THE TRIANGLE INTERSECTION PROBLEM FOR HEXAGON TRIPLE SYSTEMS

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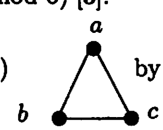
1 Introduction

A Steiner triple system (more simply *triple system*) of order n is a pair (S, T), where T is a collection of edge disjoint triangles (triples) which partitions the edge set of K_n (= the complete undirected graph on n vertices) with vertex set S. It is easy to see that $|T| = n(n-1)/6$. In 1847, T. P. Kirkman proved that the spectrum for triple systems (= the set of all n such that a triple system of order n exists) is precisely the set of all $n \equiv 1$ or $3 \pmod 6$ [3].

In what follows we will denote the triangle (triple) 

by $\{a, b, c\}$.

Example 1.1 (Two triple systems of order 7.)

$$\begin{aligned} T_1 &= \{\{1,2,3\},\{1,4,5,\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,5,6\},\{3,4,7\}\} \textit{ and} \\ T_2 &= \{\{1,2,3\},\{2,4,5\},\{2,6,7\},\{3,4,6\},\{3,5,7\},\{1,5,6\},\{1,4,7\}\}. \end{aligned}$$

Notice that $|T_1 \cap T_2| = 1$.

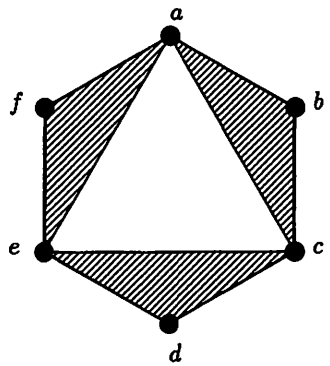
In [5], C. C. Lindner and A. Rosa gave a complete solution of the intersection problem for triple systems. Let $I(n) = \{0, 1, 2, \ldots, n(n-1)/6 = x\} \backslash \{x-1, x-2, x-3, x-5\}$ and let $Int(n)$ be the set of all k such that there exists a pair of triple systems of order n having exactly k triples in common. Lindner and Rosa proved that $Int(n) = I(n)$ for all $n \equiv 1$ or $3 \pmod 6$, except for $n = 9$. In this case $Int(9) = I(9) \backslash \{5, 8\} = \{0, 1, 2, 3, 4, 6, 12\}$ [4].

In what follows we will denote by λK_n the graph on n vertices with each pair of vertices connected by λ edges. A λ-fold triple system of order n is a pair

(S,T), where T is a collection of triples which partitions the edge set of λK_n with vertex set S. It is well known that a necessary condition for the existence of a 3-fold triple system of order n is that n is odd and the number of triples is $\binom{n}{2}$. The construction of 3-fold triple systems showing that this condition is sufficent is quite easy and whoever did this for the first time is lost to history.

Just as we looked at the intersection problem for triple systems, we can look at the intersection problem for 3-fold triple systems. It is not difficult to see that if two 3-fold triple systems of order n have k triples in common $k \in \{0,1,2,\ldots,\binom{n}{2} = x\}\backslash\{x-1, x-2, x-3, x-5\} = 3I(n)$. Several people have shown, except for $n = 5$, that this necessary condition is sufficient [1].

The graph below is called a *hexagon triple*



and will be denoted by $[a,b,c,d,e,f]$ or any cyclic 2-shift.

Notice that a hexagon triple consists of three triples.

A *hexagon triple system* is an edge disjoint partition of $3K_n$ into hexagon triples. Note that if we break up each hexagon triple, we have a 3-fold triple system. Hence we can think of a hexagon triple system as the piecing together of the triples of a 3-fold triple system into hexagon triples.

Example 1.2 (Two hexagon triple systems of order 7)

$$\begin{aligned} H_1 &= \{[4,3,2,5,1,6],[5,4,3,1,2,7],[6,5,4,7,3,2],[7,6,5,1,4,2],\\ &\quad [1,7,6,2,5,3],[2,1,7,3,6,4],[3,6,1,4,7,5]\} \textit{ and}\\ H_2 &= \{[4,6,2,5,1,7],[5,7,3,6,2,1],[6,5,4,2,3,1],[7,6,5,3,4,2],\\ &\quad [3,1,6,7,5,4],[7,3,2,5,1,4],[7,2,6,4,1,3]\} \end{aligned}$$

If we break up H_1 and H_2 into triangles (triples) we have a pair of 3-fold triple system intersecting in 10 triples.

The object of this paper is the following twist on the intersection problem for 3-fold triple systems. For which $k \in \{0,1,2,\ldots,\binom{n}{2} = x\}\backslash\{x-1, x-2, x-3, x-$

5} does there exist a pair of 3-fold triple systems intersecting in k triples, each of which can be assembled into hexagon triples. We give a complete solution of this problem modulo a few possible exceptions for $n = 13$.

2 Preliminaries

We collect together in this section some of the ideas and background material necessary to obtain the main results.

We begin with partial triple systems. A *partial* triple system of order n is a pair (X, P), where P is a collection of edge disjoint triples of the edge set of K_n with vertex set X. The difference between a partial triple system and a triple system is that the triples in a partial triple system (X, P) do not necessarily partition the edge set of K_n. Note that a triple system is also a partial triple system.

Two partial triple systems (X, P_1) and (X, P_2) are said to be balanced if the triples in P_1 and P_2 cover the same edges.

Example 2.1 (Balanced partial triple systems of order 6)

$$\begin{aligned} P_1 &= \{\{1,3,4\},\{1,5,6\},\{2,3,5\},\{2,4,6\}\} \textit{ and} \\ P_2 &= \{\{1,3,5\},\{1,4,6\},\{2,5,6\},\{2,3,4\}\} \end{aligned}$$

The following result is due to Lucia Gionfriddo.

Lemma 2.2 (Lucia Gionfriddo [2]) *Let (X, P_1) and (X, P_2) be partial triple systems that are balanced and disjoint (having no triples in common). Then using each triple in P_2 three times, we can construct a partial hexagon triple system whose inside triples are P_1.* □

Example 2.3 (Partial hexagon triple system constructed from Example 2.1)

$$P(H) = \{[3,2,4,6,1,5],[5,2,6,4,1,3],[4,1,6,5,2,3],[3,1,5,6,2,4]\}.$$

Corollary 2.4 *If $k \in Int(n)$ for Steiner triple systems, then $3k \in 3Int(n)$ for hexagon triple systems.*

Proof Let (S, T_1) and (S, T_2) be a pair of triple systems intersecting in k triples. Luc Teirlinck [7] has shown that every Steiner triple system has a disjoint mate. So let (S, T_1^*) and (S, T_2^*) be triple systems such that $T_1 \cap T_1^* = \emptyset$ and $T_2 \cap T_2^* = \emptyset$. If we place the triples of $3T_1$ on T_1^* and the triples of $3T_2$ on T_2^*, the resulting hexagon triple systems H_1 and H_2 have exactly $3k$ triples in common. □

Example 2.5 (Hexagon triple systems of order 7 intersecting in 9 triples)

$$\begin{aligned}
T_1 &= \{\{1,2,3\},\{2,4,6\},\{2,5,7\},\{3,5,6\},\{1,6,7\},\{3,4,7\},\{1,4,5\}\},\\
T_2 &= \{\{1,2,3\},\{2,4,6\},\{2,5,7\},\{1,5,6\},\{3,4,5\},\{3,6,7\},\{1,4,7\}\},
\end{aligned}$$

$$\begin{aligned}
T_1^* &= \{\{1,5,6\},\{1,2,7\},\{1,3,4\},\{3,5,7\},\{2,4,5\},\{2,3,6\},\{4,6,7\}\},\\
T_2^* &= \{\{1,4,5\},\{2,3,5\},\{5,6,7\},\{2,4,7\},\{3,4,6\},\{1,3,7\},\{1,2,6\}\},
\end{aligned}$$

$$\begin{aligned}
H_1 &= \{[6,3,5,4,1,7],[4,6,2,7,5,1],[2,5,7,6,1,3],[3,1,2,4,6,5],\\
&\quad [4,7,3,2,1,5],[4,3,7,1,6,2],[3,4,7,2,5,6]\} \text{ and}\\
H_2 &= \{[5,3,4,7,1,6],[6,7,3,5,4,2],[3,1,2,7,5,4],[7,6,3,2,1,4],\\
&\quad [6,3,7,2,5,1],[6,4,2,3,1,5],[7,5,2,6,4,1]\}.
\end{aligned}$$

3 $n = 7$

We give a complete solution of the Δ-intersection problem for $n = 7$ in this section. In what follows for each $k \in Int(7) = \{0,1,2,\dots,21\}\backslash\{20,19,18,16\}$ we list a pair of hexagon triple systems having k triples in common. This gives a complete solution. Let
H_1 = [[1, 7, 3, 5, 2, 4], [1, 6, 5, 7, 4, 2], [1, 3, 7, 2, 6, 5], [2, 7, 6, 3, 4, 1], [2, 6, 7, 4, 5, 3], [3, 1, 7, 5, 4, 6], [3, 4, 6, 1, 5, 2]]; H_2 = [[1, 4, 3, 7, 2, 6], [2, 1, 6, 3, 5, 4], [3, 5, 6, 7, 4, 1], [3, 2, 7, 1, 5, 6], [1, 7, 5, 2, 4, 3], [2, 3, 7, 6, 4, 5], [1, 5, 7, 4, 6, 2]]; H_3 = [[6, 4, 5, 2, 3, 1], [2, 3, 4, 5, 1, 7], [5, 1, 3, 4, 7, 6], [4, 7, 1, 3, 6, 2], [3, 6, 7, 1, 2, 5], [1, 2, 6, 7, 5, 4], [7, 5, 2, 6, 4, 3]]; H_4 = [[1, 3, 4, 5, 2, 6], [2, 4, 5, 6, 3, 7], [3, 5, 6, 7, 4, 1], [4, 6, 7, 1, 5, 2], [5, 7, 1, 2, 6, 3], [6, 1, 2, 3, 7, 4], [7, 2, 3, 4, 1, 5]]; H_5 = [[5, 4, 6, 7, 1, 3], [7, 1, 4, 6, 3, 2], [6, 3, 1, 4, 2, 5], [4, 2, 3, 1, 5, 7], [1, 5, 2, 3, 7, 6], [3, 7, 5, 2, 6, 4], [2, 6, 7, 5, 4, 1]];, H_6 = [[1, 7, 6, 3, 5, 4], [1, 3, 2, 5, 7, 6], [1, 5, 4, 7, 3, 2], [5, 6, 3, 4, 7, 2], [5, 1, 4, 6, 2, 7], [6, 5, 3, 1, 2, 4], [6, 2, 4, 3, 7, 1]]; H_7 = [[1, 6, 5, 2, 4, 7], [4, 6, 3, 1, 2, 5], [4, 3, 6, 2, 7, 1], [5, 3, 7, 6, 2, 4], [5, 1, 6, 4, 3, 7], [1, 4, 7, 5, 3, 2], [1, 5, 6, 7, 2, 3]]; H_8 = [[1, 5, 4, 6, 2, 3], [2, 7, 5, 6, 3, 1], [3, 5, 6, 2, 4, 7], [4, 3, 7, 2, 5, 1], [1, 7, 6, 3, 5, 4], [2, 5, 7, 1, 6, 4], [1, 6, 7, 4, 3, 2]]; H_9 = [[2, 6, 7, 5, 3, 1], [4, 3, 6, 1, 5, 2], [4, 1, 7, 2, 6, 3], [2, 4, 6, 7, 3, 5], [1, 6, 5, 7, 4, 2], [1, 6, 5, 4, 3, 7], [2, 3, 1, 4, 7, 5]], H_{10} = [[1, 4, 3, 5, 2,6], [5, 2, 4, 3, 6, 7], [3, 6, 2, 4, 7, 1], [4, 7, 6, 2, 1, 5], [2, 1, 7, 6, 5, 3], [6, 5, 1, 7, 3, 4], [7, 3, 5, 1, 4, 2]]; H_{11} = [[5, 2, 6, 4, 1, 3], [4, 1, 2, 6, 3, 7], [6, 3, 1, 2, 7, 5], [2, 7, 3, 1, 5, 4], [1, 5, 7, 3, 4, 6], [3, 4, 5, 7, 6, 2], [7, 6, 4, 5, 2, 1]], H_{12} = [[6, 3, 4, 5, 2, 1], [7, 4, 5, 6, 3, 2], [4, 1, 2, 3, 7, 6], [5, 2, 3, 4, 1, 7], [1,5, 6, 7, 4, 3], [2, 6, 7, 1, 5, 4], [3, 7, 1, 2, 6, 5]]; H_{13} = [[1, 4, 3, 2, 5, 6], [2, 5, 4, 3, 6, 7], [3, 6, 5, 4, 7, 1], [4, 7, 6, 5, 1, 2], [5, 1, 7, 6, 2, 3], [6, 2, 1, 7, 3, 4], [7, 3, 2, 1, 4, 5]]; H_{14} = [[4, 1, 5, 2, 3, 6], [2, 3, 1, 5, 6, 7], [5, 6, 3, 1, 7, 4], [1, 7, 6, 3, 4, 2], [3, 4, 7, 6, 2, 5], [6, 2, 4, 7, 5, 1], [7, 5, 2, 4, 1, 3]]; H_{15} = [[1, 6, 5, 3, 4, 7], [5, 4, 3, 1, 2, 7], [5, 1, 6, 3, 7, 2], [4, 1, 7, 5, 2, 6], [4, 2, 6, 7, 3, 5], [1, 4,

7, 6, 3, 2], [1, 5, 6, 4, 2, 3]]; H_{16} = [[5, 3, 4, 2, 1, 6], [2, 1, 3, 4, 6, 7], [4, 6, 1, 3, 7, 5], [3, 7, 6, 1, 5, 2], [1, 5, 7, 6, 2, 4], [6, 2, 5, 7, 4, 3], [7, 4, 2, 5, 3, 1]]; H_{17} = [[2, 1, 4, 7, 3, 6], [1, 6, 4, 5, 3, 7], [2, 5, 7, 3, 6, 4], [1, 7, 6, 5, 3, 2], [4, 2, 7, 1, 5, 3], [1, 4, 5, 6, 2, 3], [5, 2, 7, 4, 6, 1]]; H_{18} = [[4, 1, 6, 2, 5, 3], [2, 5, 1, 6, 3, 7], [6, 3, 5, 1, 7, 4], [1, 7, 3, 5, 4, 2], [5, 4, 7, 3, 2, 6], [3, 2, 4, 7, 6, 1], [7, 6, 2, 4, 1, 5]]; H_{19} = [[5, 7, 3, 2, 1, 6], [2, 1, 7, 3, 6, 4], [3, 6, 1, 7, 4, 5], [7, 4, 6, 1, 5, 2], [1, 5, 4, 6, 2, 3], [6, 2, 5, 4, 3, 7], [4, 3, 2, 5, 7, 1]]; H_{20} = [[5, 2, 3, 4, 1, 7], [4, 1, 2, 3, 7, 6], [3, 7, 1, 2, 6, 5], [2, 6, 7, 1, 5, 4], [1, 5, 6, 7, 4, 3], [7, 4, 5, 6, 3, 2], [6, 3, 4, 5, 2, 1]]; H_{21} = [[3, 6, 5, 4, 7, 1], [4, 7, 6, 5, 1, 2], [5, 1, 7, 6, 2, 3], [6, 2, 1, 7, 3, 4], [7, 3, 2, 1, 4, 5], [1, 4, 3, 2, 5, 6], [2, 5, 4, 3, 6, 7]]; H_{22} = [[2, 6, 7, 5, 4, 1], [6, 1, 5, 7, 4, 3], [1, 3, 7, 2, 6, 4], [1, 6, 5, 2, 3, 7], [2, 1, 4, 6, 3, 5], [2, 4, 5, 7, 1, 3], [6, 2, 7, 4, 3, 5]]; H_{23} = [[1, 5, 4, 6, 2, 3], [2, 7, 5, 6, 3, 1], [3, 5, 6,2, 4, 7], [4, 3, 7, 2, 5, 1], [1, 7, 6, 3, 5, 4], [2, 5, 7, 1, 6, 4], [1, 6, 7, 4, 3, 2]]; H_{24} = [[1, 7, 6, 3, 5, 4], [1, 6, 7, 5, 2, 3], [1, 5, 4, 7, 3, 2], [3, 4, 7, 2, 5, 6], [2, 7, 5, 1, 4, 6], [2, 4, 6, 5, 3, 1], [4, 3, 7, 1, 6, 2]].

Then $|H_1 \cap H_2| = 0$, $|H_3 \cap H_4| = 1$, $|H_5 \cap H_4| = 2$, $|H_6 \cap H_7| = 3$, $|H_8 \cap H_9| = 4$, $|H_{10} \cap H_4| = 5$, $|H_{11} \cap H_{12}| = 6$, $|H_4 \cap H_{13}| = 7$, $|H_{14} \cap H_{12}| = 8$, $|H_6 \cap H_{15}| = 9$, $|H_{16} \cap H_{17}| = 10$, $|H_{12} \cap H_{18}| = 11$, $|H_{19} \cap H_{17}| = 12$, $|H_{16} \cap H_9| = 13$, $|H_{20} \cap H_4| = 14$, $|H_{14} \cap H_{21}| = 15$, $|H_{16} \cap H_{22}| = 17$, $|H_{23} \cap H_{24}| = 21$.

This gives a complete solution to the Δ-intersection problem for hexagon triple systems of order 7.

Lemma 3.1 $3Int(7) = \{0, 1, 2, \ldots, 21\} \backslash \{20, 19, 18, 16\}$. □

4 $n = 9$

We give a complete solution of the Δ-intersection problem for $n = 9$ in this section. In what follows for each $k \in Int(9) = \{0, 1, 2, \ldots, 36\} \backslash \{35, 34, 33, 31\}$ we list a pair of hexagon triple systems having k triples in common. This gives a complete solution. Let

H_1 = [[1, 9, 3, 4, 2, 7], [1, 6, 8, 3, 5, 4], [1, 8, 6, 9, 4, 5], [1, 3, 9, 5, 7, 2], [5, 9, 7, 3, 6, 2], [2, 8, 9, 4, 6, 5], [2, 9, 8, 4, 7, 1], [2, 6, 5, 1, 4, 3], [4, 6, 9, 2, 8, 7], [3, 6, 7, 8, 4, 2], [3, 1, 9, 7, 5, 8], [3, 5, 8, 1, 6, 7]]; H_2 = [[1, 8, 5, 2, 4, 6], [1, 7, 9, 5, 3, 2], [1, 9, 7, 8, 2, 3], [1, 5, 8,3, 6, 4], [3, 4, 7, 5, 6, 8], [4, 9, 8, 2, 7, 3], [4, 8, 9, 2, 6, 1], [2, 5, 4, 7, 3, 1], [2, 6, 9, 4, 8, 7], [2, 9, 6, 7, 5, 4], [3, 6, 8, 1, 5, 9], [5, 3, 9, 1, 7, 6]]; H_3 = [[3, 6, 9, 2, 8, 7], [2, 9, 4, 5, 3, 6], [1, 2, 5, 7, 3, 9], [3, 8, 7, 1, 6, 2], [4, 2, 7, 9, 5, 3], [1, 4, 8, 5, 2, 3], [1, 3, 9, 6, 4, 7], [5, 1, 9, 4, 6, 8], [2, 8, 9, 5, 7, 4], [4, 3, 8, 1, 6, 5], [2, 7, 6, 8, 5, 1], [1, 4, 8, 9, 7, 6]]; H_4 = [[3, 6, 9, 2, 8, 7], [2, 7, 4, 8, 3, 1], [1, 2, 5, 7, 3, 9], [3, 5, 7, 1, 6, 2], [4, 1, 7, 9, 5, 3], [1, 6, 8, 5, 2, 3], [1, 5, 9, 2, 4, 8], [5, 1 , 9, 3, 6, 4], [2, 4, 9, 8, 7, 6], [4, 3, 8, 5, 6, 9], [2, 7, 6, 4, 5, 8], [1, 6, 8, 9, 7, 4]], H_5 = [[1, 9, 5, 6, 4, 7], [1, 6, 8, 4, 3, 2], [1, 4, 7, 6, 2, 3], [1, 5, 9, 3, 6, 8], [3, 5, 7, 2, 6, 9], [4, 2, 9, 8, 7, 1], [4, 3, 8, 1, 6, 5], [2, 9, 4, 8, 3, 1], [2, 4, 9, 7, 8, 5], [2, 7, 6, 4, 5, 8], [3, 6, 9, 1, 5, 7], [5, 2, 8, 9, 7, 3]]; H_6 = [[1, 5, 4, 7, 6, 9], [7, 5, 9, 3, 1, 2], [3, 6, 5, 4, 1, 7], [1, 7, 2, 5, 8, 6], [9, 6, 2, 1, 5, 7], [3, 5, 6, 4, 7, 8], [3, 2, 4, 8, 9, 1], [5, 3, 4, 1, 8, 9], [7, 9, 4, 3, 2, 8], [9, 2, 6, 1, 8, 4], [7, 3, 8, 2, 5, 6], [3, 8, 6, 4, 2, 9]]; H_7 = [[3, 6, 9, 7, 8, 4], [2, 9, 4, 8, 3, 7], [1, 8, 5, 9, 3, 2], [3, 5, 7, 2, 6, 9], [4, 1, 7, 3, 5, 2], [1, 6, 8, 5, 2, 3], [1, 5, 9, 2, 4, 3], [5, 1, 9, 4, 6, 7], [2, 8, 9, 1, 7, 6], [4, 7, 8, 3, 6, 5], [2, 1, 6, 4, 5, 8], [1, 6, 8, 9, 7, 4]]; H_8 = [[1, 4, 3, 7, 2, 6], [2, 8, 9, 3, 5, 4], [2, 9, 8, 3, 6, 1], [2, 3, 7, 8, 4, 5], [6, 4, 9, 1, 7, 5], [3, 8, 6, 9, 4, 1], [3, 2, 7, 6, 5, 9], [3, 5, 9, 2, 8, 6], [4, 7, 8, 1, 5, 2], [1, 5, 8, 4, 7, 9], [1, 7, 9, 6, 4, 3], [1, 2, 6, 7, 5, 8]]; H_9 = [[1, 9, 3, 6, 2, 5], [1, 6, 7, 2, 4, 8], [1, 3, 9, 7, 5, 2], [1, 4, 8, 5, 6, 7], [4, 9, 6, 8, 5, 3], [2, 9, 8, 6, 5, 1], [2, 4, 7, 1, 6, 3], [2, 8, 9, 6, 4, 7], [7, 5, 9, 2, 8, 3], [3, 1, 9, 4, 6, 2], [3, 7, 8, 1, 4, 5], [3, 8, 7, 9, 5, 4]]; H_{10} = [[3, 5, 9, 2, 8, 4], [2, 5, 4, 1, 3, 7], [1, 8, 5, 9, 3, 2], [3, 2, 7, 5, 6, 8], [4, 8, 7, 3, 5, 2], [1, 5, 8, 9, 2, 6], [1, 7, 9, 6, 4, 3], [5, 1, 9, 3, 6, 4], [2, 4, 9, 1, 7, 6], [4, 7, 8, 3, 6, 9], [2, 1, 6, 7, 5, 8], [1, 6, 8, 9, 7, 4]]; H_{11} = [[9, 5, 3, 8, 2, 4], [8, 5, 4, 1, 9, 6], [1, 2, 5, 3, 9, 8], [9, 8, 6, 5, 7, 2], [4, 2, 6, 9, 5, 8], [1, 5, 2, 3, 8, 7], [1, 6, 3, 7, 4, 9], [5, 1, 3, 9, 7, 4], [8, 4, 3, 1, 6, 7], [4, 6, 2, 9, 7, 3], [8, 1, 7, 6, 5, 2], [1, 7, 2, 3, 6, 4]]; H_{12} = [[9, 5, 3, 2, 8, 4], [2, 5, 4, 1, 9, 7], [1, 8, 5, 3, 9, 2], [9, 2, 7, 5, 6, 8], [4, 8, 7, 9, 5, 2], [1, 5, 8, 3, 2, 6], [1, 7, 3, 6, 4, 9], [5, 1, 3, 9, 6, 4], [2, 4, 3, 1, 7, 6], [4, 7, 8, 9, 6, 3], [2, 1, 6, 7, 5, 8], [1, 6, 8, 3, 7, 4]]; H_{13} = [[1, 5, 9, 7, 8, 4], [7, 5, 4, 3, 1, 2], [3, 8, 5, 9, 1, 7], [1, 7, 2, 5, 6, 8], [4, 8, 2, 1, 5, 7], [3, 5, 8, 9, 7, 6], [3, 2, 9, 6, 4, 1], [5, 3, 9, 1, 6, 4], [7, 4, 9, 3, 2, 6], [4, 2, 8, 1, 6, 9], [7, 3, 6, 2, 5, 8], [3, 6, 8, 9, 2, 4]]; H_{14} = [[3, 6, 2, 9, 8, 7], [9, 2, 4, 5, 3, 6], [1, 9, 5, 7, 3, 2], [3, 8, 7, 1, 6, 9], [4, 9, 7, 2, 5, 3], [1, 4, 8, 5, 9, 3], [1, 3, 2, 6, 4, 7], [5, 1, 2, 4, 6, 8], [9, 8, 2, 5, 7, 4], [4, 3, 8, 1, 6, 5], [9, 7, 6, 8, 5, 1], [1, 4, 8, 2, 7, 6]]; H_{15} = [[1, 5, 9, 2, 8, 4], [2, 5, 4, 3, 1, 7], [3, 8, 5, 9, 1, 2], [1, 2, 7, 5, 6, 8], [4, 8, 7, 1, 5, 2], [3, 5, 8, 9, 2, 6], [3, 7, 9, 6, 4, 1], [5, 3, 9, 1, 6, 4], [2, 4, 9, 3, 7, 6], [4, 7, 8, 1, 6, 9], [2, 3, 6, 7, 5, 8], [3, 6, 8, 9, 7, 4]]; H_{16} = [[3, 5, 9, 2, 8, 1], [2, 5, 1, 4, 3, 7], [4, 8, 5, 9, 3, 2], [3, 2, 7, 5, 6, 8], [1, 8, 7, 3, 5, 2], [4, 5, 8, 9, 2, 6], [4, 7, 9, 6, 1, 3], [5, 4, 9, 3, 6, 1], [2, 1, 9, 4, 7, 6], [1, 7, 8, 3, 6, 9], [2, 4, 6, 7, 5, 8], [4, 6, 8, 9, 7, 1]]; H_{17} = [[3, 5, 2, 9, 8, 4], [9, 5, 4, 1, 3, 7], [1, 8, 5, 2, 3, 9], [3, 9, 7, 5, 6, 8], [4, 8, 7, 3, 5, 9], [1, 5, 8, 2, 9, 6], [1, 7, 2, 6, 4, 3], [5, 1, 2, 3, 6, 4], [9, 4, 2, 1, 7, 6], [4, 7, 8, 3, 6, 2], [9, 1, 6, 7, 5, 8], [1, 6, 8, 2, 7, 4]]; H_{18} = [[1, 9, 5, 8, 3, 2], [1, 6, 8, 5, 2, 7], [1, 8, 6, 9, 4, 5,], [1, 3, 9, 8,7, 2], [5, 9, 7, 3, 6, 4], [2, 4, 9, 3, 6, 5], [2, 9, 8, 4, 7, 6], [2, 6, 5, 1, 4, 3], [4, 6, 9, 2, 8, 7], [3, 5, 7, 1, 4, 2], [3, 1, 9, 7, 5, 8], [3, 4, 8, 1, 6, 7]]; H_{19} = [[7, 2, 9, 6, 4, 3], [6, 9, 8, 5, 7, 2], [1, 6, 5, 3, 7, 9], [7, 4, 3, 1, 2, 6], [8, 6, 3, 9, 5, 7], [1, 8, 4, 5, 6, 7], [1, 7, 9, 2, 8, 3], [5, 1, 9, 8, 2, 4], [6, 4, 9, 5, 3, 8], [8, 7, 4, 1, 2, 5], [6, 3, 2, 4, 5, 1], [1, 8, 4, 9, 3, 2]]; H_{20} = [[3, 6, 9, 2, 8, 7], [2, 9, 4, 1, 3, 6], [5, 2, 1, 7, 3, 9], [3, 8, 7, 5, 6, 2], [4, 2, 7, 9, 1, 3], [5, 4, 8, 1, 2, 3], [5, 3, 9, 6, 4, 7], [1, 5, 9, 4, 6, 8], [2, 8, 9, 1, 7, 4], [4, 3, 8, 5, 6, 1], [2, 7, 6, 8, 1, 5], [5, 4, 8, 9, 7, 6]]; H_{21} = [[6, 5, 4, 3, 9, 2], [9, 7, 2, 1, 3, 6], [9, 8, 1, 7, 5, 4], [4, 6, 1, 8, 3, 2], [6, 8, 2, 9, 5, 3], [4, 1, 2, 5, 7, 9], [3, 4, 7, 6, 5, 2], [6, 8, 7, 3, 1, 9], [4, 2, 8, 3, 5, 1], [6, 4, 8, 9, 3 7], [1, 6, 2, 7, 8, 5], [9, 5, 8, 4, 7, 1]; H_{22} = [[1, 5, 3, 4, 2, 9], [2, 7, 9, 6, 4, 1], [2, 8, 6, 7, 5, 3], [3, 1, 6, 8, 4, 9], [1, 8, 9, 2, 5, 4], [3, 6, 9, 5, 7, 2], [4, 3, 7, 1, 5, 9], [1, 8, 7, 4, 6, 2], [3, 9, 8, 4, 5, 6], [1, 3, 8, 2, 4, 7], [6, 1, 9, 7, 8, 5], [2, 5, 8, 3, 7, 6]]; H_{23} = [[1, 6, 3, 5, 2, 4], [2, 1, 9, 6, 4, 8], [2, 7, 6, 3, 5, 9], [3, 1, 6, 7, 4, 9], [1, 2, 9, 7, 5, 4], [3, 8, 9, 5, 7, 4], [4, 3, 7, 1, 5, 8], [1, 8, 7, 2, 6, 9], [3, 9, 8, 6, 5, 2], [1, 3, 8, 2, 4, 5], [6, 4, 9, 7, 8, 5], [2, 6, 8, 1, 7, 3]]; H_{24} = [[1, 9, 3, 5, 2, 4], [1, 3, 8, 4, 5, 6], [1, 8, 6, 7, 4, 5], [4, 6, 9, 1, 7, 3], [5, 9, 7, 2, 6, 3], [2, 1, 9, 8, 6, 5], [2, 4, 8, 5, 7, 1], [2, 9, 5, 1, 4, 3], [4, 6, 9, 2, 8, 7], [1, 8, 7, 2, 3, 6], [3, 4, 9, 7, 5, 8], [3, 9, 8, 2, 6, 7]], H_{25} = [[1, 9, 5, 6, 4, 7], [1, 6, 8, 4, 3, 2], [1, 4, 7, 6, 2, 3], [1, 5, 9, 3, 6, 8], [3, 5, 7, 2, 6, 9], [4, 2, 9, 8, 7, 1], [4, 3, 8, 1, 6, 5], [2, 9, 4, 8, 3, 1], [2, 4, 9, 7, 8, 5], [2, 7, 6, 4, 5, 8], [3, 6, 9, 1, 5, 7], [5, 2, 8, 9, 7, 3]]; H_{26} = [[1, 9, 5, 8, 3, 2], [1, 6, 8, 5, 2, 3], [1, 8, 6, 5, 4, 7], [1, 3, 9, 8, 7, 4], [5, 9, 7, 2, 6, 4], [2, 4, 9, 3, 6, 7], [2, 5, 8, 9, 7, 1], [2, 6, 5, 1, 4, 9], [4, 6, 9, 2, 8, 3], [3, 5, 7, 8, 4, 2], [3, 6, 9, 1, 5, 7], [3, 4, 8, 1, 6, 7]]; H_{27} = [[1, 6, 3, 4, 2, 9], [2, 7, 9, 3, 4, 1], [2, 4, 6, 7, 5, 3], [3, 9, 6, 8, 4, 7], [1, 6, 9, 2, 5, 3], [3, 8, 9, 5, 7, 2], [4, 1, 7, 8, 5, 9], [6, 8, 9, 1, 7, 4], [3, 1, 8, 4, 5, 6], [1, 2, 8, 9, 4, 5], [1, 7, 8, 2, 6, 5], [2, 5, 8, 3, 7, 6]]; H_{28} = [[1, 4, 7, 2, 6, 8], [1, 7, 9, 6, 3, 2], [1, 5, 8, 3, 4, 6], [1, 9, 5, 4, 2, 3], [2, 4, 9, 7, 8, 5], [5, 2, 8, 1, 6, 7], [2, 9, 6, 8, 3, 1], [6, 3, 9, 8, 4, 5], [3, 9, 5, 6, 4, 7], [2, 6, 7, 1, 4, 9], [5, 1, 9, 8, 7, 3], [3, 4, 8, 2, 7, 5]]; H_{29} = [[1, 5, 3, 4, 2, 9], [2, 7, 9, 6, 4, 1], [2, 8, 6, 7, 5, 3], [3, 1, 6, 8, 4, 7], [1, 8, 9, 2, 5, 6], [3, 4, 9, 5, 7, 2], [4, 1, 7, 8, 5, 9], [6, 3, 9, 1, 7, 4], [3, 9, 8, 4, 5, 6], [1, 3, 8, 2, 4, 5], [1, 7, 8, 9, 6, 2], [2, 5, 8, 3, 7, 6]].

$|H_1 \cap H_2| = 0$, $|H_3 \cap H_2| = 1$, $|H_4 \cap H_2| = 2$, $|H_5 \cap H_2| = 3$, $|H_6 \cap H_{10}| = 4$, $|H_7 \cap H_1| = 5$, $|H_8 \cap H_9| = 6$, $|H_{10} \cap H_{11}| = 7$, $|H_3 \cap H_{12}| = 8$, $|H_8 \cap H_1| = 9$, $|H_3 \cap H_{13}| = 10$, $|H_{10} \cap H_{14}| = 11$, $|H_1 \cap H_9| = 12$, $|H_{10} \cap H_2| = 13$, $|H_{10} \cap H_4| = 14$, $|H_3 \cap H_{15}| = 15$, $|H_{16} \cap H_{17}| = 16$, $|H_4 \cap H_{18}| = 17$, $|H_8 \cap H_2| = 18$, $|H_{10} \cap H_{19}| = 19$, $|H_{20} \cap H_3| = 20$, $|H_{21} \cap H_{22}| = 21$, $|H_{10} \cap H_{15}| = 22$, $|H_{23} \cap H_{24}| = 23$, $|H_3 \cap H_{14}| = 24$, $|H_{25} \cap H_{26}| = 25$, $|H_7 \cap H_4| = 26$, $|H_7 \cap H_{26}| = 27$, $|H_4 \cap H_{26}| = 28$, $|H_{22} \cap H_{27}| = 29$, $|H_{28} \cap H_7| = 30$, $|H_{22} \cap H_{29}| = 32$, $|H_5 \cap H_5| = 36$.

This gives a complete solution of the Δ-intersection problem for hexagon triple systems of order 9.

Lemma 4.1 $3Int(9) = \{0, 1, 2, \ldots, 36\} \setminus \{35, 34, 33, 31\}$. □

5 $n = 13$

The complete solution of the Δ-intersection problem for hexagon triple systems remains elusive. To date we can show that 60 of the 75 intersection numbers are possible. Hopefully a complete solution can be obtained at a later date. Since $\{3k | k \in Int(13)\} \subseteq 3Int(13)$, we need look only at numbers in $3Int(13) \setminus \{3k | k \in Int(13)\}$. We will do this by listing 36 hexagon triple systems and 37 intersections between them.

Lemma 5.1 $3Int(13) \supseteq \{3k_n | k \in Int(13)\} \cup \{0, 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 38, 40, 43, 44, 46, 47, 50, 52, 56, 58, 62, 70, 74, 78\}$.

Proof To begin with $3Int(n) \supseteq \{3k | k \in Int(n)\}$ follows from Lemma 2.2. We will list the remaining solutions. This will be done by listing 36 hexagon triple system and then the various intersections between them for a total of 37. □

H_1 = [[2, 10, 3, 13, 1, 7], [2, 4, 6, 10, 11, 3], [11, 9, 7, 5, 4, 3], [4, 13, 8, 9, 3, 7], [3, 8, 9, 2, 6, 5], [6, 12, 10, 8, 7, 13], [7, 4, 5, 1, 8, 10], [8, 2, 12, 10, 9, 5], [9, 2, 13, 11, 10, 5], [10, 4, 1, 11, 5, 7], [5, 11, 2, 1, 12, 13], [12, 4, 11, 8, 13, 5], [13, 10, 4, 9, 1, 3], [1, 6, 9, 7, 11, 12], [2, 3, 10, 1, 4, 6], [3, 4, 11, 2, 5, 6], [4, 9, 12, 7, 6, 8], [3, 9, 13, 6, 7, 12], [6, 9, 1, 5, 8, 11], [7, 1, 2, 13, 9, 11], [8, 6, 11, 13, 10, 2], [9, 12, 4, 2, 5, 10], [10, 1, 3, 8, 12, 6], [5, 3, 6, 1, 13, 12], [12, 3, 7, 8, 1, 11], [13, 4, 8, 12, 1, 7]]

H_2 = [[2, 11, 3, 10, 1, 7], [2, 9, 6, 8, 11, 3], [11, 9, 6, 3, 4, 12], [4, 6, 8, 12, 3, 7], [3, 13, 9, 1, 6, 5], [6, 11, 10, 8, 7, 12], [7, 10, 5, 9, 8, 1], [8, 2, 12, 10, 9, 3], [9, 2, 13, 4, 10, 5], [10, 4, 1, 11, 5, 7], [5, 4, 2, 1, 12, 13], [12, 1, 11, 10, 13, 5], [13, 8, 4, 9, 1, 3], [1, 4, 9, 7, 11, 5], [2, 3, 10, 13, 4, 5], [3, 4, 11, 2, 5, 6], [4, 11, 12, 10, 6, 2], [3, 9, 13, 6, 7, 12], [6, 13, 1, 5, 8, 4], [7, 13, 2, 6, 9, 11], [8, 13, 11, 6, 10, 2], [9, 12, 4, 7, 5, 8], [10, 1, 3, 8, 12, 9], [5, 3, 6, 1, 13, 1], [12, 6, 7, 8, 1, 2], [13, 11, 8, 10, 2, 7]]

H_3 = [[2, 11, 3, 10, 1, 12], [2, 9, 6, 10, 11, 3], [11, 9, 7, 3, 4, 12], [4, 6, 8, 12, 3, 7], [3, 13, 9, 2, 6, 5], [6, 11, 10, 5, 7, 12], [7, 10, 5, 9, 8, 1], [8, 3, 12, 10, 9, 5], [9, 3, 13, 4, 10, 12], [10, 3, 1, 11, 5, 7], [5, 4, 2, 8, 12, 13], [12, 4, 11, 8, 13, 5], [13, 10, 4, 9, 1, 11], [1, 4, 9, 7, 11, 5], [2, 1, 10, 13, 4, 5], [3, 2, 11, 8, 5, 6], [4, 11, 12, 7, 6, 1], [3, 9, 13, 2, 7, 4], [6, 13, 1, 7, 8, 4], [7, 13, 2, 6, 9, 11], [8, 13, 11, 6, 10,2], [9, 8, 4, 2, 5, 1], [10, 8, 3, 1, 12, 9], [5, 3, 6, 1, 13, 12], [12, 6, 7, 8, 1, 2], [13, 6, 8, 10, 2, 7]]

H_4 = [[12, 8, 3, 13, 7, 5], [12, 10, 4, 2, 9, 8], [9, 11, 1, 12, 6, 10], [6, 2, 5, 11, 3, 4], [3, 9, 11, 13, 4, 6], [4, 12, 10, 5, 1, 13], [1, 4, 5, 9, 8, 6], [5, 6, 2, 12, 11, 10], [11, 12, 13, 1, 10, 8], [10, 3, 7, 4, 8, 2], [8, 3, 12, 5, 2, 10], [2, 7, 9, 12, 13, 8], [13, 5, 6, 10, 7, 2], [7, 6, 11, 3, 9, 1], [12, 3, 10, 9, 6, 1], [3, 6, 9, 5, 8, 4], [6, 11, 2, 9, 4, 12], [3, 5, 13, 10, 1, 2], [4, 11, 7, 8, 5, 1], [1, 7, 12, 2, 11, 8], [5, 4, 9, 13, 10, 11], [11, 7, 6, 13, 8, 1], [10, 7, 3, 1, 2, 4], [8, 7, 4, 11, 13, 6], [2, 3, 1, 9, 7, 13], [13, 3, 5, 7, 12, 9]]

H_5 = [[12, 8, 3, 10, 7, 1], [12, 6, 4, 2, 9, 13], [9, 11, 1, 12, 6, 3], [6, 13, 5, 11, 3, 9], [3, 5, 11, 13, 4, 8], [4, 12, 10, 13, 1, 5], [1, 10, 5, 9, 8, 6], [5, 12, 2, 6, 11, 10], [11, 12, 13, 9, 10, 8], [10, 6, 7, 5, 8, 2], [8, 9, 12, 5, 2, 13], [2, 7, 9, 10, 13, 8], [13, 5, 6, 11, 7, 3], [7, 4, 11, 1, 9, 2], [12, 3, 10, 7, 6, 4], [3, 11, 9, 12, 8, 4], [6, 5, 2, 10, 4, 3] , [3, 7, 13, 4, 1, 2], [4, 11, 7, 8, 5, 9], [1, 7, 12, 2, 11, 8], [5, 4, 9, 6, 10, 1], [11, 2, 6, 1, 8, 10] , [10, 12, 3, 1, 2, 4], [8, 7, 4, 1, 13, 6], [2, 3, 1, 9, 7, 13], [13, 3, 5, 7, 12, 11]]

H_6 = [[2, 5, 3, 12, 1, 8], [11, 13, 12, 7, 2, 6], [4, 2, 13, 12, 11, 3], [5, 9, 1, 7, 4, 10], [6, 11, 2, 3, 5, 12], [7, 5, 11, 2, 6, 10], [8, 6, 4, 1, 7, 3], [9, 1, 5, 13, 8, 11], [10, 7, 6, 3, 9, 2], [3, 8, 7, 6, 10, 13], [12, 10, 8, 7, 3, 1], [13, 7, 9, 4, 12, 11], [1, 11, 10, 3, 13, 6], [11, 9, 8, 2, 1, 10], [4, 12, 9, 10, 2, 13], [5, 4, 10, 1, 11, 7], [6, 9, 3, 11, 4, 8], [7, 2, 12, 6, 5, 11], [8, 5, 13, 1, 6, 4], [9, 5, 1, 4, 7, 13], [10, 9, 2, 1, 8, 12], [11, 4, 3, 6, 9, 8], [12, 9, 4, 5, 10, 8], [13, 8, 5, 2, 3, 10], [1, 13, 6, 5, 12, 3], [2, 12, 7, 9, 13, 4]]

H_7 = [[2, 8, 3, 12, 1, 5], [11, 13, 12, 7, 2, 6], [4, 2, 13, 12, 11, 1], [5, 9, 3, 7, 4, 10], [6, 11, 2, 1, 5, 12], [7, 5, 11, 2, 6, 10], [8, 6, 4, 3, 7, 1], [9, 3, 5, 13, 8, 11], [10, 7, 6, 1, 9, 2], [1, 8, 7, 6, 10, 13], [12, 10, 8, 7, 1, 3], [13, 7, 9, 4, 12, 11], [3, 11, 10, 1, 13, 6], [11, 9, 8, 2, 3, 10], [4, 12, 9, 10, 2, 13], [5, 4, 10, 3, 11, 7], [6, 9, 1, 11, 4, 8], [7, 2, 12, 6, 5, 11], [8, 5, 13, 3, 6, 4], [9, 5, 3, 4, 7, 13], [10, 9, 2, 3, 8, 12], [11, 4, 1, 6, 9, 8] , [12, 9, 4, 5, 10, 8], [13, 8, 5, 2, 1, 10], [3, 13, 6, 5, 12, 1], [2, 12, 7, 9, 13, 4]]

H_8 = [[2, 11, 3, 12, 1, 10], [2, 9, 6, 10, 11, 3], [11, 9, 7, 3, 4, 12], [4, 9, 8, 10, 3, 7], [3, 13, 9, 2, 6, 5], [6, 11, 10, 5, 7, 1], [7, 10, 5, 11, 8, 1], [8, 2, 12, 10, 9, 4], [9, 3, 13, 4, 10, 12], [10, 2, 1, 9, 5, 7], [5, 4, 2, 8, 12, 13], [12, 4, 11, 1, 13, 5], [13, 10, 4, 6, 1, 11], [1, 5, 9, 7, 11, 13], [2, 1, 10, 13, 4, 5], [3, 2, 11, 8, 5, 6], [4, 11, 12, 7, 6, 1], [3, 9, 13, 2, 7, 4], [6, 4, 1, 7, 8, 13], [7, 13, 2, 6, 9, 11], [8, 5, 11, 6, 10, 3], [9, 8, 4, 2, 5, 1], [10, 8, 3, 1, 12, 9], [5, 3, 6, 8, 13, 12], [12, 6, 7, 8, 1, 3], [13, 6, 8, 12, 2, 7]]

H_9 = [[2, 11, 3, 10, 1, 12], [2, 9, 6, 10, 11, 3], [11, 9, 7, 3, 4, 12], [4, 6, 8, 12, 3, 7], [3, 13, 9, 2, 6, 5], [6, 11, 10, 5, 7, 12], [7, 10, 5, 9, 8, 1], [8, 3, 12, 10, 9, 5], [9, 3, 13, 4, 10, 12], [10, 3, 1, 11, 5, 7], [5, 4, 2, 1, 12, 13], [12, 4, 11, 8, 13, 5], [13, 10, 4, 9, 1, 6], [1, 4, 9, 7, 11, 5], [2, 8, 10, 13, 4, 5], [3, 2, 11, 1, 5, 6], [4, 11, 12, 7, 6, 8], [3, 9, 13, 2, 7, 4], [6, 13, 1, 7, 8, 4], [7, 13, 2, 6, 9, 11], [8, 13, 11, 6, 10, 2], [9, 1, 4, 2, 5, 8], [10, 1, 3, 8, 12, 9], [5, 3, 6, 1, 13, 12], [12, 6, 7, 8, 1, 2], [13, 11, 8, 10, 2, 7]]

H_{10} = [[12, 6, 3, 2, 1, 8], [7, 13, 2, 11, 12, 5], [9, 12, 13, 2, 7, 3], [6, 4, 1, 11, 9, 10], [5, 7, 12, 3, 6, 2], [7, 6, 11, 10, 5, 12], [8, 5, 9, 1, 11, 3], [4, 1, 6, 13, 8, 7], [10, 11, 5, 3, 4, 12], [3, 8, 11, 5, 10, 13], [2, 10, 8, 11, 3, 1], [13, 11, 4, 9, 2, 7], [1, 7, 10, 3, 13, 5], [7, 4, 8, 12, 1, 10], [4, 2, 9, 13, 12, 10], [6, 9, 10, 1, 7, 11], [5, 4, 3, 7, 9, 8], [11, 12, 2, 5, 6, 7], [8, 6, 13, 1, 5, 9], [4, 6, 1, 9, 11, 13], [10, 4, 12, 1, 8, 2], [7, 9, 3, 5, 4, 8], [2, 4, 9, 6, 10, 8], [13, 8, 6, 12, 3, 10], [1, 13, 5, 6, 2, 3], [12, 2, 11, 4, 13, 9]]

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H_{30} = [[2, 10, 3, 13, 1, 7], [11, 1, 12, 6, 2, 5], [11, 10, 13, 6, 4, 3], [4, 7, 5, 6, 1, 10], [5, 3, 8, 4, 2, 11], [7, 9, 11, 6, 8, 13], [7, 10, 6, 13, 4, 5], [6, 1, 9, 10, 5, 3], [9, 5, 10, 12, 8, 1], [7, 6, 10, 2, 3, 12], [6, 2, 12, 7, 3, 9], [12, 5, 13, 2, 9, 4], [10, 11, 13, 3, 1, 4], [6, 8, 11, 12, 1, 5], [4, 12, 9, 13, 2, 8], [10, 13, 11, 2, 5, 9], [4, 2, 8, 9, 3, 11], [7, 3, 12, 13, 5, 4], [6, 4, 13, 7, 8, 11], [7, 11, 9, 8, 1, 2], [6, 7, 10, 3, 2, 12], [9, 7, 11, 4, 3, 6], [10, 8, 12, 9, 4, 1], [5, 12, 13, 1, 3, 8], [8, 10, 12, 11, 1, 5], [7, 8, 13, 9, 2, 1]]

H_{31} = [[2, 10, 3, 4, 1, 7], [11, 1, 12, 8, 2, 5], [11, 3, 13, 8, 4, 10], [4, 7, 5, 6, 1, 3], [5, 1, 6, 4, 2, 11], [7, 9, 11, 8, 6, 13], [7, 10, 8, 13, 4, 5], [8, 1, 9, 10, 5, 3], [9, 4, 12, 10, 6, 3], [7, 8, 10, 2, 3, 12], [8, 2, 12, 7, 3, 5], [10,1, 13, 2, 9, 5], [12, 5, 13, 10, 1, 11], [8, 6, 11, 12, 1, 9], [4, 12, 9, 13, 2, 6], [10, 13, 11, 2, 5, 9], [4, 2, 6, 9, 3, 11], [7, 3, 12, 13, 5, 4], [8, 4, 13, 7, 6, 11], [7, 11, 9, 8, 1, 2], [8, 7, 10, 3, 2, 12], [9, 7, 11, 13, 3, 6], [10, 6, 12, 9, 4, 11], [5, 12, 13, 1, 3, 8], [6, 12, 10, 4, 1, 5], [7, 6, 13, 9, 2, 1]]

H_{32} = [[1, 13, 3, 10, 6, 11], [7, 8, 2, 4, 6, 5], [4, 5, 11, 9, 7, 8], [3, 9, 8, 13, 4, 7], [2, 1, 9, 8, 3, 5], [11, 8, 10, 12, 2, 13], [8, 1, 5, 4, 11, 10], [9, 4, 12, 6, 8, 3], [10, 7, 13, 6, 9, 5], [6, 12, 8, 4, 13, 9], [5, 8, 1, 4, 10, 9], [12, 8, 6, 7, 5, 13], [13, 10, 7, 1, 12, 5], [1, 10, 4, 8, 13, 3], [7, 11, 9, 2, 1, 12], [4, 1, 10, 3, 6, 2], [5, 6, 7, 4, 3, 2], [2, 10, 12, 9, 4, 6], [11, 2, 13, 1, 3, 12], [8, 5, 1, 9, 2, 7], [9, 13, 6, 1, 11, 7], [10, 13, 7, 2, 8, 11], [5, 11, 4, 12, 9, 10], [12, 11, 3, 6, 10, 2], [13, 11, 2, 3, 5, 12], [1, 6, 11, 3, 12, 7]]

H_{33} = [[2, 12, 8, 4, 13, 9], [1, 13, 3, 10, 2, 7], [11, 8, 6, 4, 2, 5], [4, 5, 7, 9, 11, 3], [3, 9, 8, 13, 4, 11], [6, 1, 9, 8, 3, 5], [7, 3, 10, 12, 6, 13], [8, 1, 5, 4, 7, 10], [9, 4, 12, 2, 8, 3], [10, 11, 13, 2, 9, 5], [5, 8, 1, 4, 10, 9], [12, 3, 2, 11, 5, 13], [13, 10, 11, 1, 12,5], [1, 10, 4, 8, 13, 3], [11, 7, 9, 6, 1, 12], [4, 1, 10, 8, 2, 6], [5, 2, 11, 4, 3, 6], [6, 10, 12, 9, 4, 2], [7, 6, 13, 1, 3, 12], [8, 5, 1, 9, 6, 11], [9, 13, 2, 1, 7, 11], [10, 13, 11, 6, 8, 7], [5, 7, 4, 12, 9, 10], [12, 7, 3, 2, 10, 6], [13, 7, 6, 3, 5, 12], [1, 2, 7, 8, 12, 11]]

H_{34} = [[2, 12, 8, 4, 13, 9], [1, 13, 3, 12, 2, 7], [11, 8, 6, 4, 2, 5], [4, 5, 7, 9, 11, 13], [3, 9, 8, 13, 4, 11], [6, 1, 9, 8, 3, 5], [7, 3, 10, 12, 6, 13], [8, 1, 5, 4, 7, 10], [9, 4, 12, 7, 8, 3], [10, 11, 13, 2, 9, 5], [5, 8, 1, 4, 10, 8], [12, 3, 2, 11, 5, 13], [13, 10, 11, 1, 12, 5], [1, 10, 4, 8, 13, 3], [11, 7, 9, 6, 1, 12], [4, 1, 10, 8, 2, 6], [5, 2, 11, 4, 3, 6], [6, 10, 12, 9, 4, 2], [7, 6, 13, 1, 3, 10], [8, 5, 1, 9, 6, 11], [9, 13, 2, 1, 7, 11], [10, 13, 11, 6, 8, 2], [5, 7, 4, 12, 9, 10], [12, 7, 3, 2, 10, 6], [13, 7, 6, 3, 5, 12], [1, 2, 7, 8, 12, 11]

H_{35} = [[2, 3, 10, 13, 1, 7], [5, 1, 12, 6, 2, 11], [5, 3, 13, 6, 4, 10], [4, 7, 11, 8, 1, 3], [11, 1, 8, 4, 2, 5], [7, 9, 5, 6, 8, 13], [7, 3, 6, 13, 4, 11], [6, 1, 9, 3, 11, 10], [9, 11, 3, 12, 8, 10], [7, 6, 3, 2, 10, 12], [6, 2, 12, 7, 10, 11], [12, 11, 13, 2, 9, 4], [3, 5, 13, 10, 1, 4], [6, 8, 5, 12, 1, 9], [4, 12, 9, 13, 2, 8], [3, 13, 5, 2, 11, 9], [4, 2, 8, 9, 10, 5], [7, 10, 12, 13, 11, 4], [6, 4, 13, 7, 8, 5], [7, 5, 9, 6, 1, 2], [6, 7, 3, 10, 2, 12], [9, 7, 5, 4, 10, 8], [3, 8, 12, 9, 4, 1], [11, 12, 13, 1, 10, 6], [8, 3, 12, 5, 1, 11], [7, 8, 13, 9, 2, 1]]

H_{36} = [[1, 13, 3, 12, 2, 7], [2, 3, 12, 6, 10, 8], [4, 8, 13, 11, 10, 1], [1, 6, 5, 7, 4, 10], [2, 4, 6, 1, 5, 11], [6, 12, 10, 3, 7, 13], [4, 13, 8, 12, 7, 5], [5, 10, 9, 1, 8, 3], [6, 8, 11, 7, 9, 3], [3, 4, 11, 9, 7, 10], [3, 2, 12, 7, 8, 5], [9, 2, 13, 5, 12, 4], [1, 3, 13, 10, 11, 12], [1, 4, 10, 2, 8, 9], [2, 13, 9, 12, 4, 6], [5, 2, 11, 13, 10, 9], [3, 9, 6, 2, 4, 11], [5, 13, 12, 8, 7, 4], [6, 7, 13, 4, 8, 11], [1, 8, 9, 11, 7, 2], [2, 5, 11, 6, 8, 10], [3, 7, 10, 5, 9, 6], [4, 9, 12, 1, 11, 3], [3, 1, 13, 12, 5, 8], [1, 11, 12, 10, 6, 5], [2, 9, 13, 6, 7, 1]]

$|H_{12} \cap H_{16}| = 1$, $|H_{11} \cap H_{16}| = 2$, $|H_4 \cap H_6| = 4$, $|H_4 \cap H_7| = 5$, $|H_{12} \cap H_{20}| = 7$, $|H_6 \cap H_{18}| = 8$, $|H_3 \cap H_{14}| = 10$, $|H_{14} \cap H_{22}| = 11$, $|H_3 \cap H_{19}| = 13$, $|H_8 \cap H_{20}| = 14$, $|H_8 \cap H_{18}| = 16$, $|H_{33} \cap H_{35}| = 17$, $|H_{18} \cap H_{21}| = 19$, $|H_{31} \cap H_{32}| = 20$, $|H_9 \cap H_{17}| = 22$, $|H_{14} \cap H_{18}| = 23$, $|H_{23} \cap H_{27}| = 25$, $|H_2 \cap H_{21}| = 26$, $|H_5 \cap H_{11}| = 28$, $|H_{23} \cap H_{28}| = 29$, $|H_{16} \cap H_{34}| = 31$, $|H_{17} \cap H_{18}| = 32$, $|H_{14} \cap H_{15}| = 34$, $|H_{15} \cap H_{16}| = 35$, $|H_{17} \cap H_{36}| = 38$, $|H_{23} \cap H_{29}| = 40$, $|H_{17} \cap H_{26}| = 43$, $|H_{23} \cap H_{26}| = 44$, $|H_1 \cap H_{15}| = 46$, $|H_{17} \cap H_{21}| = 47$, $|H_{16} \cap H_{18}| = 50$, $|H_{14} \cap H_{30}| = 52$, $|H_{14} \cap H_{17}| = 56$, $|H_2 \cap H_{15}| = 58$, $|H_{21} \cap H_{33}| = 62$, $|H_{14} \cap H_{25}| = 70$, $|H_{14} \cap H_{24}| = 74$.

Combining all of the above gives the following lemma.

Lemma 5.2 $3Int(13) \supseteq$ {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 54, 56, 57, 58, 60, 62, 66, 70, 74, 78}. □

6 $n = 15$

Let (X, F) and (X, G) be two 1-factorizations of K_{2n}, where $F = \{F_1, F_2, \ldots, F_{2n-1}\}$ and $G = \{G_1, G_2, \ldots G_{2n-1}\}$. We say that (X, F) and (X, G) have k edges in common provided $\sum_{i=1}^{2n-1} |F_i \cap G_i| = k$. The intersection problem for 1-factorization of K_{2n} was solved in 1982 by C. C. Lindner and W. D. Wallis [6]. In particular, for $2n = 8$ the intersection numbers are
$\{0, 1, 2, \ldots, 28\} \backslash \{27, 26, 25, 23\}$.

We will need the following construction: the $2n+1$ Construction specialized to $2n + 1 = 15$.

Let (S, T) be a 3-fold triple system of order 7 where $S = \{1, 2, 3, 4, 5, 6, 7\}$. Let (X, F) be 1-factorization of K_8 where $S \cap X = \emptyset$ and $F = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$. Define a collection of triples T^* as follows:

(1) $T \subseteq T^*$, and

(2) for each edge $\{x, y\} \in F_i$, place three copies of $\{i, x, y\}$ in T^*.

Then $(S \cup X, T^*)$ is a 3-fold triple system.

The following lemma is immediate.

Lemma 6.1 *Let (X, F) and (X, G) have k edges in common. Then the type (2) triples have $3k$-triples in common.* □

Lemma 6.2 *If (S, T) can be organized into hexagon triples, then $(S \cup X, T^*)$ can be organized into hexagon triples.*

Proof It is only necessary to organize the triples of type (2) into hexagon triples. This is quite easy. Let (X, G) be a 1-factorization of K_8 where $G \cap F = \emptyset$. Then by Lemma 2.2 we can place the triples of type (2) on the triples $\{i, x, y\} \in G_i \in G$ to obtain a collection of hexagon triples. □

Corollary 6.3 *If $x \in 3Int(7)$ and $k \in \{0, 1, 2, \ldots, 28\} \backslash \{27, 26, 25, 23\}$, then $x + 3k \in 3Int(15)$.*

Proof Let (S, T_1) and (S, T_2) be 3-fold triple system having k triples in common which can be organized into hexagon triples and let (X, F) and (X, G) be 1-factorizations of K_8 having k edges in common. Then the 3-fold triple systems $(S \cup X, T_1^*)$ and $(S \cup X, T_2^*)$ constructed using the $2n + 1$ Construction can be organized into hexagon triple systems having $x + 3k$ triples in common. □

Lemma 6.4 $3Int(15) = \{0, 1, 2, \ldots, 78\} \backslash \{77, 76, 75, 73\}$.

Proof Each $n \in 3Int(15)$ can be written in the form $n = x + 3k$, where $x \in 3Int(7)$ and $k \in \{0, 1, 2, \ldots, 28\} \backslash \{27, 26, 25, 23\}$. □

7 The $6n + 1 \geq 19$ Construction

Since we have a solution for 7, 13 (modulo a few exceptions), and 15 we need consider only the cases $6n + 1 \geq 19$. Before giving the $6n + 1$ Construction, we need the following example.

Example 7.1 (Two decompositions of $3K_{3,3,3}$ into hexagon triples having no triples in common.)

Let $3K_{3,3,3}$ have parts $\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}$ and let

$$H_1 = \{[4,3,8,6,1,7], [5,3,7,4,1,9], [6,3,9,5,1,8], [4,1,7,6,2,9], [5,1,9,4,2,8], [6,1,8,5,2,7], [4,2,9,6,3,8], [5,2,8,4,3,7], [6,2,7,5,3,9]\}, \text{ and}$$
$$H_2 = \{[4,2,7,5,1,8], [5,2,9,6,1,7], [6,2,8,4,1,9], [4,3,9,5,2,7], [5,3,8,6,2,9], [6,3,7,4,2,8], [4,1,8,5,3,9], [5,1,7,6,3,8], [6,1,9,4,3,7]\}.$$

Corollary 7.2 *There exists a pair of $3K_{3,3,3}$ hexagon triple systems having 0 or 27 triples in common.* □

With this example in hand, we can proceed to the $6n + 1 \geq 19$ Construction.

The 6n + 1 $\geq$ 19 Construction

Let (X, G, T) be a group divisible design (GDD) of order $2n \geq 6$ with at most one group of size 4 and the remaining groups of size 2. If $2n \equiv 0$ or 2 (mod 6) all groups are of size 2 and if $2n \equiv 4$ (mod 6) exactly one group is of size 4, the others of size 2. Let $S = \{\infty\} \cup (X \times \{1, 2\})$ and define a collection of hexagon triples H as follows:

(1) For each $g \in G$, let $(\{\infty\} \cup (g \times \{1, 2, 3\}), H(g))$ be a hexagon triple system of order 7 or 13, as the case may be, and place the hexagon triples of $H(g)$ in H.

(2) For each block $t = \{a, b, c\} \in T$, let $(3K_{3,3,3}, T(t))$ be a decomposition of $3K_{3,3,3}$ with parts $\{a\} \times \{1, 2, 3\}, \{b\} \times \{1, 2, 3\}$, and $\{c\} \times \{1, 2, 3\}$ into hexagon triples and place these hexagon triples in H.

Then (S, H) is a hexagon triple system.

Lemma 7.3 *There exists a pair of hexagon triple systems of order* $6n + 1 \geq 19$ *having* k *triples in common for all* $k \in 3Int(n)$.

Proof Let $k \in 3Int(n)$. If $2n \equiv 0$ or 2 (mod 6), we can write $k = \sum_{i=1}^{n} a_i + \sum_{i=1}^{|T|} \{0, 27\}$ where the a_i's belong to $3Int(7)$. If $2n \equiv 4$ (mod 6), we can write $k = a_1 + \sum_{i=1}^{n-2} a_i + \sum_{i=1}^{|T|} \{0, 27\}$, where $a_1 \in \{0, 78\}$ (see Lemma 6.4) and $a_2, a_3, \ldots, a_{n-2} \in 3Int(7)$. □

We have the following theorem.

Theorem 7.4 $3Int(n) = 3I(n)$ *for all* $n \equiv 1$ *(mod 6), with possibly a few exceptions for* $n = 13$ *(see Section 5).* □

8 The $6n + 3$ Construction

Before giving the $6n + 3$ Construction we will need the following lemma.

Lemma 8.1 *There exist a pair of partial hexagon triple systems of order* 9 *which are disjoint, balanced, and cover the edges of* $3K_9 \backslash 3K_3$.

Proof Let (S, T_1) and (S, T_2) be a pair of triple systems of order 9 having *exactly* the one triple t^* in common. Then $(S, T_1 \backslash t^*)$ and $(S, T_2 \backslash t^*)$ are disjoint and balanced. Putting the triples of $T_1 \backslash t^*$ on the triples of $T_2 \backslash t^*$ and vice versa gives the desired pair of partial hexagon triple systems. □

Example 8.2 (A pair of partial hexagon triple systems of order 9.)

$$\begin{aligned} T_1 &= \{\{1,2,3\},\{1,4,7\},\{1,5,9\},\{1,6,8\},\{4,5,6\},\{2,5,8\},\{2,6,7\}, \\ &\quad \{4,5,6\},\{7,8,9\},\{3,6,9\},\{3,4,8\},\{3,5,7\}\}, \textit{ and} \\ T_2 &= \{\{1,2,3\},\{1,5,8\},\{1,4,6\},\{1,7,9\},\{5,6,7\},\{2,6,9\},\{2,7,8\}, \\ &\quad \{2,4,5\},\{4,8,9\},\{3,4,7\},\{3,5,9\},\{3,6,8\}\} \end{aligned}$$

$$t^* = \{1,2,3\}$$

$$\begin{aligned} T_1 \backslash t^* &= \{\{1,4,7\},\{1,5,9\},\{1,6,8\},\{4,5,6\},\{2,5,8\},\{2,6,7\},\{2,4,9\}, \\ &\quad \{7,8,9\},\{3,6,9\},\{3,4,8\},\{3,5,7\}\}, \textit{ and} \\ T_2 \backslash t^* &= \{\{1,5,8\},\{1,4,6\},\{1,7,9\},\{5,6,7\},\{2,6,9\},\{2,7,8\},\{2,4,5\}, \\ &\quad \{4,8,9\},\{3,4,7\},\{3,5,9\},\{3,6,8\}\} \end{aligned}$$

$$\begin{aligned} H_1 &= \{[5,2,8,6,1,9],[4,5,6,8,1,7],[7,8,9,5,1,4],[6,2,7,3,5,4], \\ &\quad [6,3,9,4,2,7],[7,9,8,5,2,6],[4,6,5,8,2,9],[8,7,9,2,4,3], \\ &\quad [4,1,7,5,3,8],[5,1,9,6,3,7],[6,1,8,4,3,9]\}, \textit{ and} \\ H_2 &= \{[4,3,7,9,1,6],[5,3,9,7,1,8],[6,3,8,5,1,4],[5,6,7,1,4,2], \\ &\quad [5,1,8,7,2,4],[6,5,7,8,2,9],[4,8,9,6,2,5],[8,4,9,1,7,2], \\ &\quad [6,2,9,5,3,8],[4,9,8,6,3,7],[5,6,7,4,3,9]\}. \end{aligned}$$

We can now give the $6n+3$ Construction.

Let (X,G,B) be a *GDD* of order $2n$ with *at most* one group of size 4 and the remaining groups of size 2. Set $S = \{\infty_1,\infty_2,\infty_3\} \cup (X \times \{1,2,3\})$ and define a collection of hexagon triples H as follows:

Let $G = \{g_1^*, g_2, g_3, \ldots, g_{2n}\}$ be the groups of G with $|g_1^*| = 4$ if G contains a group of size 4.

(1) Place a hexagon triple system on $\{\infty_1,\infty_2,\infty_3\} \cup (g_1^* \times \{1,2,3\})$ and place these hexagon triples in H.

(2) For each $g_i \in G$, $i \geq 2$, place a partial hexagon triple system on $\{\infty_1,\infty_2,\infty_3\} \cup \{g_i \times \{1,2,3\})$ as in Lemma 8.1

(3) For each triple $\{a,b,c\} \in B$, place a hexagon triple system on $3K_{3,3,3}$ with parts $\{a\} \times \{1,2,3\}, \{b\} \times \{1,2,3\}$, and $\{c\} \times \{1,2,3\}$.

Then (S,H) is a hexagon triple system of order $6n+3$.

Theorem 8.3 *The intersection numbers for hexagon triple systems of order* $n \equiv 3 \pmod 6$ *are precisely* $\{0,1,2,\ldots,\binom{n}{2} = x\} \backslash \{x-1,x-2,x-3,x-5\}\}$.

Proof 9 and 15 are taken care of in Sections 4 and 6. So we need only concern ourselves here with $6n+3 \geq 21$.

In case $|g_1^*| = 4$, any number in $\{0,1,2,\ldots \binom{n}{2} = x\} \backslash \{x-1,x-2,x-3,x-5\}$ can be written as a sum $a + \sum b_i + \sum\{0,27\}$, where $a \in 3Int(15)$ and $b_i \in \{0,33\}$. If $|g_i^*| = 2$ then any number in $\{0,1,2,\ldots,\binom{n}{2} = x\} \backslash \{x-1,x-2,x-3,x=5\}$ can be written as a sum $a + \sum b_i + \sum\{0,27\}$ where $a \in 3Int(9)$. □

9 Summary

Combining Lemmas 3.1, 4.1, and 5.2, along with Theorems 7.4 and 8.3 gives the following theorem:

Theorem 9.1 $3Int(n) = \{0, 1, 2, \ldots, \binom{n}{2} = x\} \backslash \{x-1, x-2, x-3, x-5\}$ *for all* $n \equiv 1$ *or* 3 (*mod 6*) *with the possible exceptions of* {37, 41, 49, 53, 44, 49, 61, 63, 64, 65, 67, 68, 69, 71, 72} *for* $n = 13$. □

References

[1] S. Ajodani-Namini and G. B. Khosrovshahi, *On a conjecture of A. Hartman*, in Combinatorics Advances (Ed. C. J. Colbourn and E. Mahmoodian), Kluwer Academic, Dordrecht, (1995), 1-12.

[2] Lucia Gionfriddo, Bullentin of the ICA, 48 (2006), 73-81.

[3] T. P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Math. J. 2, 1847, 191-204.

[4] E. S. Kramer and D. M. Messner, *Intersections among Steiner systems*, J. Combinat. Theory (A), 16 (1974), 273-285.

[5] C. C. Lindner and A. Rosa, *Construction of Steiner triple systems having a prescribed number of triples in common*, Canad. J. Math., XXVII (1975), 1166-1175.

[6] C. C. Lindner and W. D. Wallis, *A note on one-factorizations having a prescribed number of edges in common*, Annals of Discrete Mathematics 12 (1982), 203-209.

[7] L. Teirlinck, L., *On making two Steiner triple systems disjoint*, J. Combinat. Theory (A) 23, 1977, 349-350.