A Sign-Reversing Involution on Bimahonian Generating Functions

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Abstract

We explicitly evaluate the generating functions for joint distributions of pairs of the permutation statistics *inv*, *maj* and *ch* over the symmetric group when both variables are set to -1. We give a combinatorial proof by means of a sign-reversing involution that specializing the variables to -1 in these bimahonian generating functions gives the number of two-colored permutations up to sign.

1 Introduction

Permutation statistics are useful tools in the study of the symmetric group S_n. The well-known inversion statistic counts the number of inversions in a permutation and is known to determine the length of a reduced word in the symmetric group (Coxeter group of type A_n) with the generating set of adjacent transpositions. The generating function for the number of inversions is [1]:

$$\sum_{\pi \in S_n} q^{inv(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$$

Setting $q = -1$ in this generating function results in a sum over the symmetric group where each permutation is weighted with a 1 if the permutation has an *even* number of inversions (i.e. can be written as a product

of an even number of transpositions) and -1 if the permutation has an *odd* number of inversions (i.e. can be written as a product of an odd number of transpositions). The right hand side of the generating function is zero when $q = -1$ giving a natural combinatorial proof that the number of even permutations is equal to the number of odd permutations.

In this paper we consider two variable generating functions over the symmetric group of the type

$$f(q,t) = \sum_{\pi \in S_n} q^{a(\pi)} t^{b(\pi)} \tag{1}$$

where $a(\pi)$ and $b(\pi)$ are Mahonian statistics and employ combinatorial techniques to evaluate the specialization $q = t = -1$.

The main result of this paper, in Section 4, is to give a sign-reversing involution that proves that $f(q,t)$ with $q = t = -1$, $a(\pi) = inv(\pi)$ and $b(\pi) = maj(\pi)$ is equal to $2^n n!$. We use this main result to prove similar results for the pairs of statistics (inv, ch), (maj, ch), (inv, cch), and (maj, cch) in Section 5. We describe the set of fixed points of the main involution in Section 4 in terms of two-colored permutations and rotationally symmetric permutations matrices which are described in Section 3. Section 2 contains the necessary background and definitions.

2 Background and Definitions

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, define an *inversion* to be a pair (i,j) such that $i < j$ and $\pi_i > \pi_j$. Then the *inversion statistic*, $inv(\pi)$, is the total number of inversions in π.

For example, for $\pi = 3\ 2\ 8\ 5\ 7\ 4\ 6\ 1\ 9$, $inv(\pi) = 15$ since each of the pairs $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(1,7)$, $(1,8)$, $(2,3)$, $(4,5)$, $(4,7)$, $(4,8)$, $(5,8)$, $(6,7)$, $(6,8)$, $(7,8)$ is an inversion.

The generating function for the inversion statistic,

$$\sum_{\pi \in S_n} q^{inv(\pi)}$$

is known to be symmetric.

For $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, we say π has a *descent* in position j if $\pi_j > \pi_{j+1}$. The *major index* of a permutation π, written $maj(\pi)$, is the sum of the descents of π. I.e.,

$$maj(\pi) = \sum_{\pi_j > \pi_{j+1}} j.$$

For example, for $\pi = 3\ 2\ 8\ 5\ 7\ 4\ 6\ 1\ 9$, $maj(\pi) = 1 + 3 + 5 + 7 = 16$. The generating function for the major index

$$\sum_{\pi \in S_n} q^{maj(\pi)}$$

is also known to be symmetric and was proven by MacMahon [5] to be equal to the generating function for the inversion statistic. Any permutation statistic that is equidistributed with the major index is now called a Mahonian statistic.

A third Mahonian statistic which has not been as widely studied as the inversion statistic and the major index is the charge statistic. The definition of the charge statistic was first given by Lascoux and Schützenberger [4].

For each element $i \in \pi$, define the *charge contribution of* i, $chc(i)$, to be zero if $i = 1$ or i lies to the right of $i-1$ in π and to be $n - i + 1$ if i lies to the left of $i - 1$ in π. We now define the charge of π to be

$$ch(\pi) = \sum_i chc(i).$$

For the previous example, the charge contribution of each element is given below that element:

$$\begin{array}{rcccccccccc} \pi & = & 3 & 2 & 8 & 5 & 7 & 4 & 6 & 1 & 9 \\ & & 7 & 8 & 2 & 5 & 3 & 0 & 0 & 0 & 0 \end{array}$$

thus the charge of π is equal to 7+8+2+5+3=25.

The statistic *cocharge* is defined similarly to *charge*. For each element $i \in \pi$, define the *cocharge contribution of* i, $coc(i)$, to be zero if $i = 1$ or i lies to the left of $i - 1$ in π and to be $n - i + 1$ if i lies to the right of $i - 1$ in π. We now define the cocharge of π to be

$$cch(\pi) = \sum_i coc(i).$$

For the previous example, the cocharge contribution of each element is given below that element:

$$\begin{array}{rcccccccccc} \pi & = & 3 & 2 & 8 & 5 & 7 & 4 & 6 & 1 & 9 \\ & & 0 & 0 & 0 & 0 & 0 & 6 & 4 & 0 & 1 \end{array}$$

thus the cocharge of π is equal to 6+4+1=11.

The two variable generating functions for the pairs (maj, inv), (maj, ch), (maj, cch), (inv, ch) and (inv, cch) are all known to be symmetric functions. Our interest is in the specialization $q = t = -1$ in each

of these bimahonian generating functions. We will examine these specializations and their corresponding fixed points in Sections 4 and 5.

There are three maps on the symmetric group that are of interest. First, $\mathbf{i}$ is the usual *inverse* map $\mathbf{i} : \pi \to \pi^{-1}$. Second, $\mathbf{c}$ is the *complement* map $\mathbf{c} : \pi \to \pi^{\mathbf{c}}$, in which each k is replaced with $n-k+1$ to form $\pi^{\mathbf{c}}$. Third, $\mathbf{r}$ is the *reverse* map $\mathbf{r} : \pi \to \pi^{\mathbf{r}}$, which replaces each π_j with π_{n-j+1} for $\pi \in S_n$.

For example, for the permutation $\pi = 3\ 2\ 8\ 5\ 7\ 4\ 6\ 1\ 9$ we have $\pi^{\mathbf{i}} = \pi^{-1} = 8\,2\,1\,6\,4\,7\,5\,3\,9$, $\pi^{\mathbf{c}} = 7\,8\,2\,5\,3\,6\,4\,9\,1$, and $\pi^{\mathbf{r}} = 9\,1\,6\,4\,7\,5\,8\,2\,3$.

The map $\psi : S_n \to S_n$ defined by $\psi(\pi) = ((\pi^{\mathbf{r}})^{-1})^{\mathbf{r}}$ is a bijection which swaps *maj* and *charge*, i.e. $maj(\pi) = ch(\psi(\pi)) = ch(((\pi^{\mathbf{r}})^{-1})^{\mathbf{r}})$ and $ch(\pi) = maj(\psi(\pi)) = maj(((\pi^{\mathbf{r}})^{-1})^{\mathbf{r}})$.

The well-known bijection ϕ given by Foata and Schützenberger [3] has the property that for any $\pi \in S_n$, $maj(\pi) = inv(\phi(\pi))$ and consequently the inverse map has the property that $inv(\pi) = maj(\phi^{-1}(\pi))$. In addition, the bijection ϕ preserves the charge statistic, thus $ch(\pi) = ch(\phi(\pi))$.

We review Foata and Schützenberger's map, $\phi : S_n \to S_n$. We construct ϕ in steps. Let $w = \pi_1\pi_2\ldots\pi_n$ be a permutation in one line notation. We define the map ϕ in steps.

1. Define $w_1 = \pi_1$. Assume w_k has been defined for all $k < n$.

2. Consider the string w_k. If the last letter of w_k is greater than π_{k+1}, split w_k by placing a bar after each letter greater than π_{k+1}. Similarly, if w_k is less that π_{k+1}, split w_k by placing a bar after each letter less than π_{k+1}.

3. In each *block* (created by the bars in Step 2) of w_k cycle the last letter of the block to the beginning of that block. Then append π_{k+1} at the end of the string to obtain w_{k+1} and repeat Steps 2 and 3.

4. The process is complete when π_n is added, i.e. $\phi(\pi) = w_n$.

For example, if $\pi = 649275183$, the successive stages of the algorithm yield

$$w_1 = 6|$$
$$w_2 = 6|4|$$
$$w_3 = 6|4|9|$$
$$w_4 = 6|4|92|$$

$$w_5 = 6|429|7|$$
$$w_6 = 6|9|4|2|7|5|$$
$$w_7 = 6|94|2|7|5|1|$$
$$w_8 = 6|4|9|27|5|18|$$
$$w_9 = 6\,4\,9\,7\,2\,5\,8\,1\,3 = \phi(\pi)$$

so $\phi(6\,4\,9\,2\,7\,5\,1\,8\,3) = 6\,4\,9\,7\,2\,5\,8\,1\,3$. Note that $maj(6\,4\,9\,2\,7\,5\,1\,8\,3) = 23 = inv(6\,4\,9\,7\,2\,5\,8\,1\,3)$.

3 Symmetric Matrices and Two-Colored Permutations

In this section we will define the combinatorial objects we will need to describe our fixed point sets in Section 4 and construct bijections between these objects. We will consider rotationally symmetric permutation matrices, two-colored permutations and the set of permutations that will appear as fixed points in the involution in Section 4.

Define a *Rotationally Symmetric Permutation Matrix* (RSPM) of size n to be an $n \times n$ permutation matrix which is invariant under a 180 degree rotation. For example for $n = 4$ there are eight such matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is easy to check that the number of RSPM's of size $2n$ is the same as the number of RSPM's of size $2n+1$. The rotational symmetry requires that a matrix of size $2n+1$ have a 1 in the center entry, i.e. in the $(n+1, n+1)$ entry.

Define a *two-colored permutation of length* n to be a permutation $\pi \in S_n$ with each π_i either *colored* or *uncolored*. A colored π_i will be denoted by a bar. For example, the permutation $\pi = 4\,\bar{3}\,1\,\bar{2}$ is a two-colored permutation of length 4.

Proposition 1. *The number of rotationally symmetric permutation matrices of size $2n$ is $2^n n!$.*

Proof. It is well known that the number of two-colored permutations of length n is $2^n n!$. We will construct a bijection from the rotationally symmetric permutation matrices of size $2n$ and the set of two-colored permutations of length n thus proving the claim.

Let R_{2n} be the set of RSPMs of size $2n$ and let C_n be the set of two-colored permutations of length n, where a *colored* part will be denoted with a bar. We will construct a map $\phi : R_{2n} \to C_n$ in two stages, ϕ_1 and ϕ_2.

First, given a matrix $M \in R_{2n}$ consider the corresponding permutation $\pi_M \in S_{2n}$. Construct an *expanded two-colored* permutation of length $2n$, σ with entries $\{1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}\}$ in the following way. Let $1, 2, \ldots, n$ in π correspond to $1, 2, \ldots, n$ in $\sigma = \phi_1(\pi)$ and let $n+1, n+2, \ldots, 2n$ in π correspond to $\bar{n}, \overline{n-1}, \ldots, \bar{1}$ in $\sigma = \phi_1(\pi)$.

For example, if $M \in R_4$ is given below, the corresponding permutation in S_4 is $\pi = (2, 4, 1, 3)$. Then $\sigma = \phi_1(\pi) = (2, \bar{1}, 1, \bar{2})$.

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Because a rotation of 180 degrees takes entry (i, j) to $(2n+1-i, 2n+1-j)$, it is easy to verify that $\phi_1(\pi)$ will be a palindrome if the bars are disregarded.

Next, given an *expanded two-colored* permutation σ of length $2n$ create a two-colored permutation $\phi_2(\sigma)$ of length n by deleting the last n entries of σ. Because the *expanded two-colored* permutations were palindromes and consist of each integer $1, 2, \ldots n$ and its barred pair, there is no loss of information in ignoring the last n entries of σ.

For example, if $\sigma = (2, \bar{1}, 1, \bar{2})$, then $\phi_2(\sigma) = (2, \bar{1})$.

It is easy to verify that what remains is the set of two-colored permutations of $\{1, 2, \ldots, n\}$, thus establishing our bijection and proving our claim. □

It is now a simple exercise to check that the fixed points in the involution in Section 4, namely permutations $\pi \in S_{2n}$ with $2i-1$ and $2i$ in adjacent positions, are in one-to-one correspondence with colored permutations of length n and thus with RSPM's of size $2n$. (If $2i-1$ is to the left of $2i$, replace $2i-1$ and $2i$ with i and if $2i-1$ is to the right of $2i$, replace $2i-1$ and $2i$ with $\bar{i}$.)

4 A Sign-Reversing Involution for (inv, maj)

The following specialization of the two variable generating function for the statistics inv and maj strongly suggests the existence of a sign-reversing involutions on the symmetric group that gives rise to a set of fixed points counted by 2-colored permutations. We give such an involution below then explain how our methods can be extended to other specializations of bimahonian generating functions with inv, maj, ch and cch.

Theorem 1.

$$\sum_{\pi \in S_{2n}} (-1)^{inv(\pi)}(-1)^{maj(\pi)} = 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)}(-1)^{maj(\pi)}.$$

Proof. We will first prove the result by induction for S_{2n}. If $n = 1$, then there are only two permutations in S_2. These are $\pi_1 = 12$ and $\pi_2 = 21$. For π_1 both inv and maj are equal to 0 so $(-1)^{inv(\pi_1)}(-1)^{maj(\pi_1)} = 1$. For π_2, both inv and maj are equal to 1 so $(-1)^{inv(\pi_2)}(-1)^{maj(\pi_2)} = 1$ and the left side of Theorem 1 is equal to 2, which is equal to $2^1\ 1!$.

Now we will assume the result is true for S_{2n} and prove the result for S_{2n+2}. Let $\pi \in S_{2n+2}$. Then the numbers $2n+1$ and $2n+2$ can appear in π in several ways.

Case 1: Suppose $2n+1$ and $2n+2$ are in non-adjacent positions. WLOG assume $2n+1$ appearing before $2n+2$ in π. Thus,

$$\pi = \pi_1 \quad \cdots \quad \pi_{i-1} \quad 2n+1 \quad \pi_{i+1} \quad \cdots \quad \pi_{j-1} \quad 2n+2 \quad \pi_{j+1} \quad \cdots \quad \pi_{2n+2}.$$

Now form σ by interchanging $2n+1$ and $2n+2$ in π so,

$$\sigma = \pi_1 \quad \cdots \quad \pi_{i-1} \quad 2n+2 \quad \pi_{i+1} \quad \cdots \quad \pi_{j-1} \quad 2n+1 \quad \pi_{j+1} \quad \cdots \quad \pi_{2n+2}.$$

Any inversion in π between the elements 1 through $2n$ also exists as an inversion in σ since these elements appear in the same order in both permutations. In π, both $2n+1$ and $2n+2$ form an inversion with each of the elements π_{j+1} through π_{2n+2}. In addition, $2n+1$ forms an inversion with each of the elements π_{i+1} through π_{j-1}. In σ, both $2n+1$ and $2n+2$ form an inversion with each of the elements π_{j+1} through π_{2n+2}. The number $2n+1$ no longer forms an inversion with the elements π_{i+1} through π_{j-1}, but $2n+2$ now forms an inversion with each of these elements. There is one additional inversion in σ formed between $2n+1$ and $2n+2$, thus the total number of inversions in σ is one greater than the number of inversions in π.

Any descents that occur in π in positions 1 through $i-2$, $i+1$ through $j-2$ and $j+1$ through $2n+1$ also occur in σ in the same positions, since

the elements π_1 through π_{i-1}, π_{i+1} through π_{j-1} and π_{j+1} through π_{2n+2} occur in the same order in both π and σ. Since both $2n+1$ and $2n+2$ are larger then all the other elements in π and σ, there is no descent in position $i-1$ or position $j-1$ in either π or σ and there are descents in positions i and j in both π and σ, thus the positions of the descents in π and σ are the same so $maj(\pi) = maj(\sigma)$.

Then for permutations in S_{2n+2} of this type, $(-1)^{inv(\pi)}(-1)^{maj(\pi)}$ and $(-1)^{inv(\sigma)}(-1)^{maj(\sigma)}$ have opposite parity, thus they will cancel each other out in the sign reversing involution.

Case 2: Suppose that $2n+1$ and $2n+2$ are in adjacent positions in S_{2n+2}, with $2n+1$ occuring first in the permutation and with $2n+1$ occuring in position j with j even. Then

$$\pi = \pi_1 \quad \cdots \quad \pi_{j-1} \quad 2n+1 \quad 2n+2 \quad \pi_{j+2} \quad \cdots \quad \pi_{2n+2}.$$

Again we will form σ by swapping $2n+1$ and $2n+2$ so

$$\sigma = \pi_1 \quad \cdots \quad \pi_{j-1} \quad 2n+2 \quad 2n+1 \quad \pi_{j+2} \quad \cdots \quad \pi_{2n+2}.$$

Any inversions formed between elements 1 through $2n$ that occur in π also occur in σ since the order of these elements remains the same in both permutations. The elements $2n+1$ and $2n+2$ form inversions with all of the elements π_{j+2} through π_{2n+2} in both π and σ. However, in σ there is one additional inversion between $2n+2$ and $2n+1$ so the total number of inversions in σ is one greater than the number of inversions in π.

Any descents that occur in π in positions 1 through $j-2$ or $j+2$ through $2n+1$ also occur in σ in these same positions since the elements in positions 1 through $j-1$ and $j+2$ through $2n+2$ remain unaltered. Since both $2n+1$ and $2n+2$ are larger than all other numbers in π, there is no descent in position $j-1$ in either π or σ and there is a descent in both π and σ in position $j+1$. In addition, there is a descent in σ between $2n+2$ and $2n+1$ which occurs in position j. Since j is even, the maj statistic for σ differs from the maj statistic for π by an even number.

Thus $(-1)^{inv(\pi)}(-1)^{maj(\pi)}$ and $(-1)^{inv(\sigma)}(-1)^{maj(\sigma)}$ have opposite parity and will cancel each other in the sign reversing involution.

Case 3: Suppose that $2n+1$ and $2n+2$ are in adjacent positions in S_{2n+2}, with $2n+1$ occuring first in the permutation and with $2n+1$ occuring in position j with j odd. Then

$$\pi = \pi_1 \quad \cdots \quad \pi_{j-1} \quad 2n+1 \quad 2n+2 \quad \pi_{j+2} \quad \cdots \quad \pi_{2n+2}.$$

Again we will form σ by swapping $2n+1$ and $2n+2$ so

$$\sigma = \pi_1 \quad \cdots \quad \pi_{j-1} \quad 2n+2 \quad 2n+1 \quad \pi_{j+2} \quad \cdots \quad \pi_{2n+2}.$$

Consider inversions and descents in σ similarly to the considerations in Case 2. We see that since j is odd, the maj statistic for σ differs from the maj statistic for π by an odd number.

Thus $(-1)^{inv(\pi)}(-1)^{maj(\pi)}$ and $(-1)^{inv(\sigma)}(-1)^{maj(\sigma)}$ have the same parity.

For any permutation π in S_{2n+2} with $2n+1$ and $2n+2$ occuring in adjacent positions with the first occurrence in an odd position, let $\omega = \pi_1 \cdots \pi_{j-1}\pi_{j+2} \cdots \pi_{2n+2}$ denote the permutation in S_{2n} formed from π by removing $2n+1$ and $2n+2$.

If ω is not a fixed point when applying the sign reversing involution to S_{2n}, let μ denote the permutation in S_{2n} of opposite parity that is paired with ω in the sign reversing involution on S_{2n}. Now let ν be the permutation in S_{2n+2} given by $\mu_1 \cdots \mu_{j-1}$ then $2n+1$ and $2n+2$ in the same order that they appeared in π, then $\mu_j \cdots \mu_{2n}$. Since ω and μ have opposite parity, π and ν will have opposite parity and will thus cancel in the sign reversing involution.

If ω is a fixed point under the sign reversing involution on S_{2n}, then π will be a fixed point under the sign reversing involution on S_{2n+2}.

By induction, the fixed points of the sign reversing involution will be precisely those permutations $\pi \in S_{2n}$ with the property that $2i-1$ and $2i$ are in adjacent positions and we showed in Section 3 that the number of such permutations is $2^n n!$.

The proof for S_{2n+1} is similar by considering $2n$ and $2n+1$ and will be left to the reader.

□

5 Specializations for *ch* and *cch*

We note that several other bimahonian generating functions specialize like (inv, maj). In this section we give four specializations involving *ch* and *cch* and use Theorem 1 to prove the results.

Theorem 2.

$$\sum_{\pi \in S_{2n}} (-1)^{inv(\pi)}(-1)^{ch(\pi)} = 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)}(-1)^{ch(\pi)}.$$

Proof. The map $\psi : S_n \to S_n$ given by $\psi(\pi) = (((\pi^r)^{-1})^r)$ is a bijection and has the property that $maj(\pi) = ch(\psi(\pi))$. In addition, $inv(\pi^r) = \binom{n}{2} - inv(\pi)$ and since $inv(\pi) = inv(\pi^{-1})$ we have that

$inv(\pi) = inv((((\pi^r)^{-1})^r))$. Thus

$$2^n n! = \sum_{\pi \in S_{2n}} (-1)^{inv(\pi)}(-1)^{maj(\pi)} = \sum_{\pi \in S_{2n}} (-1)^{inv(\psi(\pi))}(-1)^{ch(\psi(\pi))}$$
$$= \sum_{\sigma \in S_{2n}} (-1)^{inv(\sigma)}(-1)^{ch(\sigma)}$$

and

$$2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)}(-1)^{maj(\pi)} = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\psi(\pi))}(-1)^{ch(\psi(\pi))}$$
$$= \sum_{\sigma \in S_{2n+1}} (-1)^{inv(\sigma)}(-1)^{ch(\sigma)}.$$

□

Theorem 3.

$$\sum_{\pi \in S_{2n}} (-1)^{maj(\pi)}(-1)^{ch(\pi)} = 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\pi)}(-1)^{ch(\pi)}.$$

Proof. The Foata-Schützenberger bijection $\phi : S_n \to S_n$ has the property that $maj(\pi) = inv(\phi(\pi))$ so $inv(\pi) = maj(\phi^{-1}(\pi))$ and that $ch(\pi) = ch(\phi(\pi))$. Thus

$$2^n n! = \sum_{\pi \in S_{2n}} (-1)^{inv(\pi)}(-1)^{ch(\pi)} = \sum_{\pi \in S_{2n}} (-1)^{maj(\phi^{-1}(\pi))}(-1)^{ch(\phi^{-1}(\pi))}$$
$$= \sum_{\sigma \in S_{2n}} (-1)^{maj(\sigma)}(-1)^{ch(\sigma)}$$

and

$$2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)}(-1)^{ch(\pi)} = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\phi^{-1}(\pi))}(-1)^{ch(\phi^{-1}(\pi))}$$
$$= \sum_{\sigma \in S_{2n+1}} (-1)^{maj(\sigma)}(-1)^{ch(\sigma)}.$$

□

Theorem 4.

$$\sum_{\pi \in S_{2n}} (-1)^{maj(\pi)}(-1)^{cch(\pi)} = (-1)^n 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\pi)}(-1)^{cch(\pi)}.$$

Proof. From the definition of *charge* and *cocharge* we have that $ch(\pi) = \binom{n}{2} - cch(\pi)$ for $\pi \in S_n$. Then

$$2^n n! = \sum_{\pi \in S_{2n}} (-1)^{maj(\pi)}(-1)^{ch(\pi)} = \sum_{\pi \in S_{2n}} (-1)^{maj(\pi)}(-1)^{\binom{2n}{2} - cch(\pi)}$$
$$= \sum_{\pi \in S_{2n}} (-1)^{\binom{2n}{2}}(-1)^{maj(\pi)}(-1)^{cch(\pi)}.$$

Since $\binom{2n}{2}$ is even if n is even and is odd if n is odd, then

$$(-1)^n 2^n n! = \sum_{\pi \in S_{2n}} (-1)^{maj(\pi)}(-1)^{cch(\pi)}.$$

Also,

$$2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\pi)}(-1)^{ch(\pi)} = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\pi)}(-1)^{\binom{2n+1}{2} - cch(\pi)}$$
$$= \sum_{\pi \in S_{2n+1}} (-1)^{\binom{2n+1}{2}}(-1)^{maj(\pi)}(-1)^{cch(\pi)}.$$

Since $\binom{2n+1}{2}$ is even if n is even and is odd if n is odd, then

$$(-1)^n 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\pi)}(-1)^{cch(\pi)}.$$

□

Theorem 5.

$$\sum_{\pi \in S_{2n}} (-1)^{inv(\pi)}(-1)^{cch(\pi)} = (-1)^n 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)}(-1)^{cch(\pi)}.$$

Proof. We will again make use of properties of the Foata-Schützenberger bijection $\phi : S_n \to S_n$. Since $ch(\pi) = ch(\phi(\pi))$ and $cch(\pi) = \binom{n}{2} - ch(\pi)$, we have that $cch(\pi) = \binom{n}{2} - ch(\pi) = \binom{n}{2} - ch(\phi(\pi)) = cch(\phi(\pi))$.

$$(-1)^n 2^n n! = \sum_{\pi \in S_{2n}} (-1)^{maj(\pi)}(-1)^{cch(\pi)} = \sum_{\pi \in S_{2n}} (-1)^{inv(\phi(\pi))}(-1)^{cch(\phi(\pi))}$$
$$= \sum_{\sigma \in S_{2n}} (-1)^{inv(\sigma)}(-1)^{cch(\sigma)}$$

and

$$(-1)^n 2^n n! = \sum_{\pi \in S_{2n+1}} (-1)^{maj(\pi)}(-1)^{cch(\pi)} = \sum_{\pi \in S_{2n+1}} (-1)^{inv(\phi(\pi))}(-1)^{cch(\phi(\pi))}$$
$$= \sum_{\sigma \in S_{2n+1}} (-1)^{inv(\sigma)}(-1)^{cch(\sigma)}.$$

□

6 Remarks and Future Work

While our primary objective was to understand the combinatorics of the fixed point set of the sign-reversing involution in Section 4, recent work by Barcelo, Reiner and Stanton gives more general results for specializing bimahonian generating functions at roots of unity [2].

In light of the work of the authors in [2], [6] the results in Section 5 can be given a group theoretic interpretation. Consider the four element group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the operations on S_n, $\mathbf{c}$ = complement and $\mathbf{r}$ = reverse. It is not difficult to check that the complement operation is simply multiplication on the left by w_0, the longest word in the Coxeter group with the standard generators. (In permutation terms $w_0 = n\ n-1\ \ldots 2\ 1$.) Similarly, the reverse operation is multiplication on the right by w_0. Thus $\mathbf{cr} = \mathbf{rc}$ is 180 degree rotation and can be thought of as conjugation by w_0.

We can now restate our main result in the language of Barcelo, Reiner and Stanton [2] by noticing that evaluating the bimahonian generating functions at $q = t = -1$ is simply verification of the bicyclic sieving phenomenon for

$$\begin{aligned} C &= \mathbb{Z}_2 \times \mathbb{Z}_2 \\ X &= S_n \quad \text{with} \\ X(q,t) &= f(q,t) = \sum_{\pi \in S_n} q^{a(\pi)} t^{b(\pi)} \end{aligned}$$

There are several generalizations of the inversion statistic and the major index for other Coxeter groups that give rise to interesting two-variable generating functions. We consider specializations $q = t = -1$ in these generating functions in another paper.

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