APPLICATIONS OF LIU'S q-SERIES EXPANSION FORMULA

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ABSTRACT. In this paper, we show new proofs of some important formulas by means of the Liu's expansion formula. Our results include a new proof of identity for sums of two squares, a new proof of Gauss's identity, a new proof of Euler's identity and a new proof of the identity for sums of four squares.

1. Introduction

We shall follow the notation and terminology in [3]. For two complex q and x, the shifted factorial of order n with base q is defined by

$$(x;\ q)_n = \begin{cases} 1, & \text{n=0}, \\ (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}), & \text{n=1, 2, \dots}. \end{cases} \quad (1.1)$$

$$(x;\ q)_\infty = \lim_{n\to\infty}(x;\ q)_n = \prod_{k=0}^{\infty}(1-xq^k),\ |\, q\,| < 1. \quad (1.2)$$

Where $|\, q\,| < 1$ in order for the infinite products to be convergent. It is easy to check that the shifted factorial whit negative integer order is give by

$$(x;\ q)_{-n} = \frac{(-q/x)^n q^{n(n-1)/2}}{(q/x;\ q)_n}. \quad (1.3)$$

The product and fractional forms of shifted factorial are abbreviated compactly to

$$(\alpha, \beta, \cdots, \gamma; q)_n = (\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n.$$

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Following Bailey [2] and Slater [6], the basic hypergeometric series ${}_{r+1}\phi_s$ is defined by

$$ {}_{r+1}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \,\middle|\, q;\ x\right] = \sum_{n=0}^{\infty}\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix} \,\middle|\, q\right]_n \times \left((-1)^n q^{n(n-1)/2}\right)^{r-s} x^n. \quad (1.4) $$

The q-differential operator about x is defined by

$$ D_{q,\,x}\{f(x)\} = \frac{f(x)-f(xq)}{x}. \quad (1.5) $$

By convention, $D^0_{q,\,x}$ is understood as the identity.

The Leibniz rule for $D_{q,\,x}$ is the following identity [5, p.233]

$$ D^n_{q,\,x}\{g(x)h(x)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D^k_{q,\,x}\{g(x)\} D^{n-k}_{q,\,x}\{h(q^k x)\}. \quad (1.6) $$

It is easy verify the following property of $D_{q,\,x}$:

$$ D^n_{q,\,x}\left\{\frac{(tx;\ q)_\infty}{(sx;\ q)_\infty}\right\} = s^n (t/s;\ q)_n \frac{(tq^n x;\ q)_\infty}{(sx;\ q)_\infty}. \quad (1.7) $$

On stetting $t=0$, we have

$$ D^n_{q,\,x}\left\{\frac{1}{(sx;\ q)_\infty}\right\} = \frac{s^n}{(sx;\ q)_\infty}. \quad (1.8) $$

In [4], Zhi-Guo Liu utilizes Leibniz formula for the q-difference operator to obtain the following q-expansion formula:

$$ f(b) = \sum_{n=0}^{\infty} \frac{(1-aq^{2n})(aq/b;\ q)_n b^n}{(q, b;\ q)_n} \left[D^n_{q,\,x}\{f(x)(x;\ q)_{n-1}\}\right]_{x=aq}, \quad (1.9) $$

where $f(b)$ is a formal series in b. This expansion formula leads to new proofs of the Rogers-Fine identity, the nonterminating ${}_6\phi_5$ summation formula, and Watson's q-analog of Whipple's theorem. Andrew's identities for sums of three squares and suns of three triangular numbers are also derived, etc.

By Jacobi's triple product identity and Ramanujan's ${}_1\psi_1$-bilateral formula, the identities for sums of two squares and sums of four square are proved [3]. In this paper, we shall derive two q-expansion formulas using (1.9). By these new q-expansion formulas, we will provide new proofs of identity for sums of two squares, Gauss's identity , Euler's identity and identity for sums of four squares. By these q-expansion formulas, we give also ${}_6\phi_4$-series, ${}_7\phi_5$-series summation formulas.

2. New Proof of Identity for Sums of Two Squares

By the definition of $D_{q,\,x}$ and the q-expansion formula (1.9), we immediately have the following lemma.

Lemma 2.1.

$$\frac{(bt, aq;\ q)_\infty}{(b, atq;\ q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-aq^{2n})(a, aq/b, tq^{1-n};\ q)_n}{(1-a)(q, b, atq;\ q)_n} b^n q^{n(n-1)}. \tag{2.1}$$

Proof. Now, we apply (1.9) to the function

$$f(b) = \frac{(bt;\ q)_\infty}{(b;\ q)_\infty}. \tag{2.2}$$

Using (1.7) , we find that

$$\begin{aligned}
[D^n_{q,\,x}\{f(x)(x;\ q)_{n-1}\}]_{x=aq} &= \left[D^n_{q,\,x}\left\{\frac{(tx;\ q)_\infty}{(xq^{n-1};\ q)_\infty}\right\}\right]_{x=aq} \\
&= \left[q^{n(n-1)}(tq^{1-n};\ q)_n \frac{(tq^n x;\ q)_\infty}{(xq^{n-1};\ q)_\infty}\right]_{x=aq} \\
&= \frac{(atq;\ q)_\infty (aq;\ q)_{n-1} (tq^{1-n};\ q)_n}{(aq;\ q)_\infty (atq;\ q)_n} q^{n(n-1)}.
\end{aligned}$$

Substituting this into (1.9), we obtain (2.1). □

Now we give our new proof of the identity for sums of two squares.

Theorem 2.2. *(Identity for sums of two squares* [1]*)*

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n r_2(n) q^n &= \left[\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\right]^2 \\
&= \frac{(q;\ q)^2_\infty}{(-q;\ q)^2_\infty} \\
&= 1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+q^n}.
\end{aligned} \tag{2.3}$$

Proof. In (2.1), setting $a = 1,\ t = -1,\ b = -q$, we have

$$\frac{(q;\ q)_\infty^2}{(-q;\ q)_\infty^2} = 1 + \sum_{n=1}^{\infty} \frac{(1 - q^{2n})(q,\ q)_{n-1}(-1, -q^{1-n};\ q)_n}{(q, -q, -q;\ q)_n}(-q)^n q^{n(n-1)}$$

$$= 1 + 4\sum_{n=1}^{\infty} \frac{(-q;\ q)_{n-1}}{q^{n(n-1)/2}(-q;\ q)_n}(-1)^n q^{n^2}$$

$$= 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}.$$

Hence Theorem 2.2 is proved. □

3. New Proofs of Gauss's Identity and Euler's Identity

In (2.1), setting $t = 0$, then

$$\frac{(aq;\ q)_\infty}{(b;\ q)_\infty} = \sum_{n=0}^{\infty} \frac{(1 - aq^{2n})(a, aq/b;\ q)_n}{(1-a)(q, b;\ q)_n} b^n q^{n(n-1)}. \tag{3.1}$$

Hence, we have

Theorem 3.1. *(Gauss's identity* [1]*)*

$$\frac{(q;\ q)_\infty}{(-q;\ q)_\infty} = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2}. \tag{3.2}$$

Proof. In (3.1), setting $a = 1,\ b = -q$, we have

$$\frac{(q;\ q)_\infty}{(-q;\ q)_\infty} = 1 + \sum_{n=1}^{\infty} \frac{(1 - q^{2n})(q,\ q)_{n-1}(-1;\ q)_n}{(q, -q;\ q)_n}(-q)^n q^{n(n-1)}$$

$$= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}$$

$$= \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2}.$$

Or, in (2.1), setting $a = q,\ t = -q^{-1}$, letting $b \to 0$, we have

$$\frac{(q^2;\ q)_\infty}{(-q;\ q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(q;\ q)_n(-q^{-n};\ q)_n(-1)^n q^{n(n+3)/2}}{(1-q)(q,-q;\ q)_n} q^{n(n-1)}$$

$$= \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(-q;\ q)_n(-1)^n q^{-n(n+1)/2} q^{n(n+3)/2}}{(1-q)(-q;\ q)_n} q^{n(n-1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(1-q^{2n+1})}{(1-q)} q^{n^2}$$

$$= \frac{1}{1-q} \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2}.$$

Hence Theorem 3.1 is proved. □

Theorem 3.2. *(Euler's identity* [3]*)*

$$(q;\ q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{3.3}$$

Proof. In (3.1), setting $a = q,\ b \to 0$, we have

$$\begin{aligned}(q^2;\ q)_\infty &= \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(q;\ q)_n(-1)^n q^{n(n+3)/2}}{(1-q)(q;\ q)_n} q^{n(n-1)} \\ &= \sum_{n=0}^{\infty} \frac{(1-q^{2n+1})(-1)^n}{(1-q)} q^{n(3n+1)/2} \\ &= \frac{1}{1-q} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.\end{aligned}$$

Hence Theorem 3.2 is proved. □

4. New Proof of Identity for Sums of Four Squares

In this section, we will provide a new proof of the identities for sums of four squares. First,we have the following lemma.

Lemma 4.1.

$$\frac{(bt, bu, aq, avq;\ q)_\infty}{(b, bv, atq, auq;\ q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-aq^{2n})(a, aq/b;\ q)_n b^n}{(1-a)(q, b, auq;\ q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \times \frac{(tq^{1-n}, avq;\ q)_k (u/v;\ q)_{n-k}}{(atq;\ q)_k} q^{k(n-1)} v^{n-k}. \tag{4.1}$$

Proof Now, we apply (1.9) to the function

$$f(b) = \frac{(bt, bu;\ q)_\infty}{(b, bv;\ q)_\infty}. \tag{4.2}$$

Taking

$$g(x) = \frac{(x;\ q)_{n-1}(tx;\ q)_\infty}{(x;\ q)_\infty} = \frac{(tx;\ q)_\infty}{(xq^{n-1};\ q)_\infty},\quad h(x) = \frac{(ux;\ q)_\infty}{(vx;\ q)_\infty},$$

in the Leibniz formula and using (1.7), we find that

$$[D^n_{q,\ x}\{f(x)(x;\ q)_{n-1}\}]_{x=aq} = \Bigg[\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D^k_{q,\ x}\left\{\frac{(tx;\ q)_\infty}{(xq^{n-1};\ q)_\infty}\right\} \times D^{n-k}_{q,\ x}\left\{\frac{(uq^k x;\ q)_\infty}{(vq^k x;\ q)_\infty}\right\}\Bigg]_{x=aq}.$$

$$\left[D^k_{q,\ x}\left\{\frac{(tx;\ q)_\infty}{(xq^{n-1};\ q)_\infty}\right\}\right]_{x=aq} = \left[q^{k(n-1)}(tq^{1-n};\ q)_k\frac{(tq^k x;\ q)_\infty}{(xq^{n-1};\ q)_\infty}\right]_{x=aq} = \frac{(atq;\ q)_\infty(aq;\ q)_{n-1}(tq^{1-n};\ q)_k}{(aq;\ q)_\infty(atq;\ q)_k}q^{k(n-1)}. \tag{4.3}$$

$$\left[D^{n-k}_{q,\ x}\left\{\frac{(uq^k x;\ q)_\infty}{(vq^k x;\ q)_\infty}\right\}\right]_{x=aq} = \left[v^{n-k}q^{k(n-k)}(u/v;\ q)_{n-k}\frac{(uq^n x;\ q)_\infty}{(vq^k x;\ q)_\infty}\right]_{x=aq} = \frac{(auq;\ q)_\infty(avq;\ q)_k(u/v;\ q)_{n-k}}{(avq;\ q)_\infty(auq;\ q)_n} \times v^{n-k}q^{k(n-k)}. \tag{4.4}$$

So

$$[D^n_{q,\ x}\{f(x)(x;\ q)_{n-1}\}]_{x=aq} = \Bigg[\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D^k_{q,\ x}\left\{\frac{(tx;\ q)_\infty}{(xq^{n-1};\ q)_\infty}\right\} \times D^{n-k}_{q,\ x}\left\{\frac{(uq^k x;\ q)_\infty}{(vq^k x;\ q)_\infty}\right\}\Bigg]_{x=aq}$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(atq, auq;\ q)_\infty(aq;\ q)_{n-1}(tq^{1-n};\ q)_k(avq;\ q)_k(u/v;\ q)_{n-k}}{(aq, avq;\ q)_\infty(atq;\ q)_k(auq;\ q)_n} \times q^{k(n-1)}v^{n-k}.$$

Substituting this into (1.9), we obtain (4.1). □

Theorem 4.2. *(Identity for sums of four squares* [3]*)*

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n r_4(n)q^n &= \left[\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\right]^4 \\ &= \frac{(q;\ q)_\infty^4}{(-q;\ q)_\infty^4} \\ &= 1+8\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{(1+q^n)^2}. \qquad (4.5)\end{aligned}$$

Proof In (4.1), setting $a=v=1,\ t=u=-1,\ b=-q$, we have

$$\begin{aligned}\frac{(q;\ q)_\infty^4}{(-q;\ q)_\infty^4} &= 1+\sum_{n=1}^{\infty}\frac{(1-q^{2n})(q;\ q)_{n-1}(-1;\ q)_n(-q)^n}{(q,-q,-q;\ q)_n} \\ &\quad\times\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q,-q^{1-n};\ q)_k(-1;\ q)_{n-k}}{(-q;\ q)_k}q^{k(n-1)} \\ &= 1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{(-q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q,-q^{1-n};q)_k(-q;q)_{n-k-1}}{(-q;q)_k}q^{k(n-1)} \\ &= 1+4\sum_{n=1}^{\infty}(-1)^n q^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q,-q^{n-k};q)_k(-q;q)_{n-k}}{(-q;q)_k(-q^{n-k};q)_{k+1}}q^{k(k-1)/2} \\ &= 1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+q^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q;\ q)_k}{(-q;\ q)_k}q^{k(k-1)/2}. \qquad (4.6)\end{aligned}$$

We now apply the q-Chu-Vandermonde identity [3]

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k\left(\frac{c}{b}\right)^k\frac{(b;\ q)_k}{(c;\ q)_k}q^{k(k-1)/2}=\frac{(c/b;\ q)_n}{(c;\ q)_n}. \qquad (4.7)$$

In (4.7), taking $c=-q, b=q$, then

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q;\ q)_k}{(-q;\ q)_k}q^{k(k-1)/2}=\frac{2}{1+q^n}. \qquad (4.8)$$

On substituting (4.8) into (4.6), We obtain (4.5). Hence Theorem 4.2 is proved. □

In (2.1), replacing q by $\sqrt{q}$, by (1.4) we obtain a summation formula for ${}_7\phi_5$-series.

Corollary 4.3.

$$
{}_7\phi_5\left[\begin{matrix} a, & \sqrt{aq}, & -\sqrt{aq}, & a\sqrt{q}/b, & tq^{(1-n)/2}, & -, & - \\ & & & \sqrt{a}, & -\sqrt{a}, & b, & at\sqrt{q} \end{matrix}\,\middle|\, \sqrt{q};\ -b\right]
= \frac{(bt, a\sqrt{q};\ \sqrt{q})_\infty}{(b, at\sqrt{q};\ \sqrt{q})_\infty}. \tag{4.9}
$$

In (3.1), replacing q by $\sqrt{q}$, by (1.4) we obtain also a summation formula for ${}_6\phi_4$-series.

Corollary 4.4.

$$
{}_6\phi_4\left[\begin{matrix} a, & \sqrt{aq}, & -\sqrt{aq}, & a\sqrt{q}/b, & -, & - \\ & & & \sqrt{a}, & -\sqrt{a}, & b \end{matrix}\,\middle|\, \sqrt{q};\ -b\right]
= \frac{(a\sqrt{q};\ \sqrt{q})_\infty}{(b;\ \sqrt{q})_\infty}. \tag{4.10}
$$

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