Bordered magic squares: Elements for a comprehensive approach *

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Abstract

General methods for the construction of magic squares of any order have been searched for centuries. Several 'standard strategies' have been found for this purpose, such as the 'knight movement', or the construction of bordered magic squares, which played an important role in the development of general methods.

What we try to do here is to give a general and comprehensive approach to the construction of magic borders, capable of assuming methods produced in the past as particular cases. This general approach consists of a transformation of the problem of constructing magic borders to a simpler - almost trivial - form. In the first section, we give some definitions and notation. The second section consists of the exposition and proof of our method for the different cases that appear (Theorems 1 and 2). As an application of this method, in the third section we caracterize magic borders of even order, giving therefore a first general result for bordered magic squares.

Although methods for the construction of bordered magic squares have always been presented as individual succesful attempts to solve the problem, we will see that a common pattern underlies the fundamental mechanisms that lead to the construction of such squares. This approach provides techniques for constructing many magic bordered squares of any order, which is a first step to construct all of them,and finally know how many bordered squares are for any order. These may be the first elements of a general theory on bordered magic squares.

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1 Previous considerations

Let us begin with some useful definitions:

Definition 1 *Magic square of order n: An $n \times n$ array containing the integers 1 through n^2, each one once, arranged in such a way that all rows, columns and the two main diagonals have the same sum $S_n = \frac{n}{2}(n^2+1)$, which is called its magic constant.*

Definition 2 *Border of a magic square: The set consisting of its first and last row and column.*

Definition 3 *Complementaries: In a magic square of order n, a pair of numbers a, b so that $a + b = n^2 + 1$ we will denote $\bar{\alpha}$ the complementary of α.*

Definition 4 *Magic border: A border that verifies that every line (row or column) sums S_n and that every two opposite numbers are complementaries. Thus, a magic border consists of two subsets, one containing the complementaries of the other, which will be denoted A and A' respectively.*

2 A method for the construction of magic borders

Since the beginning of the history of bordered magic squares, many particular methods for their construction have been produced, references on this topic can be seen at [1, 2, 3, 4, 5, 6, 7]. What we try to do here is to give a general approach to these methods.

2.1 Exposition of the general method for the construction of Magic Borders

As we said before, the method we propose consists of a simplification of the problem of building magic borders. We will put the problem in terms that will lead us to an easy way of resolution.

First of all, we name the numbers in the border as follows:

$$\begin{array}{cccccc} v & b_1 & b_2 & \ldots & b_n & w \\ c_1 & & & & & \bar{c}_1 \\ c_2 & & & & & \bar{c}_2 \\ \ldots & & & & & \ldots \\ c_n & & & & & \bar{c}_n \\ \bar{w} & \bar{b}_1 & \bar{b}_2 & \ldots & \bar{b}_n & \bar{v} \end{array}$$

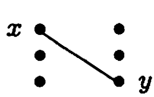
where $\bar{\alpha}$ is the *complementary* of α. Note that $\alpha + \bar{\alpha} = (n+2)^2 + 1 = n^2 + 4n + 5$, which is the difference between the magic constant of the big square and that of the inner one (The magic constant of a magic square of order n is $S_n = \frac{n}{2}(n^2+1)$. So, the constant of the big square is $S_{n+2} = \frac{(n+2)}{2}((n+2)^2+1)$ and the constant of the inner square —with it's numbers incremented in $2n+2$— is $S_n + n(2n+2)$. Hence, $S_{n+2} - S_n - n(2n+2) = n^2 + 4n + 5$). So, we see that every row, column, and diagonal sums the same, the magic constant, except the first and last row and column. We just need then to verify the magic condition placing the numbers we have: $\{1, \ldots, 2n+2\} \cup \{n^2+2n+3, \ldots, (n+2)^2\}$ so that the following equalities stand

$$\sum_{i=1}^{n} b_i + v + w = \sum_{i=1}^{n} \bar{b}_i + \bar{v} + \bar{w} = \sum_{i=1}^{n} b_c + v + \bar{w} = \sum_{i=1}^{n} \bar{c}_i + \bar{v} + w = S_{n+2}$$

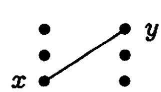
To make the placement of these numbers easier, we arrange them in two columns, as follows

1 •		• $(n+2)^2$
2 •		• $(n+2)^2 - 1$
⋮		⋮
$2n+2$ •		• n^2+2n+3

With this arrangement, if we match two numbers with a line, say x and y, we can assign to that matching line, the number that expresses the difference between the sum of those two numbers and the sum of two complementary numbers, i.e. $x + y - (n^2 + 4n + 5)$. So, for example



equals -2



equals $+2$

We denote this number by d_{xy}.

Next, we will show different procedures for the construction of the magic border if the square has even or odd order.

— Even order:

If the order of the square is even, we proceed as follows:

We choose two numbers v and w to be placed in the corners [1]. After this, we take n numbers from one column and the other alternately, beginning in the opposite column to that containing the number named v, and taking

[1] In section 3 we will discuss what numbers can or cannot be chosen to be v and w

care not to have chosen any pair of complementary numbers. These will be our $b_i's$. The way of choosing these $b_i's$ has to be one in which we have some permutation σ of $\{1, \ldots, n\}$ and a way to put them in pairs,

$$\beta = \{(b_{\sigma(1)}, v), (b_{\sigma(2)}, b_{\sigma(3)}), \ldots, (b_{\sigma(n)}, w)\}$$

satisfying

$$\sum_{(x,y)\in\beta} d_{xy} = 0.$$

After this, we have to choose n numbers, from one column and the other alternately, in such a way that not two of them are complementary. These will be the $c_i's$ and they have to be so that there exists some permutation τ of $\{1, \ldots, n\}$ and a way to put them in pairs,

$$\gamma = \{(c_{\tau(1)}, c_{\tau(2)}), \ldots, (c_{\tau(n-1)}, c_{\tau(n)})\}$$

satisfying

$$\sum_{(x,y)\in\gamma} d_{xy} = -d_{v\bar{w}}.$$

Making such an election of the numbers,we assure that putting $b_i's$ in the top (or bottom) row, except the corners, in which we will put v and w; and putting $c_i's$ in the column between v and $\bar{w}$ the border obtained is magic, regardless the order in which $b_i's$ and $c_i's$ have been placed in their respective rows or columns.

— Odd order:

If the order of the square is odd, the procedure is a little different. First, we choose a number for one of the squares, v, and denote $d_v := v - \frac{(n+2)^2+1}{2}$. After that, as we have done in the even case, we choose $n+1$ numbers (which will be our $b_i's$ and w) from one column and the other, alternately, in such a way that we can match them in pairs, being $\beta = \{(b_{\sigma(1)}, v), (b_{\sigma(2)}, b_{\sigma(3)}), \ldots, (b_{\sigma(n)}, w)\}$ satisfying

$$\sum_{(x,y)\in\beta} d_{xy} + d_v = 0.$$

After this, we have to choose n numbers, from one column and the other alternately, in such a way that not two of them are complementary. These will be the $c_i's$ and they have to be so that there exists any way to put them in pairs, $\gamma = \{(c_{\tau(1)}, c_{\tau(2)}), \ldots, (c_{\tau(n-1)}, c_{\tau(n)})\}$ satisfying

$$\sum_{(x,y)\in\gamma} d_{xy} + d_v = 0.$$

Like in the precedent case, we assert that the border obtained is magic, regardless the order in which $b_i's$ and $c_i's$ have been placed in their respective rows or columns.

The following theorem expresses the fact that this choice of the numbers leads to a magic border.

Theorem 1 *Let* $A = \{v, w\} \cup \{b_1, \ldots, b_n\} \cup \{c_1, \ldots, c_n\}$ *and* $B = A \cup A'$ *a border, built as we have just seen, then* B *is a magic border.*

Proof

—Case n even:

Each row and column of the border must sum the magic constant of a square of order $n+2$, that is $S_{n+2} = \frac{(n+2)}{2}((n+2)^2+1)$; and we know that any pair of complementary numbers sum $(n+2)^2+1$, so in each row and column of the border we have the same sum as if we had $\frac{(n+2)}{2}$ pairs of complementary numbers. With the election done, we have

$$\sum_{(x,y)\in\beta} d_{xy} = 0$$

so the total difference between these $\frac{(n+2)}{2}$ pairs and $\frac{(n+2)}{2}$ pairs of complementary numbers is zero, so for the rows of the border, the magic condition stands (we have seen it just for one row, but as the other is formed by the complementary numbers of our $b_i's$ and those of v, w the condition stands for the other row).

Now we have one column formed by $\{c_1, \ldots, c_n\} \cup \{v, \bar{w}\}$ and we have

$$\sum_{(x,y)\in\{c_1,\ldots,c_n\}} d_{xy} = -d_{v\bar{w}} \Rightarrow \sum_{(x,y)\in\{c_1,\ldots,c_n\}} d_{xy} + d_{v\bar{w}} = 0$$

so columns also verify magic condition and hence, the border is magic.

—Case n odd:

Now we must have in each row and column of the border the same sum as if we had $\frac{(n+1)}{2}$ pairs of complementary numbers, plus a half of the sum of one of these pairs, that is $\frac{(n+2)^2+1}{2}$. With the election done, we have

$$v + \sum_{i=1}^{n} b_i + w = v + \frac{(n+1)}{2}((n+2)^2+1) + \sum_{(x,y)\in\beta} d_{xy}$$

$$= v + \frac{(n+1)}{2}((n+2)^2+1) - d_v = v + \frac{(n+1)}{2}((n+2)^2+1) - v + \frac{(n+2)^2+1}{2}$$

$$= \frac{(n+1)}{2}((n+2)^2+1) + \frac{((n+2)^2+1)}{2}$$

so for the rows of the border, the magic condition stands.

Now we have one column formed by $\{c_1, \ldots, c_n\} \cup \{v, \bar{w}\}$ and we have

$$v + \sum_{i=1}^{n} c_i + \bar{w} = v + \frac{(n+1)}{2}((n+2)^2+1) + \sum_{(x,y)\in\gamma} d_{xy}$$

$$= v + \frac{(n+1)}{2}((n+2)^2+1) - d_v = v + \frac{(n+1)}{2}((n+2)^2+1) - v + \frac{(n+2)^2+1}{2}$$

$$= \frac{(n+1)}{2}((n+2)^2+1) + \frac{((n+2)^2+1)}{2},$$

so columns also verify magic condition and, hence, the border is magic. □

Now we know that the choice we presented before lead us to a magic border. The next question is if that kind of choice is possible for any n. If so, we have a method for constructing magic squares of any order. The answer of this question is in the next theorem:

Theorem 2 *For any positive integer $n > 2$, it is possible to make a choice of the kind we have already seen , in order to build a magic border.*

Proof

The proof will consist on a constructive method. We will give an actual example of choice satisfying the needed properties. Some generalizations of this method are possible.

—Case n even:

The way to prove this is to give (in a simple way) some election (among the various valid) that works for any n even. We will proceed giving one for $n = 4k, k \geq 1$ and other for $n = 2(2k+1), k \geq 1$.

· Case $n = 4k, k \geq 1$

We have the following arrangement of the numbers in two columns:

$$\begin{array}{rccl}
1 & \bullet & \bullet & (n+2)^2 \\
2 & \bullet & \bullet & (n+2)^2-1 \\
3 & \bullet & \bullet & (n+2)^2-2 \\
4 & \bullet & \bullet & (n+2)^2-3 \\
5 & \bullet & \bullet & (n+2)^2-4 \\
6 & \bullet & \bullet & (n+2)^2-5 \\
7 & \bullet & \bullet & (n+2)^2-6 \\
8 & \bullet & \bullet & (n+2)^2-7 \\
9 & \bullet & \bullet & (n+2)^2-8 \\
10 & \bullet & \bullet & (n+2)^2-9 \\
 & \vdots & \vdots & \\
2n+2 & \bullet & \bullet & n^2+2n+3
\end{array}$$

where $(2n+2) \in \{10, 18, 26, \ldots\}$.

Now, we make our choice as follows: In a first part of the choice, we take:

$$v = (n+2)^2 - 1, \qquad \bar{w} = 5,$$

$$b_1 = 1, \quad b_2 = 3, \quad b_3 = (n+2)^2 - 3, \quad b_4 = (n+2)^2 - 4,$$

$$c_1 = 6, \quad c_2 = (n+2)^2 - 7, \quad c_3 = 9, \quad c_4 = (n+2)^2 - 9.$$

Once this part of the choice is made (and we have finished if $n = 4$), we have:

$$d_{v\bar{w}} = (n+2)^2 - 1 + 5 - (n^2 + 4n + 5) = 3,$$

$$d_{vb_1} + d_{b_2b_3} + d_{b_4w} = -1 - 1 + 2 = 0,$$

$$d_{c_1c_2} + d_{c_3c_4} = -3.$$

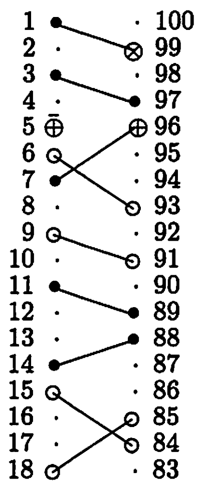
And so, the magic property of the border stands in this first part.

Now we have to place the other $2n-8$ numbers and its complementaries; $n-4$ of them will be b_i's and the other $n-4$ will be c_i's. Note that $n-4 = 4k-4 = 4(k-1)$, so we can divide these numbers into $k-1$ sets of 4 numbers that we must place in a way in which the magic property keeps standing. An easy way of doing it is choosing number four by four in one of the following dispositions

or equivalently

so, if we choose this way the remainding $b_i's$ and $c_i's$, we obtain a magic border (of course, once $b_i's$ have been chosen, we can put them in the border in any order, and the same stands for the $c_i's$, what makes the two dispositions above equivalent).

Example 1 *For example, we can form a magic border for* $n = 8$ *with the numbers* $1, 2, \dots, 18$ *and* $83, 84, \dots, 100$*:*



where:
$\bullet \to b_i$
$\circ \to c_i$
$\otimes \to v$
$\oplus \to w$
$\bar{\oplus} \to \bar{w}$

That produces the border

99	1	3	97	7	11	89	14	88	96
6									95
93									8
9									92
91									10
15									86
85									16
84									17
18									83
5	100	98	4	94	90	12	87	13	2

· Case $n = 2(2k+1), k \geq 1$.

In this case, we have a similar arrangement of the numbers in two columns so, making analogous considerations to those of the precedent case, and noting that $2n+2 \in \{14, 22, \ldots\}$, we can make the following first part of the choice:

$$v = 1, \qquad \bar{w} = (n+2)^2 - 3,$$

$$b_1 = (n+2)^2 - 1, \qquad b_2 = (n+2)^2 - 2, \qquad b_3 = 5,$$

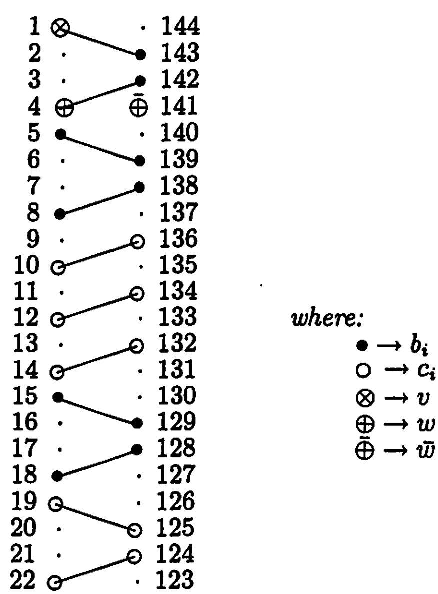
$$b_4 = (n+2)^2 - 5, \qquad b_5 = (n+2)^2 - 6, \qquad b_6 = 8,$$

$$c_1 = 10, \qquad c_2 = (n+2)^2 - 8, \qquad c_3 = (n+2)^2 - 10,$$

$$c_4 = 12, \qquad c_5 = (n+2)^2 - 12, \qquad c_6 = 14,$$

And for the second part, we proceed as before.

Example 2 *As an example, we can form a magic border for* $n = 10$ *with the numbers* $1, 2, \ldots, 22$ *and* $123, 124, \ldots, 144$*:*



That produces the border

1	143	142	5	139	138	8	15	129	128	18	4
136											9
10											135
134											11
12											133
132											13
14											131
19											126
125											20
124											21
22											123
141	2	3	140	6	7	137	130	16	17	127	144

—Case n odd:

In this case we shall proceed in a somewhat different way. In this case, the method is general for all n odd, but there are some details to observe. As in the even case, there are many ways to fill the border, among which we choose a simple one. We part from an arrangement of the $n+2$ pairs of complementary numbers as seen before. But now, we divide this columns in three parts:

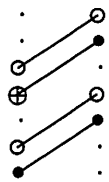
We call A to the first part, containing the first $n-3$ pairs, that is $1 \rightarrow (n+2)^2$ to $n-3 \rightarrow n^2+3n+8$.

Part B contains seven pairs, from $n-2 \rightarrow n^2+3n+7$ to $n+4 \rightarrow n^2+3n+1$.

Finally, part C contains the remaining $n-2$ pairs.

Once we have these three sets, we select a number for the corner named v. We shall select the number $(n+7)$ and put it in v (we must note that this $(n+7)$ must be between $(n+5)$ and $(2n+2)$, and this happens if $n \geq 5$ so we must consider the case $n=3$ special. The reader can construct a special method for $n=3$ easily from the general method for n odd). After this, we name $b_i's$ or $c_i's$ the numbers in the first column of part C and in the second column of part A. Beginning with part C we have two numbers above $(n+7)$, that is $(n+5)$ and $(n+6)$; we can name one of them b_i for some i, and the other c_j for some j, we match them with $(n+2)^2$ and $(n+2)^2-1$ respectively (labeled with the same letter of its pair). So we have that each of these pairs contributes with $(n+4)$ to $\sum d_{xy}$. The $(n-5)$ numbers under v will be matched with the $n-5$ numbers under $(n+2)^2-1$, half of these pairs (note $(n-5)$ is even) being $b_i's$ and the other half $c_i's$ in any order. Each of these $n-5$ pairs contributes with $n+5$ to $\sum d_{xy}$.

Finally, numbers in set B are matched following this scheme:



where:
$\bullet \to b_i$
$\circ \to c_i$
$\oplus \to w$

So, with this choice of the numbers, we must have

$$\sum_{(x,y)\in\beta} d_{xy} + d_v = 0$$

$$\sum_{(x,y)\in\gamma} d_{xy} + d_v = 0$$

and we have

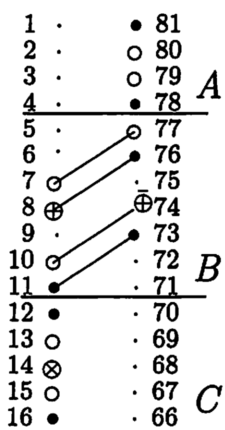
$$d_v = (n+7) - \frac{n^2+4n+5}{2} = \frac{-(n^2+2n-9)}{2},$$

$$\sum_{(x,y)\in\beta} d_{xy} = (n+4) + \frac{(n-5)(n+5)}{2} + 4 = \frac{n^2+2n-9}{2},$$

$$\sum_{(x,y)\in\gamma} d_{xy} = (n+4) + \frac{(n-5)(n+5)}{2} + 4 = \frac{n^2+2n-9}{2}.$$

Hence, our choice leads to a magic border for all n odd.

Example 3 *As an example, we can form a magic border for* $n = 7$ *with the numbers* $1, 2, \ldots, 16$ *and* $66, 67, \ldots, 81$*:*



where:
$\bullet \to b_i$
$\circ \to c_i$
$\otimes \to v$
$\oplus \to w$
$\bar{\oplus} \to \bar{w}$

That produces the border

14	81	78	12	16	76	73	11	8
80								2
79								3
13								69
15								67
77								5
7								75
10								72
74	1	4	70	66	6	9	71	68

Remark 1 *Let us insist in the fact that all the choices made in both even and odd cases were only particular ones, we have many possibilities that lead to other magic borders.*

2.2 Generalization of the method.

We can make a generalization of the method exposed in theorems 1 and 2 in three ways (apart from those transformations of the square by its symetry group):

Remark 2 *Any permutation of the b_i's and c_i's in a magic border keeps the border magic.*

We have already said that b_i's and c_i's can be placed in any order in their respective rows or columns when constructing the magic border. Of course, when we change the disposition of b_i's or c_i's we must change at the same time their respective complementary numbers so that each pair of complementary numbers are one opposite the other. As an example we can make a permutation of both b_i's and c_i's in the magic border constructed in example 1:

99	88	14	89	11	7	97	3	1	96
93									8
6									95
91									10
9									92
85									16
15									86
18									83
84									17
5	13	87	12	90	94	4	98	100	2

which is still magic. So, this method produces, if we consider this generalitation, $2 \cdot n!$ magic borders with the same four numbers in the corners.

Remark 3 *Changes in the construction of the inner square do not affect the validity of the method.*

Changes in the construction of the inner square may have or not an effect on our method. If we construct the inner square using numbers $2n+3\ldots, n^2+2n+2$ there is no effect on our method whatever method we choose to construct this square. But if we use different numbers in this construction, it will affect the construction of the magic border(for example, we can use $\{1, \ldots \frac{n^2}{2}\} \cup \{(n+2)^2 - \frac{n^2}{2} - 1, \ldots, (n+2)^2\}$). Anyway, we will have, for our border, $2n+1$ pairs of complementary numbers. So, if we arrange them in two columns as we saw before, we can apply the method in the same way, making the appropiate changes.

Remark 4 *There are many possibilities of changes in the scheme used in theorem 2 to verify the conditions of magic border that keep the magic of the border.*

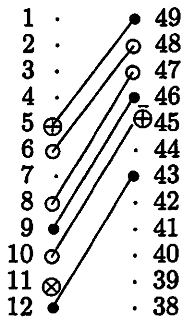
The schemes used before in the proofs of the method for each case are, as we said, only a valid choice among many correct ones. We shall now have a closer look on this statement:

In any of the cases, we can say that the method consists of two (more or less explicit) parts: in the first part we make the choice of the corners, which leads to an *unbalance situation*, and in the second part we try to *re-establish the balance*, where the rest of the numbers in the border are involved, being put by pairs of complementaries.

We have also seen in the different cases considered before two ways of solving this *re-establishment* part: In the even cases, we used a minimum number of pairs to re-establish the balance (first step of the methods) and then, we pass to a second step that holds the *inner balance* (second step of the methods). In the odd case we have seen a way to re-establish the balance little by little (which involved the pairs in A and C) having a situation that we can control, i.e. a *standard* step (scheme of the numbers in B).

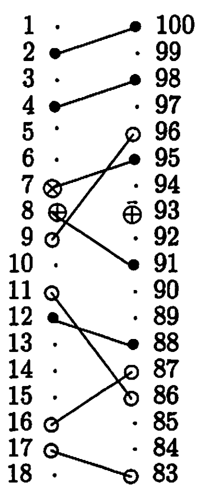
Any of these parts and steps can be changed, what leads us to different methods (which are suitable to be generalised in the sense we have seen). We shall give now a pair of examples to illustrate these affirmations.

Example 4 *This example gives a magic border of order* 5 *in which the election of the corner is also changed, and pairs are not distributed in three sets* A, B, C.



11	9	12	49	46	43	5
48						2
47						3
6						44
8						42
10						40
45	41	38	1	4	7	39

Example 5 *This second example gives a magic border of order* 8 *in which we have changed the difference between the numbers in the corners, and the distinction between the two parts of the method (in the even case).*



7	100	2	98	4	95	91	12	88	8
96									5
9									92
11									90
87									14
86									15
16									85
17									84
83									18
93	1	99	3	97	6	10	89	13	94

3 Even bordered magic squares: some results.

In this section we give answer, with the help of the method we have just exposed, and for the even case, to some questions one may pose about magic borders.

We can see, from what we said in the *generalization of the method* that a magic border depends only on the number in the corners, and it is enough to consider two contiguous corners, say the two upper corners, which we denoted v and w. So, a magic border can be represented as an element of $\mathbf{Z}^2 \times \mathbf{Z}^n \times \mathbf{Z}^n$ being the first two numbers $\{v, w\}$, the second n numbers the $b_i's$ and the last numbers the $c_i's$. We will denote by $\Omega^k_{v,w}$ the set of magic borders of order k having v and w as upper corners; Ω^k will stand for the set of magic borders of order k.

Remark 5 *If we apply the symmetry group of the square to a given border, we obtain the following eight*

$$
\begin{array}{ccccc}
v & b_1 & \dots & b_n & w \\
c_1 & & & & \bar{c}_1 \\
\dots & & & & \dots \\
c_n & & & & \bar{c}_n \\
\bar{w} & \bar{b}_1 & \dots & \bar{b}_n & \bar{v}
\end{array}
\qquad
\begin{array}{ccccc}
w & b_n & \dots & b_1 & v \\
\bar{c}_1 & & & & c_1 \\
\dots & & & & \dots \\
\bar{c}_n & & & & c_n \\
\bar{v} & \bar{b}_1 & \dots & \bar{b}_n & \bar{w}
\end{array}
\qquad
\begin{array}{ccccc}
\bar{w} & \bar{b}_1 & \dots & \bar{b}_n & \bar{v} \\
c_n & & & & \bar{c}_n \\
\dots & & & & \dots \\
c_1 & & & & \bar{c}_1 \\
v & b_1 & \dots & b_n & w
\end{array}
$$

$$
\begin{array}{ccccc}
\bar{w} & c_n & \dots & c_1 & v \\
\bar{b}_1 & & & & b_1 \\
\dots & & & & \dots \\
\bar{b}_n & & & & b_n \\
\bar{v} & \bar{c}_n & \dots & \bar{c}_1 & w
\end{array}
\qquad
\begin{array}{ccccc}
\bar{v} & \bar{b}_n & \dots & \bar{b}_1 & \bar{w} \\
\bar{c}_n & & & & c_n \\
\dots & & & & \dots \\
\bar{c}_1 & & & & c_1 \\
w & b_1 & \dots & b_n & v
\end{array}
\qquad
\begin{array}{ccccc}
w & \bar{c}_1 & \dots & \bar{c}_n & \bar{v} \\
b_n & & & & \bar{b}_n \\
\dots & & & & \dots \\
b_1 & & & & \bar{b}_1 \\
v & c_1 & \dots & c_n & \bar{w}
\end{array}
$$

$$
\begin{array}{ccccc}
\bar{v} & \bar{c}_n & \dots & \bar{c}_1 & w \\
\bar{b}_n & & & & b_n \\
\dots & & & & \dots \\
\bar{b}_1 & & & & b_1 \\
\bar{w} & c_n & \dots & c_1 & v
\end{array}
\qquad
\begin{array}{ccccc}
v & c_1 & \dots & c_n & \bar{w} \\
b_1 & & & & \bar{b}_1 \\
\dots & & & & \dots \\
b_n & & & & \bar{b}_n \\
w & \bar{c}_1 & \dots & \bar{c}_n & \bar{v}
\end{array}
$$

So if we know $\Omega^k_{v,w}$ we can obtain, permuting its elements (considered in $\mathbf{Z}^{2n+2}$) all the elements in $\Omega^k_{w,v}$, $\Omega^k_{\bar{w},\bar{v}}$, $\Omega^k_{\bar{w},v}$, $\Omega^k_{\bar{v},\bar{w}}$, $\Omega^k_{w,\bar{v}}$, $\Omega^k_{\bar{v},w}$ and $\Omega^k_{v,\bar{w}}$.

With this notation we can now set the following theorem:

Theorem 3 *Let n be a positive even number; let $v, w \in \{1, \dots, 2n+2\}$, then*

$$\Omega^n_{v,w} \neq \emptyset \iff v \text{ and } w \text{ have not the same parity.}$$

Proof:

· We shall first prove the direct statement. To do this, we will see that if v and w have the same parity then $\Omega^n_{v,w} = \emptyset$:

We give here the proof for the case $n = 4k$, $k \in \mathbf{N}$; for the other case ($n = 4k+2$, $k \in \mathbf{N}$) we can make analogous reasoning.

If $n = 4k$ then $S_n = 2k((4k)^2+1)$ is even, $S_{n+2} = (2k+1)((4k+2)^2+1)$ is odd, and $\sum_{i=1}^{2n+2} i = (2n+3)\frac{(2n+2)}{2}$ is odd. If we arrange the numbers for the border in two columns (each number opposite its complementary) then we can take, with no loose of generality v and w from the first column.

Note that $\sum_i^n b_i = S_{4k+2} - v - w$ is odd, and $\sum_i^n c_i = S_{4k+2} - v - \bar{w}$ is even.

Let x be the sum of the $b_i's$ in the first column (they are $\frac{n-2}{2}$ numbers) let y be the sum of the $b_i's$ in the second column and let x' and y' be the sum of their respective complementaries. As $\sum_i^n b_i$ is odd, x and y cannot have the same parity; and neither can x and x' nor y and y', because $x' = \frac{n-2}{2}((n+2)^2+1) - x$, $y' = \frac{n+2}{2}((n+2)^2+1) - x$ being $\frac{n-2}{2}((n+2)^2+1)$ and $\frac{n+2}{2}((n+2)^2+1)$ odd numbers.

Now, let t be the sum of the $c_i's$ in the first column (they are $\frac{n}{2}$ numbers) let s be the sum of the $c_i's$ in the second column and let t' and s' be the sum of their respective complementaries. As $\sum_i^n c_i$ is even, and so is $\frac{n}{2}((n+2)^2+1)$, we have t, t', s and s' have the same parity.

Finally, the sum of the numbers in the first column is $\sum_{i=1}^{2n+2} i$ which is odd, but this sum equals $v + w + x + y' + t + s'$ which, as we have just seen, is even, so if v and w have the same parity, then $\Omega_{v,w}^n = \emptyset$.

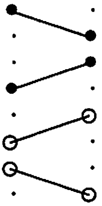
· Let us see now that if v and w have not the same parity, it is always possible to complete a magic border.

First, note that, as we have already seen, it is enough to work the following cases:

$$(1,2), \ldots, (1, 2n+2)$$
$$(2,3), \ldots, (2, 2n+1)$$
$$\cdots\cdots\cdots$$
$$(2n, 2n+1)$$
$$(2n+1, 2n+2)$$

Note also that we can complete the borders of order $n+2$ by using those of order n, in the following way:

We use the numbers 1 to $2n+2$ and their complementaries to complete the borders of order $4k$; now, for the order $4(k+1)$ we need 8 new pairs of numbers. We shall match these numbers to have two pairs of $b_i's$ and two pairs of $c_i's$; let us do the matching so that $\sum d_{xy} = 0$ for both the $b_i's$ and the $c_i's$; let us also avoid crossing lines in these matchings. We can make for instance, a matching like this:



where:
$\bullet \longrightarrow b_i$
$\circ \longrightarrow c_i$

Now, we can put this scheme at the bottom of the scheme of a border of order $4k$ with v and w as upper corners and we have a border of order $4(k+1)$ with the same upper corners. If we put the first two pairs at the top of the scheme of a magic border of order n with v and w as upper corners, and the other six pairs at the bottom, we obtain a scheme for a magic border of order $n+2$ with $v+2$ and $w+2$ as upper corners. With the same procedure, starting from an element of $\Omega^{4k}_{v,w}$ we have elements of $\Omega^{4(k+1)}_{v,w}$, $\Omega^{4(k+1)}_{v+2,w+2}$, $\Omega^{4(k+1)}_{v+4,w+4}$, $\Omega^{4(k+1)}_{v+6,w+6}$ and $\Omega^{4(k+1)}_{v+8,w+8}$.

So, starting from the magic borders of order $4k$ we can obtain elements of $\Omega^{4(k+1)}_{v,w}$ for all suitable pairs (v,w), except for the following:

$$(1, 2m-4), (1, 2m-2), (1, 2m), (1, 2m+2)$$
$$(2, 2m-5), (2, 2m-3), (2, 2m-1), (2, 2m+1)$$
$$(3, 2m-2), (3, 2m), (3, 2m+2)$$
$$(4, 2m-3), (4, 2m-1), (4, 2m+1)$$
$$(5, 2m), (5, 2m+2)$$
$$(6, 2m-1), (6, 2m+1)$$
$$(7, 2m+2)$$
$$(8, 2m+1)$$

where m=4(k+1).

Thus, what we need is to verify if $\Omega^{4}_{v,w} \neq \emptyset$ for all suitable pairs (v,w) and $\Omega^{4(k+1)}_{v,w} \neq \emptyset$ for the 20 pairs (v,w) that cannot be solved starting from elements of $\Omega^{4(k)}$.

This can be easily done using the method exposed in the first section, and this completes the proof. As it would be too long to draw here all the schemes,we give only the solutions, in two tables. In the first one, we write an element of $\Omega^{4}_{v,w}$ for every one of the 25 pairs needed, and in the second, an element of $\Omega^{m}_{v,w}$ for every one of the pairs needed when $m = 4(k+1)$, being in this second table

$$B = \bigcup_{i=1}^{\frac{m-8}{4}} \{11+8i, 16+8i, m^2+2m-2+8i, m^2+2m+1+8i\}$$

$$C = \bigcup_{i=1}^{\frac{m-8}{4}} \{13+8i, 14+8i, m^2+2m+3+8i, m^2+2m+4+8i\}$$

This completes the proof of our theorem.

v	w	$b_i's$	$c_i's$
1	2	34,33,32,9	6,30,29,10
1	4	35,32,31,8	34,7,9,27
1	6	35,34,5,30	33,8,28,10
1	8	35,3,33,31	32,30,9,10
1	10	35,4,31,30	34,32,8,9
3	2	36,32,30,8	4,31,28,10
3	4	36,35,5,28	6,30,29,10
3	6	36,35,4,27	32,7,29,9
3	8	36,5,31,28	35,4,30,10
3	10	1,35,32,30	33,31,8,9
5	2	36,34,4,30	6,29,9,27
5	4	36,2,34,30	6,29,28,10
5	6	36,2,34,28	33,7,8,27
5	8	1,34,33,30	35,6,9,27
5	10	1,34,33,28	35,6,30,8
7	2	36,3,32,31	4,29,9,27
7	4	36,2,34,28	5,31,8,27
7	6	1,35,34,28	33,5,8,27
7	8	36,5,28,27	2,34,33,6
7	10	1,33,32,28	35,3,31,8
9	2	36,3,32,29	4,6,30,27
9	4	36,2,31,29	3,32,7,27
9	6	1,35,33,27	3,32,7,29
9	8	1,35,31,27	34,4,5,30
9	10	1,32,30,29	2,34,33,6

Table 1: $\Omega^4_{v,w}$

v	w	$b_i's = B \cup$	$c_i's = C \cup$
1	2m+2	$\{10, 14, 16, m^2+2m+4, (m+2)^2-15, (m+2)^2-12, (m+2)^2-11, (m+2)^2-1\}$	$\{7, 8, 9, 11, (m+2)^2-5, (m+2)^2-4, (m+2)^2-3, (m+2)^2-2\}$
3	2m+2	$\{7, 13, 14, m^2+2m+4, (m+2)^2-15, (m+2)^2-11, (m+2)^2-7, (m+2)^2-1\}$	$\{4, 9, 10, 15, (m+2)^2-10, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
5	2m+2	$\{2, 13, 2m-1, m^2+2m+4, m^2+2m+5, (m+2)^2-8, (m+2)^2-7, (m+2)^2-2\}$	$\{4, 10, 11, 14, (m+2)^2-11, (m+2)^2-6, (m+2)^2-5, (m+2)^2\}$
7	2m+2	$\{3, 5, 15, m^2+2m+4, (m+2)^2-10, (m+2)^2-9, (m+2)^2-5, (m+2)^2-3\}$	$\{2, 12, 13, 16, (m+2)^2-13, (m+2)^2-8, (m+2)^2-7, (m+2)^2\}$
2	2m+1	$\{7, 13, 16, m^2+2m+3, (m+2)^2-13, (m+2)^2-11, (m+2)^2-7, (m+2)^2-2\}$	$\{4, 9, 10, 15, (m+2)^2-10, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
4	2m+1	$\{3, 13, 2m-1, m^2+2m+3, m^2+2m+5, (m+2)^2-8, (m+2)^2-7, (m+2)^2\}$	$\{5, 10, 11, 14, (m+2)^2-11, (m+2)^2-6, (m+2)^2-5, (m+2)^2-1\}$
6	2m+1	$\{3, 9, 16, m^2+2m+3, (m+2)^2-13, (m+2)^2-9, (m+2)^2-4, (m+2)^2-3\}$	$\{2, 11, 12, 15, (m+2)^2-12, (m+2)^2-7, (m+2)^2-6, (m+2)^2\}$
8	2m+1	$\{4, 14, 15, m^2+2m+3, (m+2)^2-15, (m+2)^2-10, (m+2)^2-6, (m+2)^2-5\}$	$\{2, 5, 12, 13, (m+2)^2-9, (m+2)^2-8, (m+2)^2-2, (m+2)^2\}$
1	2m	$\{8, 9, 2m-1, m^2+2m+4, m^2+2m+8, (m+2)^2-11, (m+2)^2-3, (m+2)^2-2\}$	$\{5, 10, 11, 2m+2, m^2+2m+7, (m+2)^2-6, (m+2)^2-5, (m+2)^2-1\}$
3	2m	$\{7, 12, 2m+2, m^2+2m+4, m^2+2m+6, (m+2)^2-13, (m+2)^2-7, (m+2)^2-1\}$	$\{4, 9, 10, 13, (m+2)^2-10, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
5	2m	$\{2, 14, 2m+1, m^2+2m+3, m^2+2m+6, (m+2)^2-12, (m+2)^2-11, (m+2)^2-1\}$	$\{3, 10, 11, 13, (m+2)^2-11, (m+2)^2-6, (m+2)^2-5, (m+2)^2\}$
2	2m-1	$\{8, 13, 2m+2, m^2+2m+4, m^2+2m+5, (m+2)^2-11, (m+2)^2-6, (m+2)^2-3\}$	$\{3, 9, 10, 14, (m+2)^2-10, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
4	2m-1	$\{8, 12, 2m+2, m^2+2m+4, m^2+2m+5, (m+2)^2-13, (m+2)^2-6, (m+2)^2-2\}$	$\{2, 9, 10, 13, (m+2)^2-10, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
6	2m-1	$\{3, 11, 14, m^2+2m+4, (m+2)^2-12, (m+2)^2-11, (m+2)^2-4, (m+2)^2-1\}$	$\{4, 9, 10, 2m+2, (m+2)^2+2m+5, (m+2)^2-7, (m+2)^2-6, (m+2)^2\}$
1	2m-2	$\{5, 10, 2m, m^2+2m+3, m^2+2m+8, (m+2)^2-8, (m+2)^2-3, (m+2)^2-1\}$	$\{6, 11, 12, 2m+1, m^2+2m+6, (m+2)^2-7, (m+2)^2-6, (m+2)^2-2\}$
3	2m-2	$\{7, 2m-1, 2m+2, m^2+2m+4, m^2+2m+5, m^2+2m+9, (m+2)^2-7, (m+2)^2-3\}$	$\{2, 9, 10, 2m-3, m^2+2m+10, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
2	2m-3	$\{7, 11, 2m+2, m^2+2m+5, m^2+2m+7, (m+2)^2-11, (m+2)^2-7, (m+2)^2\}$	$\{4, 9, 10, 2m+1, m^2+2m+6, (m+2)^2-5, (m+2)^2-4, (m+2)^2-2\}$
4	2m-3	$\{11, 2m-1, 2m+1, m^2+2m+3, m^2+2m+5, m^2+2m+7, (m+2)^2-11, (m+2)^2-1\}$	$\{3, 7, 8, 12, (m+2)^2-8, (m+2)^2-5, (m+2)^2-4, (m+2)^2\}$
1	2m-4	$\{2m-3, 2m, 2m+1, m^2+2m+3, m^2+2m+6, m^2+2m+7, m^2+2m+11, (m+2)^2-1\}$	$\{6, 7, 8, 2m-5, m^2+2m+12, (m+2)^2-4, (m+2)^2-3, (m+2)^2-2\}$
2	2m-5	$\{8, 2m, 2m+2, m^2+2m+4, m^2+2m+7, m^2+2m+8, (m+2)^2-11, (m+2)^2-1\}$	$\{5, 6, 9, 2m-1, m^2+2m+9, (m+2)^2-6, (m+2)^2-3, (m+2)^2-2\}$

Table 2: $\Omega^m_{v,w}$; $m = 4(k$ [illegible]$1)$ for the missing pairs.

4 Conclusions.

A method for the construction of magic bordered squares was presented, which consists of a simplification of the problem of construcing magic borders. The method allows the construction of such borders in an easy way by arranging numbers in a two-column diagram. The method was used to give a complete description of magic borders of even order.

In addition, the method can be used (among other utilities) to classify the existing methods for constructing magic squares (which are known since the 10th century) and to give a complete enumeration of magic bordered squares for some orders. This is work in progress.

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