The hyper-Wiener index of unicyclic graphs with given matching number

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Abstract

Let G be a connected simple graph. The hyper-Wiener index $WW(G)$ is defined as $WW(G) = \frac{1}{2}\sum_{u,v\in V(G)} (d(u,v) + d^2(u,v))$, with the summation going over all pairs of vertices in G. In this paper, we determine the extremal unicyclic graphs with given matching number and minimal hyper-Wiener index.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Wiener index, defined as $W(G) = \sum_{u,v\in V(G)} d(u,v)$, is perhaps the most studied topological index from application and theoretical viewpoints [2]. The hyper-Wiener index of acyclic graphs was introduced by Milan Randić in 1993 [12]. Then Klein et al. [10] extend Randić's definition for all connected graphs as a generalization of the Wiener index. In parallel with the symbol W for the Wiener index, the hyper–Wiener index is traditionally denoted by WW. The *hyper–Wiener index* of a graph G is defined as

$$WW(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v) + \frac{1}{2}\sum_{u,v\in V(G)} d^2(u,v),$$

where and hereafter the summation going over all pairs of vertices in G.

Set $S(G) = \sum_{u,v\in V(G)} d^2(u,v)$. Then $WW(G) = \frac{1}{2}\ W(G) + \frac{1}{2}S(G)$. We denote by $D_G(u) = \sum_{v\in V(G)} d(u,v)$, $\quad DD_G(u) = \sum_{v\in V(G)} d^2(u,v)$, then

$$W(G) = \frac{1}{2}\sum_{u\in V(G)} D_G(u), \quad S(G) = \frac{1}{2}\sum_{u\in V(G)} DD_G(u).$$

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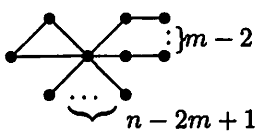
We encourage the reader to consult [3, 5, 6, 7, 10, 11, 14, 15] and the references therein for the mathematical properties of hyper-Wiener index and its applications in chemistry.

A matching M of the graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. A matching M of G is said to be maximum, if for any other matching M' of G, $|M'| \leq |M|$. The matching number of G is the number of edges of a maximum matching in G. If M is a matching of a graph G and vertex $v \in V(G)$ is incident with an edge of M, then v is said to be M-saturated, and if every vertex of G is M-saturated, then M is a perfect matching.

A unicyclic graph is a connected graph which has equal vertex number and edge number. For $u \in V(G)$, let $d_G(u)$ be the degree of u in G, and the eccentricity of u, denoted by $ecc(u)$, is the maximum distance from u to all other vertices in G. A pendent vertex is a vertex of degree one. Let C_n be a cycle with n vertices. For a unicyclic graph G with cycle C_s, the forest formed from G by deleting the edges of C_s consists of s vertex disjoint subtrees, each containing a vertex on C_s, which is called the root of this tree in G. These subtrees are called the branches of G.

For integers n and m with $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$, let $\mathbb{U}(n,m)$ be the set of unicyclic graphs with n vertices and matching number m. Obviously, if $G \in \mathbb{U}(n,1)$, then G is the triangle. In the following we assume that $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Let $U_{n,m}$ be the unicyclic graph obtained by attaching a pendent vertex to $m-2$ noncentral vertices and adding an edge between two other noncentral vertices of the star S_{n-m+2}. Obviously, $U_{n,m} \in \mathbb{U}(n,m)$.



$U_{n,m}$

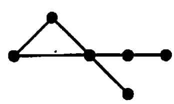

$U_{6,3}$

Figure 1. The unicyclic graph $U_{n,m}$.

It can be checked that $WW(U_{n,m}) = \frac{1}{2}(3n^2+6mn-19n+m^2-23m+42)$ and $WW(U_{2m,m}) = \frac{1}{2}(25m^2 - 61m + 42)$.

Recently, the hyper-Wiener index of trees with various parameters and unicyclic graphs with given girth was studied in [14] and [5], respectively. In this paper, we completely determine the extremal unicyclic graphs with given matching number and minimal hyper–Wiener index.

2 Some Lemmas

Lemma 2.1 *[1] Let $G \in \mathbb{U}(2m,m)$, where $m \geq 3$, and let T be a branch of G with root r. If $u \in V(T)$ is a pendent vertex furthest from the root r*

with $d_G(u,r) \geq 2$, then u is adjacent to a vertex of degree two.

Lemma 2.2 *[13] Let $G \in \mathbb{U}(n,m)$ where $n > 2m$, and $G \neq C_n$. Then there is a maximum matching M and a pendent vertex u of G such that u is not M-saturated.*

Lemma 2.3 *Let H be a connected graph. Let X and Y be two stars, with centers $v' \in V(X)$ and $u' \in V(Y)$. Suppose that u and v are two vertices from H. Let G be the graph obtained from H, X, Y by identifying v with v' and u with u', respectively. Let G' be the graph obtained from H, X, Y by identifying the vertices v, v', u', and G'' be the graph obtained from H, X, Y by identifying the vertices u, v', u'. Then we have*
(i). [5] either $S(G') < S(G)$ or $S(G'') < S(G)$ holds.
(ii). [4] either $W(G') < W(G)$ or $W(G'') < W(G)$ holds.

Lemma 2.4 *Let G be an n-vertex connected graph with a pendent vertex u being adjacent to vertex v, and let w be a neighbor of v different from u, where $n \geq 4$. Then $WW(G) - WW(G-u) \geq -3d_G(v) + 6n - 8$, with equality if and only if $ecc(v) = 2$.*

Moreover, if $d_G(v) = 2$, then $WW(G) - WW(G-u-v) \geq -7d_G(w) + 16n - 36$, with equality if and only if $ecc(w) = 2$.

Proof. From the definition, we have $WW(G) = WW(G-u) + \frac{1}{2}\sum_{x \in V(G-u)}\Big(d(u,x) + d^2(u,x)\Big) = WW(G-u) + \frac{1}{2}\sum_{x \in V(G-u)}\Big(3d(v,x) + d^2(v,x) + 2\Big) = WW(G-u) + \frac{1}{2}\Big(3D_{G-u}(v) + DD_{G-u}(v) + 2(n-1)\Big) = WW(G-u) + \frac{1}{2}\Big(3D_G(v) + DD_G(v)\Big) + n - 3 \geq WW(G-u) + \frac{1}{2}\Big(3(d_G(v) + 2(n-1-d_G(v))) + (d_G(v) + 4(n-1-d_G(v)))\Big) + n - 3 = WW(G-u) - 3d_G(v) + 6n - 8$, with equality if and only if $ecc(v) = 2$.

Similarly, we have $WW(G) = WW(G-u-v) + \frac{1}{2}\Big[\sum_{x \in V(G-u-v)}\Big(d(u,x) + d^2(u,x)\Big) + \sum_{x \in V(G-u-v)}\Big(d(v,x) + d^2(v,x)\Big) + 2\Big] = WW(G-u-v) + \frac{1}{2}\Big[\sum_{x \in V(G-u-v)}\Big(8d(w,x) + 2d^2(w,x) + 8\Big) + 2\Big] = WW(G-u-v) + 4D_{G-u-v}(w) + DD_{G-u-v}(w) + 4(n-2) + 1 = WW(G-u-v) + 4D_G(w) + DD_G(w) + 4n - 24 \geq WW(G-u-v) + 4\Big(d_G(w) + 2(n-1-d_G(w))\Big) + \Big(d_G(w) + 4(n-1-d_G(w))\Big) + 4n - 24 = WW(G-u-v) - 7d_G(w) + 16n - 36$, with equality if and only if $ecc(w) = 2$. ■

3 Main results

For a vertex $u \in V(C_k)$ (see [9]), we have $D_{C_k}(u) = \lfloor \frac{k^2}{4} \rfloor$, and

$$DD_{C_k}(u) = \begin{cases} \frac{k}{12}(k^2+2), & \text{if } k \text{ is even;} \\ \frac{k}{12}(k^2-1), & \text{if } k \text{ is odd.} \end{cases}$$

We denote by $H_{n,k}$ the graph obtained from $C_k = v_1 \dots v_k v_1$ by adding $n-k$ pendent vertices to a vertex of C_k. Then it is shown in [5] that if k is even, $WW(H_{n,k}) = \frac{1}{48}\Big\{72n^2 + (2k^3 + 18k^2 - 92k - 72)n - (k^4 + 15k^3 - 22k^2 - 72k)\Big\}$;
if k is odd, $WW(H_{n,k}) = \frac{1}{48}\Big\{72n^2 + (2k^3 + 18k^2 - 98k - 90)n - (k^4 + 15k^3 - 25k^2 - 87k)\Big\}$.

Let $U_n(k)$ be the unicyclic graph obtained from $C_k = v_1 \dots v_k v_1$ by attaching a pendent vertex and $n-k-1$ pendent vertices to v_k and v_1, respectively, where $3 \le k \le n-2$. It can be checked that $WW(U_n(k)) = WW(H_{n-1,k}) + \frac{1}{2}(DD_{C_k}(v_1) + 3D_{C_k}(v_1) + 2k + 12(n-k-1))$.

Hence, if k is even, $WW(U_n(k)) = \frac{1}{48}\Big(72n^2 + (2k^3 + 18k^2 - 92k + 72)n - (k^4 + 15k^3 - 22k^2 + 72k + 144)\Big)$. If k is odd, $WW(U_n(k)) = \frac{1}{48}\Big(72n^2 + (2k^3 + 18k^2 - 98k + 54)n - (k^4 + 15k^3 - 25k^2 + 57k + 144)\Big)$.

Lemma 3.1 *Suppose that $m+1 \le k \le 2m-2$. If $m \ge 5$ or $(m,k) = (4,6)$, then $WW(U_{2m}(k)) > \frac{1}{2}(25m^2 - 61m + 42)$.*

Proof. If k is even, then $WW(U_{2m}(k)) = f(k)$, where $f(k) = \frac{1}{48}\Big(288m^2 + (4k^3 + 36k^2 - 184k + 144)m - (k^4 + 15k^3 - 22k^2 + 72k + 144)\Big)$.

It is easy to check that

$f'(k) = \frac{1}{48}\Big((12k^2 + 72k - 184)m - (4k^3 + 45k^2 - 44k + 72)\Big)$,

$f''(k) = \frac{1}{48}\Big((24k + 72)m - (12k^2 + 90k - 44)\Big)$,

$f'''(k) = \frac{1}{48}\Big(24m - (24k + 90)\Big)$, $f^{(4)}(k) = \frac{1}{48}\Big(-24\Big) < 0$. So $f'''(k) < f'''(m+1) < 0$, $-4(15m - 44) = f''(2m-2) < f''(k) < f''(m+1) = 2(6m^2 - 9m - 29)$.

Let k_0 be the positive root of $f''(k) = 0$. It can be checked that $k_0 > m+1$. Then $f''(k)$ is positive in $(m+1, k_0)$, negative in $(k_0, 2m-2)$. Hence $f'(k)$ is increasing in $(m+1, k_0)$ and decreasing in $(k_0, 2m-2)$. Thus, $f'(k)$ takes its minimal value at $k = m+1$ or $k = 2m-2$.

$f'(m+1) = \frac{1}{48}(8m^3 + 39m^2 - 158m - 77) > 0$ for $m \ge 4$. $f'(2m-2) = \frac{1}{12}(4m^3 - 9m^2 + 18m - 77) > 0$ for $m \ge 4$.

So we get that $f'(k) > 0$ for $m+1 \le k \le 2m-2$. So we have $f(k) \ge f(m+1)$.

It is easy to see that $f(m+1) - \frac{1}{2}(25m^2 - 61m + 42) = \frac{1}{48}\Big(3m^4 + 29m^3 + 159m^2 - 77m - 210\Big) - \frac{1}{2}(25m^2 - 61m + 42) = \frac{1}{48}(m-3)\Big(3m^3 + 38m^2 - 327m + 406\Big)$.

Let $g(m) = 3m^3 + 38m^2 - 327m + 406$. Then $g'(m) = 9m^2 + 76m - 327$, $g''(m) = 18m + 76 > 0$ for $m \ge 4$, thus $g'(m)$ is strictly increasing for $m \ge 4$. Note that $g'(4) = 121 > 0$, so $g(m)$ is strictly increasing for $m \ge 4$. since $g(5) = 96 > 0$, so $g(m) > 0$ for $m \ge 5$.

Similarly, if k is odd and $m \ge 6$, we can also get the result.

If $m = 5$, then $6 \le k \le 8$. Hence $k = 7$ since it is odd. For $(m, k) = (5, 7)$, then $WW(U_{2m}(k)) = 202 > WW(U_{2m,m}) = 181$.

If $(m, k) = (4, 6)$, then it can be checked that $WW(U_{2m}(k)) = 106 > WW(U_{2m,m}) = 99$.

Combining the above cases, we complete the proof. ■

For integer $m \ge 3$, let $\mathcal{U}_1(m)$ be the set of graphs in $\mathbb{U}(2m, m)$ containing a pendent vertex whose neighbor is of degree two. Let $\mathcal{U}_2(m) = \mathbb{U}(2m, m) \setminus \mathcal{U}_1(m)$. Let H_8 be the graph obtained by attaching three pendent vertices to three consecutive vertices of C_5. It is easy to see that $WW(H_8) = 99$.

Lemma 3.2 *Let $G \in \mathcal{U}_2(m)$ with $m \ge 4$. If $G = H_8$, then $WW(G) = 99 = \frac{1}{2}(25m^2 - 61m + 42)$, and if $G \ne H_8$, then $WW(G) > \frac{1}{2}(25m^2 - 61m + 42)$.*

Proof. If $G = H_8$, then the result follows easily. If $G \ne H_8$, then by Lemma 2.1, $G \in \mathcal{U}_2(m)$ implies that $G = C_{2m}$ or G is a graph of maximum degree three obtained by attaching some pendent vertices to a cycle. If $G = C_{2m}$, then from [10], $WW(C_{2m}) = \frac{1}{6}(2m^4 + 3m^3 + m^2) > \frac{1}{2}(25m^2 - 61m + 42)$.

Suppose that $G \ne C_{2m}$. Then G is a graph of maximum degree three obtained by attaching some pendent vertices to a cycle C_k, where $m \le k \le 2m - 1$.

If $k = m$, then every vertex on the cycle has degree three, and for any pendent vertex x and its neighbor y :
if $m \ge 4$ is even, then $WW(G) = \frac{m}{4}\Big(D_G(x) + D_G(y) + DD_G(x) + DD_G(y)\Big) = \frac{m}{4}\Big(\frac{1}{2}(m^2 + 6m - 4) + \frac{1}{2}(m^2 + 2m) + \frac{1}{6}(m^3 + 9m^2 + 32m - 24) + \frac{1}{6}(m^3 + 3m^2 + 8m)\Big) = \frac{1}{12}(m^4 + 9m^3 + 32m^2 - 18m) > \frac{1}{2}(25m^2 - 61m + 42)$.

If $m \ge 5$ is odd, then as above $WW(G) = \frac{m}{4}\Big(\frac{1}{2}(m^2 + 6m - 5) + \frac{1}{2}(m^2 + 2m - 1) + \frac{1}{6}(m^3 + 9m^2 + 29m - 33) + \frac{1}{6}(m^3 + 3m^2 + 5m - 3)\Big) = \frac{1}{12}(m^4 + 9m^3 + 29m^2 - 27m) > \frac{1}{2}(25m^2 - 61m + 42)$.

If $m+1 \leq k \leq 2m-2$, then $m \geq 5$ or $(m,k)=(4,6)$ since $G \neq H_8$, by Lemmas 2.3 and 3.2, for some $U_{2m}(k)$, we have $WW(G) \geq WW(U_{2m}(k)) > \frac{1}{2}(25m^2-61m+42)$.

If $k=2m-1$, then G is the graph obtained from C_k by attaching a pendent vertex, By direct computation, we have $WW(G)=\frac{1}{6}(2m^4+m^3+4m^2+5m-6) > \frac{1}{2}(25m^2-61m+42)$ for $m \geq 4$. ■

In the following, if G is a graph in $\mathcal{U}_1(m)$ with a perfect matching M, then u is a pendent vertex whose neighbor v is of degree two in G, and w is the neighbor of v different from u. Obviously, $uv \in M$. Since $|M|=m$, we have $d_G(w) \leq m+1$.

Let H_6 be the graph obtained by attaching a pendent vertex to C_5. Let H_6' be the graph obtained by attaching a pendent vertex to every vertex of a triangle. Let H_6'' be the graph obtained by attaching two pendent vertices to two adjacent vertices of a quadrangle. It may be easily verified that the following lemma holds.

Lemma 3.3 *Among the graphs in* $\mathbb{U}(6,3)$, H_6 *is the unique graph with minimum hyper-Wiener index 39, and* H_6', H_6'', $U_{6,3}$ *and* C_6 *are the only graphs with the second minimum hyper-Wiener index 42.*

Lemma 3.4 *Let* $G \in \mathbb{U}(8,4)$. *Then* $WW(G) \geq 99$ *with equality if and only if* $G=H_8$ *or* $U_{8,4}$.

Proof. If $G \in \mathcal{U}_2(4)$, then by Lemma 3.2, $WW(G) \geq 99$ with equality if and only if $G=H_8$.

If $G \in \mathcal{U}_1(4)$, then $G-u-v \in \mathbb{U}(6,3)$. If $G-u-v \neq H_6$, then by Lemma 2.4 $WW(G) \geq WW(G-u-v)-7d_G(w)+16n-36 \geq 42-35+128-36=99$, with equality if and only if $G-u-v=H_6'$, H_6'', $U_{6,3}$ or C_6, $d_G(w)=5$, $ecc(w)=2$, i.e., $G=U_{8,4}$.

If $G-u-v=H_6$, then $d_G(w) \leq 4$, and by Lemma 2.4, $WW(G) \geq WW(H_6)-7d_G(w)+16n-36 \geq 39-28+128-36=103>99$. The result follows. ■

Lemma 3.5 *Let* $G \in \mathbb{U}(10,5)$. *Then* $WW(G) \geq 181$ *with equality if and only if* $G=U_{10,5}$.

Proof. If $G \in \mathcal{U}_2(5)$, then by Lemma 3.2, $WW(G) > 181$. If $G \in \mathcal{U}_1(5)$. Then $G-u-v \in \mathbb{U}(8,4)$. By Lemma 2.4 $WW(G) \geq WW(G-u-v)-7d_G(w)+16n-36 \geq 99-42+160-36=181$, with equality if and only if $G-u-v=U_{8,4}$, $d_G(w)=6$, $ecc(w)=2$, i.e., $G=U_{10,5}$. ■

Theorem 3.6 *Let* $G \in \mathbb{U}(2m,m)$, *where* $m \geq 2$.
(i) If $m=3$, *then* $WW(G) \geq 39$ *with equality if and only if* $G=H_6$.
(ii) If $m \neq 3$, *then* $WW(G) \geq \frac{1}{2}(25m^2-61m+42)$ *with equality if and*

only if $G = C_4$, $U_{4,2}$ *for* $m = 2$; $G = H_8$, $U_{8,4}$ *for* $m = 4$; *and* $G = U_{2m,m}$ *for* $m \geq 5$.

Proof. The case $m = 2$ is obvious since $\mathbb{U}(4,2) = \{C_4, U_{4,2}\}$, $WW(C_4) = WW(U_{4,2}) = 10$. The cases $m = 3$ and $m = 4$ follow from Lemmas 3.3 and 3.4, respectively.

Suppose that $m \geq 5$. Let $g(m) = \frac{1}{2}(25m^2 - 61m + 42)$. We prove the result by induction on m. If $m = 5$, then the result follows from Lemma 3.5. Suppose that $m \geq 6$ and the result holds for graphs in $\mathbb{U}(2m-2, m-1)$. Let $G \in \mathbb{U}(2m, m)$. If $G \in \mathcal{U}_2(m)$, then by Lemma 3.2, $WW(G) > g(m)$. If $G \in \mathcal{U}_1(m)$, then $G - u - v \in \mathbb{U}(2m-2, m-1)$, and thus by Lemma 2.4 and the induction hypothesis, it is easily seen that $WW(G) \geq WW(G - u - v) - 7d_G(w) + 16n - 36 \geq g(m-1) - 7(m+1) + 32m - 36 = g(m)$, with equality if and only if $G - u - v = U_{2(m-1),m-1}$, $d_G(w) = m+1$, $ecc(w) = 2$, i.e., $G = U_{2m,m}$. ■

Let H_7 be the graph obtained by attaching two pendent vertices to a vertex of C_5.

Theorem 3.7 *Let* $G \in \mathbb{U}(n,m)$, *where* $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$.
(i) If $(n,m) = (6,3)$, *then* $WW(G) \geq 39$ *with equality if and only if* $G = H_6$.
(ii) If $(n,m) \neq (6,3)$, *then* $WW(G) \geq \frac{1}{2}(3n^2 + 6mn - 19n + m^2 - 23m + 42)$ *with equality if and only if* $G = C_n$, $U_{n,2}$ *for* $(n,m) = (4,2), (5,2)$; $G = H_7$, $U_{7,3}$ *for* $(n,m) = (7,3)$; $G = H_8$, $U_{8,4}$ *for* $(n,m) = (8,4)$; *and* $G = U_{n,m}$ *for* $m \geq 5$.

Proof. The case $(n,m) = (6,3)$ follows from Lemma 3.3. Suppose that $(n,m) \neq (6,3)$. Let $g(n,m) = \frac{1}{2}(3n^2 + 6mn - 19n + m^2 - 23m + 42)$.

For C_7, we have $WW(C_7) = 70 > g(7,3) = 61$. For C_n with $n \geq 8$, we have either $n = 2m$, bear in mind that $(n,m) \neq (6,3)$, $WW(C_n) = \frac{1}{6}(2m^4 + 3m^3 + m^2) > g(n,m) = \frac{1}{2}(25m^2 - 61m + 42)$; or $n = 2m+1$, $WW(C_n) = \frac{1}{6}(2m^4 + 7m^3 + 7m^2 + 2m) > g(n,m) = \frac{1}{2}(25m^2 - 43m + 26)$.

If $G \neq C_n$ with $n > 2m$, then by Lemma 2.2, there exists a pendent vertex x and a maximum matching M such that x is not M-saturated in G, and thus $G - x \in \mathbb{U}(n-1, m)$. Let y be the unique neighbor of x. Since M contains one edge incident with y, and there are $n - m$ edges of G outside M, we have $d_G(y) \leq n - m + 1$.

Case 1. $m = 2$. The result for $n = 4$ is obvious as in previous theorem. The result for $n = 5$ may be checked directly as there are only five possibilities for G. For $n \geq 6$, it is known in [5] that $U_{n,2}$ is the unique unicyclic graph on n vertices with minimum hyper-Wiener index, and thus the unique graph in $\mathbb{U}(n,2)$ with minimum hyper-Wiener index.

Case 2. $m = 3$. If $n = 7$, then $G - x \in \mathbb{U}(6,3)$. If $G - x = H_6$, then $d_G(y) \leq 4$, and by Lemma 2.4, $WW(G) \geq WW(G - x) - 3d_G(y) + 6n - 8 \geq$

$39-12+42-8=61$, with equalities if and only if $d_G(y)=4$ and $ecc(y)=2$, i.e., $G=H_7$.

If $G-x\neq H_6$, then by Lemma 2.4 we have $WW(G)\geq WW(G-x)-3d_G(y)+6n-8\geq 42-15+42-8=61$, with equalities if and only if $G-x=H_6'$, H_6'', $U_{6,3}$ or C_6. $d_G(y)=5$ and $ecc(y)=2$, i.e., $G=U_{7,3}$. Thus, for $n=7$, we have $WW(G)\geq 61$ with equality if and only if $G=H_7$ or $U_{7,3}$.

For $n\geq 8$, we prove the result by induction on n. If $n=8$, then $G-x\in \mathbb{U}(7,3)$. By Lemma 2.4, $WW(G)\geq WW(G-x)-3d_G(y)+6n-8\geq 61-18+48-8=83$, with equalities if and only if $G-x=H_7$ or $U_{7,3}$, $d_G(y)=6$ and $ecc(y)=2$, i.e., $G=U_{8,3}$.

Suppose that $n\geq 9$ and the result holds for graphs in $\mathbb{U}(n-1,3)$. By Lemma 2.4 and the induction hypothesis, $WW(G)\geq WW(G-x)-3d_G(y)+6n-8\geq \frac{1}{2}(3n^2-7n-14)-3(n-2)+6n-8=\frac{1}{2}(3n^2-n-18)$, with equalities if and only if $G-x=U_{n-1,3}$, $d_G(y)=n-2$ and $ecc(y)=2$, i.e., $G=U_{n,3}$.

Case 3. $m=4$. The case $n=8$ follows from Lemma 3.4. For $n\geq 9$, we prove the result by induction on n. If $n=9$, then $G-x\in\mathbb{U}(8,4)$, and by Lemmas 3.4 and 2.4, $WW(G)\geq WW(G-x)-3d_G(y)+6n-8\geq 99-18+54-8=127$, with equalities if and only if $G-x=H_8$ or $U_{8,4}$, $d_G(y)=6$ and $ecc(y)=2$, i.e., $G=U_{9,4}$.

Suppose that $n\geq 10$ and the result holds for graphs in $\mathbb{U}(n-1,4)$. By Lemma 2.4 and the induction hypothesis, $WW(G)\geq WW(G-x)-3d_G(y)+6n-8\geq \frac{1}{2}(3n^2-n-36)-3(n-3)+6n-8=\frac{1}{2}(3n^2+5n-34)$, with equalities if and only if $G-x=U_{n-1,4}$, $d_G(y)=n-3$ and $ecc(y)=2$, i.e., $G=U_{n,4}$.

Case 4. $m\geq 5$. We prove the result by induction on n. If $n=2m$, then the result follows from Theorem 3.6. Suppose that $n>2m$ and the result holds for graphs in $\mathbb{U}(n-1,m)$. Let $G\in\mathbb{U}(n,m)$. By Lemma 2.4 and the induction hypothesis, it is easily seen that $WW(G)\geq WW(G-x)-3d_G(y)+6n-8\geq g(n-1,m)-3(n-m+1)+6n-8=g(n,m)$, with equality if and only if $G-x=U_{n-1,m}$, $d_G(y)=n-m+1$, $ecc(y)=2$, i.e., $G=U_{n,m}$. ■

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