k}{dx^k}\left[x^{A+I}\frac{d}{dx}Z_n^{(A,\lambda)}(x;k)\right]\\ =\ & -\lambda^{kI}k\,((kn-k+1)I+A)_k\,x^A\frac{\Gamma((k(n-1)+1)I+A)}{(n-1)!}\\ &\times\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\Gamma^{-1}(A+(kj+1))I)(\lambda x)^{kj}\\ =\ & -\lambda^{kI}k\,((kn-k+1)I+A)_k\,x^A Z_{n-1}^{(A,\lambda)}(x;k).\end{aligned}\tag{4.14}$$

From (4.6) and (4.14), we find

$$\frac{d^k}{dx^k}\left[x^{A+I}\frac{d}{dx}Z_n^{(A,\lambda)}(x;k)\right]=\lambda^{kI}x^{A+I}\frac{d}{dx}Z_n^{(A,\lambda)}(x;k)-nk\lambda^{kI}x^AZ_n^{(A,\lambda)}(x;k)\tag{4.15}$$

Thus, matrix differential equation of $(k+1)$ order satisfied by $Z_n^{(A,\lambda)}(x;k)$ Konhauser matrix polynomials is found. For $k=1$, (4.15) reduces to the matrix differential equation for the $L_n^{(A,\lambda)}(x)$ Laguerre matrix polynomials [8]. That is, it is hold that

$$x\frac{d^2}{dx^2}L_n^{(A,\lambda)}(x)+(A+I-\lambda xI)\frac{d}{dx}L_n^{(A,\lambda)}(x)+\lambda nL_n^{(A,\lambda)}(x)=0.$$

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